ELSEVIER Linear Algebra and its Applications 318 (2000) 145–172

LINEAR ALGEBRA AND ITS APPLICATIONS

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Symmetric failures in symmetric control systems

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Received 12 January 1999; accepted 5 May 2000

Submitted by R.A. Brualdi

Abstract

This paper discusses the fault-tolerance of symmetric systems with respect to controllability, which is a fundamental characteristic of control systems. In particular, we reveal the underlying mathematical mechanism of the loss of controllability for symmetric systems induced by failures. Based on the decomposition of the symmetric systems into subsystems under the symmetry, the controllability of the entire system can be discussed by checking that of each subsystem. The analysis of the fault-tolerance in this paper is an extension of this idea with the aid of the chain-adapted transformation matrix for the decomposition. The result is shown as a necessary condition for symmetric systems to retain the controllability despite some symmetric failures. We also discuss sufficient conditions. © 2000 Published by Elsevier Science Inc. All rights reserved.

Keywords: Symmetry; Fault-tolerance; Controllability; Group representation

1. Introduction

Mathematical treatment of the symmetry found in various phenomena is systematized based on the group theory. From this standpoint, great deal of researches in diverse fields have been considerably developed, for instance, in bifurcation theory [3,15], quantum mechanics [23], crystallography [25], chemical molecular sys-

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tems [10], structural engineering [7] and so on. Also in the field of control theory, scattered researches have been carried out concerning the control of group-theoretic symmetric systems [4,5,8,14,16]. In [5], it is shown that the characteristics of the symmetric systems can be investigated by those of subsystems obtained by the decomposition based on the group-theoretic symmetry. In addition, Refs. [12,18,19,24] have considered decentralized control systems composed of identical modules that are connected with certain regularity. In such systems, the symmetry of the entire system results from the homogeneity of the constituent modules and the regularity of their connections.

For control systems in general, practical importance of the fault-tolerance has been fully recognized. That is, the systems are desired to retain some characteristics in spite of some failures. There are largely two approaches in fault-tolerant control systems design. The first is by the synthesis of controllers for a given plant to make the entire system fault-tolerant [2,17,22]. The second is by the appropriate design of the plants themselves before the design of the controllers, in order to achieve the fault-tolerance as a whole. Relating to the latter idea, there are some studies on the adjustment of design parameters of plants to increase the fault-tolerance [11].

As for the fault-tolerance of symmetric control systems, there are only studies on the graph-theoretic connectivity [1,6,9] and few researches on the control theoretic characteristics. This paper will discuss the fault-tolerance of symmetric systems with respect to controllability, which is a fundamental characteristic of control systems. We consider the controllability of a system as a characteristic that should be retained in spite of failures in some control channels, and clarify those failures which cause the symmetric system to lose its controllability. A first attempt in this direction is found in [19], where the fault-tolerance of some symmetric systems has been evaluated. The analysis has revealed the failure patterns that make the systems uncontrollable. Whereas Ref. [19] has dealt with the restricted class of symmetric systems whose symmetry as a whole originates in the identity of the modules and certain regularity of their connections, the present paper will be concerned with systems with more general symmetry. Based on the decomposition of the symmetric systems into subsystems under the symmetry, the controllability of the entire system can be discussed by checking that of each subsystem [5]. The analysis of the fault-tolerance in this paper is an extension of this idea with the aid of the chain-adapted transformation matrix for the decomposition.

An interesting relationship between the symmetry and the fault-tolerance has been observed in [19]. That is, when some failures cause a symmetric system to be uncontrollable, the system after the failures has certain symmetry as well. For example, consider a system that consists of nine identical modules connected in a ring as shown in Fig. 1(a). It is symmetric with respect to $2\pi/9$ rotations. The arrows in the figure represent effective inputs. Then, the failures shown in Fig. 1(b) turn out to cause the system to be uncontrollable, where the modules without the arrow are in the outage. As can be seen from the figure, the system after the failures retains a partial symmetry, being symmetric with regard to $2\pi/3$ rotations. Conversely, if the system

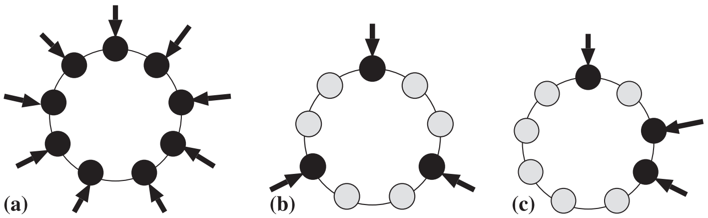

Fig. 1. (a) A symmetric system consisting of nine identical modules connected in ring-type. The arrows represent the effective control inputs. (b) The system is also symmetric but becomes uncontrollable because of the failures. (c) The system is not symmetric and retains its controllability in spite of the failures.

after failure is completely nonsymmetric, the entire system keeps its controllability (Fig. 1(c)). Note that the partial symmetry in the symmetric system results from the symmetric failures. From the observation above, we can deduce that symmetric failure patterns tend to cause the symmetric systems to lose their controllability. Now, a question comes about if all the symmetric failures cause general symmetric systems to be uncontrollable or not.

The example below (Fig. 2) illustrates our main result in this paper: Consider a symmetric spherical diamond system. According to our result, the system shown in Fig. 2(a) turns out to be uncontrollable because of the symmetric failures, whereas the system shown in Fig. 2(b) turns out to retain its controllability despite the symmetry in the failures. Note that the both systems shown in Fig. 2 are symmetric regarding $2\pi/3$ rotations. Based on the group representation theory, the present paper will reveal the underlying mathematical mechanism of the loss of controllability for symmetric systems induced by failures.

The outline of this paper is as follows. In Section 2, we show the main results that reveal the mechanism of the loss of controllability for systems with general symmetry. There, the standard results in the group representation theory are properly

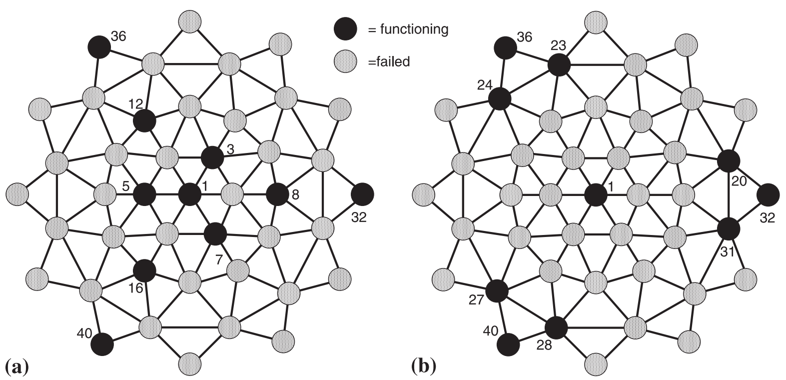

Fig. 2. Two examples of D_3-symmetric failures. (a) Uncontrollable because of D_3-symmetric failures. (b) Controllable in spite of D_3-symmetric failures.

applied to the discussion of the control theoretic characteristics, i.e., the controllability specifically. The result is shown as a necessary condition for symmetric systems to retain the controllability. Section 3 provides some examples of the main results for systems with other symmetries. In Section 4, we discuss sufficient conditions for the controllability of symmetric systems after some symmetric failures. Finally, concluding remarks are given in Section 5.

2. Group theoretic treatment of failure in symmetric systems

2.1. Symmetric failures in symmetric systems

As explained in Section 1, the main concern of this paper is to reveal the mathematical mechanism of the loss of controllability for symmetric systems caused by symmetric failures. In this section, we will formulate the notions of symmetric systems and symmetric failure patterns in precise terms.

Consider a linear time-invariant system $\mathscr{S}$ that consists of m control modules $\{\mathscr{S}_1, \mathscr{S}_2, \ldots, \mathscr{S}_m\}$, each of which has its own control channel. The entire system $\mathscr{S}$ is then described by

$$\begin{bmatrix} \dot{\boldsymbol{x}}_1 \\ \dot{\boldsymbol{x}}_2 \\ \vdots \\ \dot{\boldsymbol{x}}_m \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ A_{21} & A_{22} & \cdots & A_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mm} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \vdots \\ \boldsymbol{x}_m \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1m} \\ B_{21} & B_{22} & \cdots & B_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ B_{m1} & B_{m2} & \cdots & B_{mm} \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \\ \vdots \\ \boldsymbol{u}_m \end{bmatrix}, \tag{1}$$

where $\boldsymbol{x}_i(t) \in \mathbb{R}^{n_i}$ and $\boldsymbol{u}_i(t) \in \mathbb{R}^{r_i}$ denote the state of $\mathscr{S}_i$ and the input from its control channel, respectively, with $\mathbb{R}^n$ being the set of real vectors of dimension n. By denoting the state and the input of the entire system $\mathscr{S}$ as

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \vdots \\ \boldsymbol{x}_m \end{bmatrix} \in \mathbb{R}^n, \quad \boldsymbol{u} = \begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \\ \vdots \\ \boldsymbol{u}_m \end{bmatrix} \in \mathbb{R}^r, \tag{2}$$

respectively, Eq. (1) can be given in the standard form of a state transition equation:

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t). \tag{3}$$

Among the systems described by (3), we are interested in ones with group-theoretic symmetry. We say that system (3) is symmetric with respect to a finite group G if

$$T(g)A = AT(g), \quad T(g)B = BS(g), \qquad g \in G, \tag{4}$$

where T and S are unitary representations of G on $\mathbb{R}^n$ and $\mathbb{R}^r$, respectively (see e.g., [13] for group representation theory). Eq. (4) often reflects the underlying geometric symmetry in the system structure. It should be mentioned here that we are interested in the characteristics of a system determined by its symmetric structure and not by the numerical information of the system matrices A and B.

Example 1. The formulation above is illustrated for a ring-type homogeneous system, as shown in Fig. 1(a), consisting of nine identical modules ($m = 9$) with $n_i = n_0$ and $r_i = r_0$ ($1 \leqslant i \leqslant 9$). The matrices A and B in (3) are given as

$$\begin{aligned} A &= \begin{bmatrix} P & Q & O & O & O & O & O & O & Q \\ Q & P & Q & O & O & O & O & O & O \\ O & Q & P & Q & O & O & O & O & O \\ O & O & Q & P & Q & O & O & O & O \\ O & O & O & Q & P & Q & O & O & O \\ O & O & O & O & Q & P & Q & O & O \\ O & O & O & O & O & Q & P & Q & O \\ O & O & O & O & O & O & Q & P & Q \\ Q & O & O & O & O & O & O & Q & P \end{bmatrix}, \\ B &= \begin{bmatrix} K & O & O & O & O & O & O & O & O \\ O & K & O & O & O & O & O & O & O \\ O & O & K & O & O & O & O & O & O \\ O & O & O & K & O & O & O & O & O \\ O & O & O & O & K & O & O & O & O \\ O & O & O & O & O & K & O & O & O \\ O & O & O & O & O & O & K & O & O \\ O & O & O & O & O & O & O & K & O \\ O & O & O & O & O & O & O & O & K \end{bmatrix}, \end{aligned} \tag{5}$$

where the modules $\{\mathscr{S}_i\}$ ($1 \leqslant i \leqslant 9$) are indexed clockwise from an arbitrary module. The matrices P and Q in (5) are $n_0 \times n_0$ and K is $n_0 \times r_0$. The system is therefore symmetric with respect to the dihedral group D_9. The dihedral group D_9, of order 18, is defined by

$$\mathrm{D}_9 = \{e, \rho, \rho^2, \ldots, \rho^8; \sigma, \sigma\rho, \ldots, \sigma\rho^8\} \tag{6}$$

with $\rho^9 = \sigma^2 = (\sigma\rho)^2 = e$ (e is the identity element). The group D_9 generally represents the geometric symmetry of a regular nonagon. The representations $T(g)$ in (4) for D_9 are given by

$$T(\rho)=\begin{bmatrix} O & O & O & O & O & O & O & O & I \\ I & O & O & O & O & O & O & O & O \\ O & I & O & O & O & O & O & O & O \\ O & O & I & O & O & O & O & O & O \\ O & O & O & I & O & O & O & O & O \\ O & O & O & O & I & O & O & O & O \\ O & O & O & O & O & I & O & O & O \\ O & O & O & O & O & O & I & O & O \\ O & O & O & O & O & O & O & I & O \end{bmatrix},$$

$$T(\sigma)=\begin{bmatrix} I & O & O & O & O & O & O & O & O \\ O & O & O & O & O & O & O & O & I \\ O & O & O & O & O & O & O & I & O \\ O & O & O & O & O & O & I & O & O \\ O & O & O & O & O & I & O & O & O \\ O & O & O & O & I & O & O & O & O \\ O & O & O & I & O & O & O & O & O \\ O & O & I & O & O & O & O & O & O \\ O & I & O & O & O & O & O & O & O \end{bmatrix}, \tag{7}$$

where I denotes the unit matrix of order n_0.

In order to discuss the fault-tolerance of the symmetric systems, we restrict the failure to that of the control channels. Generally, if a control channel of the module, say $\mathscr{S}_i$, is in the outage or replacement, the control input $\boldsymbol{u}_i(t)$ has no influence on state $\boldsymbol{x}(t)$. This situation is described in the mathematical model (1) by

$$\boldsymbol{u}_i(t) = \boldsymbol{0} \quad (\mathscr{S}_i \text{ is in the outage}). \tag{8}$$

According to the failure defined in (8), let M and N denote the index sets of the functioning modules and of the modules in the outage, respectively ($M \cap N = \emptyset$, $M \cup N = \{1, \ldots, m\}$). The failure pattern of the system is thus described by the pair of M and N. In addition, we introduce the failure matrix F of order r in such a way that the matrix BF has zero column blocks that correspond to the control channels in the outage. Such a matrix F is given by $F = \bigoplus_{i=1}^{m} F_i$ with

$$F_i = \begin{cases} I_{r_i} & (i \in M), \\ O_{r_i} & (i \in N), \end{cases}$$

where the matrices I_k and O_k denote, respectively, the unit matrix and the zero matrix of order k in general. This means

$$F = F_M \oplus F_N = I_f \oplus O_{r-f} \tag{9}$$

with $F_M = \bigoplus_{i \in M} F_i = I_f$, $F_N = \bigoplus_{i \in N} F_i = O_{r-f}$, $f = \sum_{i \in M} r_i$, by an appropriate permutation of the indices of the modules. Note that any failure pattern can be given in the form of (9) and that the system after the failures is denoted as (A, BF).

The symmetry of a failure pattern can then be formulated similarly to (4) for the matrix F in (9). A failure pattern F is said to be symmetric with respect to a subgroup H of G if

$$S(h)F = FS(h), \quad h \in H. \tag{10}$$

Note that the symmetry of failure patterns is defined with respect to a subgroup H of G. Given a subgroup H of G, we have a set of failure patterns F that satisfy (10). Conversely, for a given F,

$$H(F) = \{g \in G \mid S(g)F = FS(g)\} \tag{11}$$

is a subgroup of G and can be chosen as the subgroup H in (10).

From the assumption above, the form of the unitary representation matrices $S(h)$ satisfying (10) is restricted to

$$S(h) = S_M(h) \oplus S_N(h), \quad h \in H, \tag{12}$$

where $S_M(h)$ is of order f and $S_N(h)$ is of order $(r - f)$, corresponding to the blocks of F in (9). Namely, the representation matrices $S(h)$ in (10) splits into diagonal blocks for all $h \in H$.

Example 2. The formulation above is illustrated for the system shown in Example 1 (Fig. 1(b)). The matrices A and B are given in (5) and the corresponding matrix representations $T(g)$ in (4) are given in (7). Similarly, representation matrices $S(g)$ for (4) are given as (7), with I denoting the unit matrix of order r_0.

The failure pattern of the system is described by $M = \{1, 4, 7\}$ and $N = \{2, 3, 5, 6, 8, 9\}$. Whereas the system (A, B) is symmetric with respect to the dihedral group D_9, the failure is symmetric with respect to $\mathrm{D}_3 = \{e, \rho^3, \rho^6; \sigma, \sigma\rho^3, \sigma\rho^6\}$, which is a subgroup of D_9.

The failure matrix is given by

$$F = \begin{array}{c|ccccccccc|}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \hline
1 & I & O & O & O & O & O & O & O & O \\
2 & O & O & O & O & O & O & O & O & O \\
3 & O & O & O & O & O & O & O & O & O \\
4 & O & O & O & I & O & O & O & O & O \\
5 & O & O & O & O & O & O & O & O & O \\
6 & O & O & O & O & O & O & O & O & O \\
7 & O & O & O & O & O & O & I & O & O \\
8 & O & O & O & O & O & O & O & O & O \\
9 & O & O & O & O & O & O & O & O & O \\ \hline
\end{array}, \tag{13}$$

which satisfies (10) for $H = \mathrm{D}_3$ with

$$
S(\rho^3) = \begin{array}{c|ccccccccc|}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \hline
1 & O & O & O & O & O & O & I & O & O \\
2 & O & O & O & O & O & O & O & I & O \\
3 & O & O & O & O & O & O & O & O & I \\
4 & I & O & O & O & O & O & O & O & O \\
5 & O & I & O & O & O & O & O & O & O \\
6 & O & O & I & O & O & O & O & O & O \\
7 & O & O & O & I & O & O & O & O & O \\
8 & O & O & O & O & I & O & O & O & O \\
9 & O & O & O & O & O & I & O & O & O \\ \hline
\end{array},
$$

$$
S(\sigma) = \begin{array}{c|ccccccccc|}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \hline
1 & I & O & O & O & O & O & O & O & O \\
2 & O & O & O & O & O & O & O & O & I \\
3 & O & O & O & O & O & O & O & I & O \\
4 & O & O & O & O & O & O & I & O & O \\
5 & O & O & O & O & O & I & O & O & O \\
6 & O & O & O & O & I & O & O & O & O \\
7 & O & O & O & I & O & O & O & O & O \\
8 & O & O & I & O & O & O & O & O & O \\
9 & O & I & O & O & O & O & O & O & O \\ \hline
\end{array}. \tag{14}
$$

A permutation of the modules makes the matrix F in (13) into the form of (9) as

$$
F = \begin{array}{c|ccc|cccccc|}
 & 1 & 4 & 7 & 2 & 3 & 5 & 6 & 8 & 9 \\ \hline
1 & I & O & O & O & O & O & O & O & O \\
4 & O & I & O & O & O & O & O & O & O \\
7 & O & O & I & O & O & O & O & O & O \\ \hline
2 & O & O & O & O & O & O & O & O & O \\
3 & O & O & O & O & O & O & O & O & O \\
5 & O & O & O & O & O & O & O & O & O \\
6 & O & O & O & O & O & O & O & O & O \\
8 & O & O & O & O & O & O & O & O & O \\
9 & O & O & O & O & O & O & O & O & O \\ \hline
\end{array}. \tag{15}
$$

Accordingly, S in (14) takes the form of (12):

$$
\begin{aligned}
S(\rho^3) &= \begin{array}{c|ccc|cccccc|}
 & 1 & 4 & 7 & 2 & 3 & 5 & 6 & 8 & 9 \\ \hline
1 & O & O & I & O & O & O & O & O & O \\
4 & I & O & O & O & O & O & O & O & O \\
7 & O & I & O & O & O & O & O & O & O \\ \hline
2 & O & O & O & O & O & O & O & I & O \\
3 & O & O & O & O & O & O & O & O & I \\
5 & O & O & O & I & O & O & O & O & O \\
6 & O & O & O & O & I & O & O & O & O \\
8 & O & O & O & O & O & I & O & O & O \\
9 & O & O & O & O & O & O & I & O & O \\ \hline
\end{array}\,, \\
S(\sigma) &= \begin{array}{c|ccc|cccccc|}
 & 1 & 4 & 7 & 2 & 3 & 5 & 6 & 8 & 9 \\ \hline
1 & I & O & O & O & O & O & O & O & O \\
4 & O & O & I & O & O & O & O & O & O \\
7 & O & I & O & O & O & O & O & O & O \\ \hline
2 & O & O & O & O & O & O & O & O & I \\
3 & O & O & O & O & O & O & O & I & O \\
5 & O & O & O & O & O & O & I & O & O \\
6 & O & O & O & O & O & I & O & O & O \\
8 & O & O & O & O & I & O & O & O & O \\
9 & O & O & O & I & O & O & O & O & O \\ \hline
\end{array}\,.
\end{aligned}
\tag{16}
$$

Remark 3. From Eqs. (4) and (10), the system after the failures (A, B_F) with $B_F = BF$ is also symmetric with respect to H in the sense of (4), since

$$T(h)A = AT(h), \quad T(h)B_F = B_F S(h), \quad h \in H.$$

Therefore, the symmetry of the original system and that of the failure patterns yield a partial symmetry of the system after the failures.

Our concern in the present paper is whether the controllability is retained in the symmetric system after some symmetric failures. The main result shown below will clarify that the groups G and H and the representation S of G play a crucial role to determine the rank of the matrix $\mathscr{C}(A, BF)$, where $\mathscr{C}(A, B) = [B\ AB \cdots A^{n-1}B]$ is the controllability matrix of a system (A, B).

2.2. Main result

The main result of this paper is stated in this section in the form of Theorem 4, which gives a group-theoretic condition for the controllability after an H-symmetric failure in a G-symmetric system.

In the following, we consider the state space $\mathscr{X} \simeq \mathbb{C}^n$ and the input space $\mathscr{U} \simeq \mathbb{C}^r$ for the simplicity of mathematical treatment,[1] although the systems formulated

[1] Namely, we consider the complexifications of $\mathscr{X}$ and $\mathscr{U}$.

above are described on real spaces. The family of all nonequivalent absolutely irreducible matrix representations of G is denoted by $\{D_G^\mu \mid \mu \in R(G)\}$, where D_G^μ is a unitary irreducible matrix representation, of dimension N^μ, over $\mathbb{C}$ and $R(G)$ is the index set for the absolutely irreducible representations of G. Let the representations $T(g)$ and $S(g)$ of G be decomposed into diagonal blocks of irreducible representations (see (27)) as

$$T = \sum_{\mu \in R(G)} a^\mu \mu, \quad S = \sum_{\mu \in R(G)} b^\mu \mu, \tag{17}$$

where the nonnegative integers a^μ and b^μ are the multiplicities of μ in T and S, respectively.

Consider an H-symmetric failure F, where H is a subgroup of G. The family of all nonequivalent irreducible matrix representations of H is denoted by $\{D_H^\nu \mid \nu \in R(H)\}$, where the dimension of D_H^ν is denoted by N^ν. Let the representations $S_M(h)$ and $S_N(h)$ of H, defined in (12), be decomposed into diagonal blocks of irreducible representations as

$$S_M = \sum_{\nu \in R(H)} b_M^\nu \nu, \quad S_N = \sum_{\nu \in R(H)} b_N^\nu \nu \tag{18}$$

with the multiplicities b_M^ν and b_N^ν of ν in S_M and S_N, respectively.

An important technical ingredient in our argument is the use of chain-adapted bases with respect to G and its subgroup H. Let the restriction of irreducible representations μ of G to H be described as

$$\mu \downarrow H = \sum_{\nu \in R(H)} \alpha_\mu^\nu \nu, \quad \mu \in R(G), \tag{19}$$

with nonnegative integers α_μ^ν representing the multiplicity of ν in $\mu \downarrow H$, the restriction of μ to H.

Then, a necessary condition of group-theoretic nature for the controllability is obtained as follows. The proof will be given in Section 2.3.

Theorem 4. *A G-symmetric system (A, B) retains its controllability in spite of an H-symmetric failure F only if there exists no pair of $\mathbb{C}$-irreducible representation μ of G and ν of H such that*

$$a^\mu \neq 0, \quad \alpha_\mu^\nu \neq 0, \quad b_M^\nu = 0, \tag{20}$$

where a^μ, α_μ^ν and b_M^ν are defined by (17), (19) *and* (18), *respectively.*

The three conditions in (20) are concerned with T, the pair of G and H, and S, respectively. Condition (20) often turns out to be sufficient, as we will see later in Section 4. Therefore, as a rule of thumb, we may hopefully expect that the system retains its controllability if condition (20) is not satisfied by any $\mu \in R(G)$ and $\nu \in R(H)$.

Remark 5. Whereas the conditions in (20) are given for $\mathbb{C}$-irreducible representations μ of G and ν of H, it is noted that the same statement of Theorem 4 holds true when $\mathbb{C}$-irreducibility is replaced by $\mathbb{R}$-irreducibility.

Moreover, the following theorem is derived concerning the rank deficiency of the controllability matrix by H-symmetric failures.

Theorem 6. *By an H-symmetric failure in a G-symmetric system, the rank of the controllability matrix is reduced at least by* $\sum_{(\mu,\nu)\in\mathbf{F}} a^{\mu}\alpha_{\mu}^{\nu}N^{\nu}$, *where* $\mathbf{F}$ *denotes the family of all the pairs* (μ,ν) *which satisfy* (20), *i.e.*,

$$\mathbf{F} = \left\{(\mu,\nu) \in R(G) \times R(H) \mid a^{\mu} \neq 0, \alpha_{\mu}^{\nu} \neq 0, b_M^{\nu} = 0\right\}. \tag{21}$$

Theorem 6 implies that a G-symmetric system (A, B) becomes uncontrollable by an H-symmetric failure F if the subset $\mathbf{F}$ is nonempty, which is the statement of Theorem 4.

2.3. Proofs

We prove Theorems 4 and 6. The spaces $\mathscr{X}$ and $\mathscr{U}$ are decomposed as direct sums of the invariant subspaces corresponding to D_G^{μ}, which is (often) abbreviated to D^{μ}, as

$$\mathscr{X} = \bigoplus_{\mu\in R(G)} \bigoplus_{i=1}^{a^{\mu}} \mathscr{X}_i^{\mu}, \quad \mathscr{U} = \bigoplus_{\mu\in R(G)} \bigoplus_{j=1}^{b^{\mu}} \mathscr{U}_j^{\mu}, \tag{22}$$

where the nonnegative integers a^{μ} and b^{μ} are the multiplicities. We define two unitary matrices Z of order n and W of order r as

$$Z = \left(Z^{\mu} \mid \mu \in R(G)\right), \quad W = \left(W^{\mu} \mid \mu \in R(G)\right), \tag{23}$$

where

$$\begin{aligned} Z^{\mu} &= \left(Z_i^{\mu} \mid 1 \leqslant i \leqslant a^{\mu}\right) \in \mathbb{C}^{n\times a^{\mu}N^{\mu}}, \\ W^{\mu} &= (W_j^{\mu} \mid 1 \leqslant j \leqslant b^{\mu}) \in \mathbb{C}^{r\times b^{\mu}N^{\mu}}, \end{aligned} \tag{24}$$

with $Z_i^{\mu} \in \mathbb{C}^{n\times N^{\mu}}$ and $W_j^{\mu} \in \mathbb{C}^{r\times N^{\mu}}$ being sets of bases of $\mathscr{X}_i^{\mu}$ and $\mathscr{U}_j^{\mu}$, respectively. Note that the unitarity of T and S allows us to choose the matrices Z and W to be unitary over $\mathbb{C}$, i.e., $Z^*Z = I_n$ and $W^*W = I_r$, where Z^* and W^* are the transposed conjugate matrices of Z and W, respectively. Since Z_i^{μ} and W_j^{μ} are bases of $\mathscr{X}_i^{\mu}$ and $\mathscr{U}_j^{\mu}$, respectively,

$$T(g)Z_i^{\mu} = Z_i^{\mu}D^{\mu}(g), \quad S(g)W_j^{\mu} = W_j^{\mu}D^{\mu}(g), \quad g \in G, \tag{25}$$

holds and thus

$$\begin{aligned} T(g)Z^{\mu} &= Z^{\mu}\left(\bigoplus_{i=1}^{a^{\mu}} D^{\mu}(g)\right), \\ S(g)W^{\mu} &= W^{\mu}\left(\bigoplus_{j=1}^{b^{\mu}} D^{\mu}(g)\right), \quad g \in G, \end{aligned} \tag{26}$$

where $\bigoplus_{i=1}^{a^{\mu}} D^{\mu}(g)$ denotes a block-diagonal matrix consisting of a^{μ} identical diagonal blocks $D^{\mu}(g)$. Then the representations $T(g)$ and $S(g)$ of G are decomposed by Z and W into diagonal blocks of irreducible representations as

$$\begin{aligned} Z^{*}T(g)Z &= \bigoplus_{\mu\in R(G)} \bigoplus_{i=1}^{a^{\mu}} D^{\mu}(g), \\ W^{*}S(g)W &= \bigoplus_{\mu\in R(G)} \bigoplus_{j=1}^{b^{\mu}} D^{\mu}(g), \quad g \in G. \end{aligned} \tag{27}$$

From (4) and (27), the system matrices A and B are also block diagonalized by the same matrices Z and W, from Schur's lemma, as

$$\tilde{A} = Z^{*}AZ = \bigoplus_{\mu\in R(G)} \bigoplus_{k=1}^{N^{\mu}} A^{\mu}, \quad \tilde{B} = Z^{*}BW = \bigoplus_{\mu\in R(G)} \bigoplus_{k=1}^{N^{\mu}} B^{\mu} \tag{28}$$

with matrices $A^{\mu} \in \mathbb{C}^{a^{\mu}\times a^{\mu}}$ and $B^{\mu} \in \mathbb{C}^{a^{\mu}\times b^{\mu}}$. Note that $\bigoplus_{k=1}^{N^{\mu}} A^{\mu}$ in (28) denotes a block-diagonal matrix consisting of N^{μ} identical diagonal blocks A^{μ}.

We can take the matrix W_j^{μ} $(1 \leqslant j \leqslant b^{\mu})$ in (24) compatibly with (19) so that it is further decomposed as

$$W_j^{\mu} = (W_j^{\mu\nu} \mid \nu \in R(H)) \tag{29}$$

with each $W_j^{\mu\nu} \in \mathbb{C}^{r\times\alpha_{\mu}^{\nu}N^{\nu}}$ being a set of bases of the invariant subspace corresponding to $\nu \in R(H)$. Such basis $W = (W_j^{\mu\nu} \mid \mu \in R(G), 1 \leqslant j \leqslant b^{\mu}, \nu \in R(H))$ is said to be chain-adapted with respect to G and its subgroup H, which is a key technical ingredient of the proof. Consequently, with the chain-adapted bases W_j^{μ} in (29), the irreducible matrix representation $D_G^{\mu}(g)$ in the second expression of (27) is decomposed as

$$D_G^{\mu}(h) = \bigoplus_{\nu\in R(H)} \bigoplus_{l=1}^{\alpha_{\mu}^{\nu}} D_H^{\nu}(h), \quad h \in H, \tag{30}$$

that is,

$$W^{*}S(h)W = \bigoplus_{\mu\in R(H)} \bigoplus_{j=1}^{b^{\mu}} \left(\bigoplus_{\nu\in R(H)} \bigoplus_{l=1}^{\alpha_{\mu}^{\nu}} D_H^{\nu}(h)\right), \quad h \in H. \tag{31}$$

Since the rank of the controllability matrix is invariant under state transformations,

$$\operatorname{rank} \mathscr{C}(A, B_F) = \operatorname{rank} \mathscr{C}(Z^* A Z, Z^* B_F) \tag{32}$$

holds. From (28) together with the unitarity of W leading to $Z^* B_F = Z^* BW \cdot W^* F$, rank (32) is calculated as

$$\operatorname{rank} \mathscr{C}(Z^* A Z, Z^* B_F) = \operatorname{rank} \left\{ [\tilde{B}\ \tilde{A}\tilde{B} \cdots \tilde{A}^{n-1}\tilde{B}] \left(\bigoplus_{i=1}^{n} W^* F \right) \right\}. \tag{33}$$

Therefore, according to decomposition (28), the controllability of the G-symmetric system (A, B) after the H-symmetric failure F is equivalent to that of the system

$$(\tilde{A}, \tilde{B} W^* F) = \left(\bigoplus_{\mu \in R(G)} \bigoplus_{k=1}^{N^\mu} A^\mu, \bigoplus_{\mu \in R(G)} \left(\left(\bigoplus_{k=1}^{N^\mu} B^\mu \right) (W^\mu)^* F \right) \right). \tag{34}$$

The matrix $(W^\mu)^* F$ in (34) is denoted as

$$(W^\mu)^* F = [(W_M^\mu)^* O] \tag{35}$$

with the failure matrix F in (9), where W_M^μ denotes the first f rows of W^μ as

$$W^\mu = \begin{pmatrix} W_M^\mu \\ W_N^\mu \end{pmatrix}. \tag{36}$$

From (24) and (29), W_M^μ and W_N^μ in (36) are described as

$$W_M^\mu = \left(W_{jM}^{\mu\nu} \mid 1 \leqslant j \leqslant b^\mu, \nu \in R(H) \right),$$

$$W_N^\mu = \left(W_{jN}^{\mu\nu} \mid 1 \leqslant j \leqslant b^\mu, \nu \in R(H) \right),$$

where $W_{jM}^{\mu\nu} \in \mathbb{C}^{f \times \alpha_\mu^\nu N^\nu}$ and $W_{jN}^{\mu\nu} \in \mathbb{C}^{(r-f) \times \alpha_\mu^\nu N^\nu}$. Moreover, with reference to

$$W_M^{\mu\nu} = \left(W_{jM}^{\mu\nu} \mid 1 \leqslant j \leqslant b^\mu \right) \in \mathbb{C}^{f \times (b^\mu \alpha_\mu^\nu N^\nu)}, \tag{37}$$

the matrix W_M^μ is described as

$$W_M^\mu = \left(W_M^{\mu\nu} \mid \nu \in R(H) \right). \tag{38}$$

Consequently, if $(W_M^\mu)^*$ has a zero row block compatible with the block structure of $\tilde{B}$, the controllability matrix is not of full row-rank, and hence the system becomes uncontrollable. Namely, we obtain the following lemma.

Lemma 7. *A G-symmetric system (A, B) becomes uncontrollable by an H-symmetric failure F if* [2] $W_M^{\mu\nu} = O$ *for some* $\mu \in R(G)$, $\nu \in R(H)$ *with* $a^\mu \alpha_\mu^\nu \neq 0$.

---

[2] We use the convention that "$W = O$" includes the cases where the column-set or the row-set of a matrix W is empty.

Proof. If $W_M^{\mu\nu} = O$ for a pair of $\mu \in R(G)$ and $\nu \in R(H)$ with $a^\mu \alpha_\mu^\nu \neq 0$, the product $(\bigoplus_{k=1}^{N^\mu} B^\mu)(W^\mu)^* F$ in (34) has a zero row block compatible with the decomposition

$$\bigoplus_{k=1}^{N^\mu} B^\mu = \bigoplus_{\nu \in R(H)} \bigoplus_{l=1}^{\alpha_\mu^\nu N^\nu} B^\mu,$$

corresponding to $N^\mu = \sum_{\nu \in R(H)} \alpha_\mu^\nu N^\nu$. $\square$

We are now to clarify a group-theoretic mechanism that produces a zero row block in $(W_M^\mu)^*$, that is, $W_M^{\mu\nu} = O$.

Lemma 8. $W_M^{\mu\nu} = O$ *if* $b_M^\nu = 0$.

Proof. The matrix $(W^\mu)^* F$ satisfies

$$\begin{aligned}(W^\mu)^* F &= (W^\mu)^* S(h)^* S(h) F \\ &= \left(\bigoplus_{j=1}^{b^\mu} D_G^\mu(h) \right)^* (W^\mu)^* F S(h), \quad h \in H, \end{aligned} \tag{39}$$

from (10) and (26). Therefore,

$$\left(\bigoplus_{j=1}^{b^\mu} D_G^\mu(h) \right) (W^\mu)^* F = (W^\mu)^* F S(h), \quad h \in H, \tag{40}$$

holds by the unitarity of D_G^μ. Using the decomposition of $D_G^\mu(h)$ in (30), Eq. (40) is rewritten as

$$\left(\bigoplus_{j=1}^{b^\mu} \bigoplus_{\nu \in R(H)} \bigoplus_{l=1}^{\alpha_\mu^\nu} D_H^\nu(h) \right) (W^\mu)^* F = (W^\mu)^* F S(h), \quad h \in H. \tag{41}$$

Therefore, with the matrix W_M^μ in (38), Eq. (41) is rewritten as

$$\left(\bigoplus_{\nu \in R(H)} \bigoplus_{j=1}^{b^\mu} \bigoplus_{l=1}^{\alpha_\mu^\nu} D_H^\nu(h) \right) (W_M^\mu)^* = (W_M^\mu)^* S_M(h), \quad h \in H. \tag{42}$$

Furthermore, $S_M(h)$ and $S_N(h)$ are decomposed into irreducible representations by unitary matrices, say, P_M of order f and P_N of order $r - f$, respectively. Note that they are defined in a similar way as for Z and W in (23), and S_M and S_N are decomposed as

$$
\begin{aligned}
P_M^* S_M(h) P_M &= \bigoplus_{\nu \in R(H)} \bigoplus_{p=1}^{b_M^\nu} D_H^\nu(h), \quad h \in H, \\
P_N^* S_N(h) P_N &= \bigoplus_{\nu \in R(H)} \bigoplus_{p=1}^{b_N^\nu} D_H^\nu(h), \quad h \in H,
\end{aligned} \tag{43}
$$

with multiplicities b_M^ν and b_N^ν defined in (18). Substituting (43) into (42) leads to

$$
\begin{aligned}
&\left(\bigoplus_{\nu \in R(H)} \bigoplus_{j=1}^{b^\mu} \bigoplus_{l=1}^{\alpha_\mu^\nu} D_H^\nu(h) \right) (W_M^\mu)^* P_M \\
&\quad = (W_M^\mu)^* P_M \left(\bigoplus_{\nu' \in R(H)} \bigoplus_{p=1}^{b_M^{\nu'}} D_H^{\nu'}(h) \right), \quad h \in H.
\end{aligned} \tag{44}
$$

In this expression, the matrix $(W_M^\mu)^* P_M$ is naturally divided into some blocks as

$$
(W_M^\mu)^* P_M = \left(\left((W_M^\mu)^* P_M \right)_{\nu\nu'} \mid \nu \in R(H), \nu' \in R(H) \right), \tag{45}
$$

where $\left((W_M^\mu)^* P_M \right)_{\nu\nu'}$ is of size $b^\mu \alpha_\mu^\nu N^\nu \times b_M^{\nu'} N^{\nu'}$. Then, an application of Schur's lemma to (44) clarifies that for a pair of μ and ν satisfying $b_M^\nu = 0$,

$$
\left((W_M^\mu)^* P_M \right)_{\nu\nu'} = O \tag{46}
$$

holds for all $\nu' \in R(H)$. This is equivalent to $(W_M^{\mu\nu})^* = O$. $\square$

Combination of Lemmas 7 and 8 results in Theorem 6. Moreover, if the subset **F** defined in (21) is nonempty, the system becomes uncontrollable. Hence Theorem 4.

3. Examples

3.1. Spherical diamond system

Consider a spherical diamond system as shown in Fig. 3. It is symmetric with respect to D_6. The dihedral group D_m $(m = 1, 2, \ldots)$ of order $2m$ is defined as

$$
\mathrm{D}_m = \{e, \rho, \ldots, \rho^{m-1}; \sigma, \sigma\rho, \ldots, \sigma\rho^{m-1}\}, \tag{47}
$$

where $\rho^m = \sigma^2 = (\sigma\rho)^2 = e$. The index set of all the irreducible representations of D_m, denoted as $R(\mathrm{D}_m)$, is given by

$$
R(\mathrm{D}_m) = \begin{cases} \{A_1, A_2, B_1, B_2, E_1, E_2, \ldots, E_{(m/2)-1}\} & (m \text{ is even}), \\ \{A_1, A_2, E_1, E_2, \ldots, E_{(m-1/2)}\} & (m \text{ is odd}), \end{cases} \tag{48}
$$

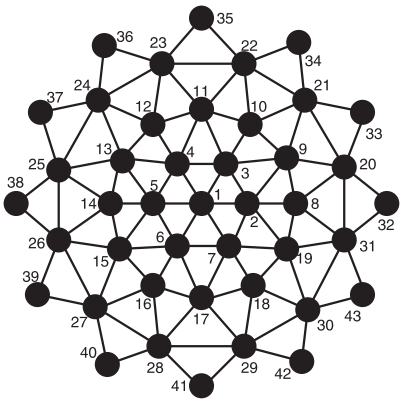

Fig. 3. D_6-symmetric spherical diamond system consisting of identical $m = 43$ modules.

where A_1, A_2, B_1 and B_2 are one-dimensional irreducible matrix representations, and E_i $(i = 1, 2, \ldots)$ are two-dimensional ones. The group D_m generally represents the geometric symmetry of a regular m-gon.

The symmetry condition (4) holds for T and S defined naturally as representations of permutations. In this example, we assume that $n_0 = r_0 = 1$, hence $T = S$ follows. The irreducible representation decomposition of S of D_6 is described as

$$S = 7A_1 \oplus A_2 \oplus 4B_1 \oplus 3B_2 \oplus 7E_1 \oplus 7E_2. \tag{49}$$

The following show D_3-, D_2- and D_6-symmetric failures, each of which consists of two examples: one causes the system to be uncontrollable and the other keeps the system controllable. The system before the failures is D_6-symmetric and controllable. It is worth mentioning that D_3-symmetric failures in D_6-symmetric ring-type homogeneous system does not cause the system to be uncontrollable [19].

Two examples of the D_3-symmetric failures are shown in Fig. 2. The restrictions of the irreducible representations of D_6 to D_3 are given as

$$\begin{aligned} A_1 \downarrow \mathrm{D}_3 &= A_1, \quad & A_2 \downarrow \mathrm{D}_3 &= A_2, \\ B_1 \downarrow \mathrm{D}_3 &= A_1, \quad & B_2 \downarrow \mathrm{D}_3 &= A_2, \\ E_1 \downarrow \mathrm{D}_3 &= E_1, \quad & E_2 \downarrow \mathrm{D}_3 &= E_1. \end{aligned} \tag{50}$$

For the failure pattern shown in Fig. 2(a) with $M = \{1, 3, 5, 7, 8, 12, 16, 32, 36, 40\}$, the representations S_M and S_N of D_3 are decomposed as

$$S_M = 4A_1 \oplus 3E_1, \quad S_N = 7A_1 \oplus 4A_2 \oplus 11E_1. \tag{51}$$

Therefore, condition (20) holds for $(\mu, \nu) = (A_2, A_2), (B_2, A_2)$. Consequently, Theorem 4 reveals that the system is uncontrollable.

The system shown in Fig. 2(b) has the D_3-symmetric failure pattern described as $M = \{1, 20, 23, 24, 27, 28, 31, 32, 36, 40\}$. Note that the number of the functioning modules, i.e., the size of M, is same as that of the system in Fig. 2(a). The representations S_M and S_N are decomposed as

$$S_M = 3A_1 \oplus A_2 \oplus 3E_1, \quad S_N = 8A_1 \oplus 3A_2 \oplus 11E_1. \tag{52}$$

Since condition (20) does not hold for any pair of $\mu \in R(\mathrm{D}_6)$ and $\nu \in R(\mathrm{D}_3)$, the system is likely to be controllable. In fact, Theorem 12 in Section 4 reveals that the system is controllable.

Two examples of the D_2-symmetric failures are shown in Fig. 4. The restrictions of the irreducible representations of D_6 to D_2 are given as

$$\begin{aligned} &A_1 \downarrow \mathrm{D}_2 = A_1, \quad A_2 \downarrow \mathrm{D}_2 = A_2,\\ &B_1 \downarrow \mathrm{D}_2 = B_1, \quad B_2 \downarrow \mathrm{D}_2 = B_2,\\ &E_1 \downarrow \mathrm{D}_2 = B_1 + B_2, \quad E_2 \downarrow \mathrm{D}_2 = A_1 + A_2. \end{aligned} \tag{53}$$

For the failure pattern shown in Fig. 4(b) with $M = \{1, 2, 5, 8, 11, 14, 17, 32, 38\}$, the representations S_M and S_N of D_2 are decomposed as

$$S_M = 5A_1 \oplus 3B_1 \oplus B_2, \quad S_N = 9A_1 \oplus 8A_2 \oplus 8B_1 \oplus 9B_2. \tag{54}$$

Therefore, condition (20) holds for $(\mu, \nu) = (A_2, A_2), (E_2, A_2)$. Consequently, Theorem 4 reveals that the system is uncontrollable.

The system shown in Fig. 4(a) has the D_2-symmetric failure pattern described by $M = \{1, 2, 5, 22, 23, 28, 29, 35, 41\}$. Note that the number of the functioning modules, i.e., the size of M, is the same as that of the system in Fig. 4(b). The representations S_M and S_N are decomposed as

$$S_M = 4A_1 \oplus A_2 \oplus 2B_1 \oplus 2B_2, \quad S_N = 10A_1 \oplus 7A_2 \oplus 9B_1 \oplus 8B_2. \tag{55}$$

Therefore, condition (20) does not hold for any pair of $\mu \in R(\mathrm{D}_6)$ and $\nu \in R(\mathrm{D}_2)$. The system is revealed to be controllable by Theorem 12 in Section 4.

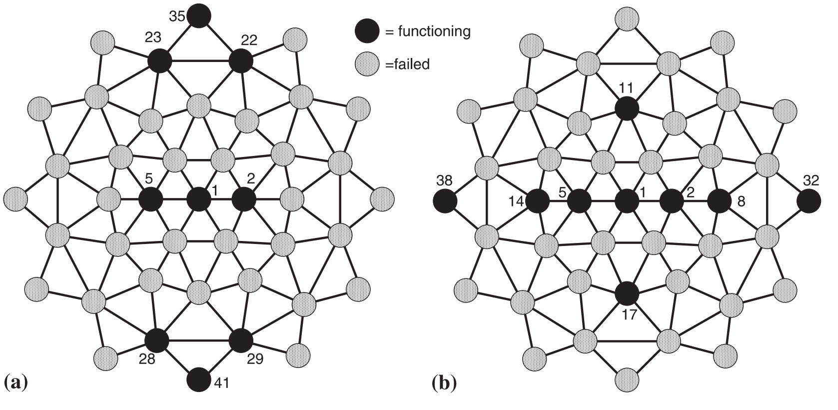

Fig. 4. Two examples of D_2-symmetric failures. (a) Controllable in spite of D_2-symmetric failures. (b) Uncontrollable because of D_2-symmetric failures.

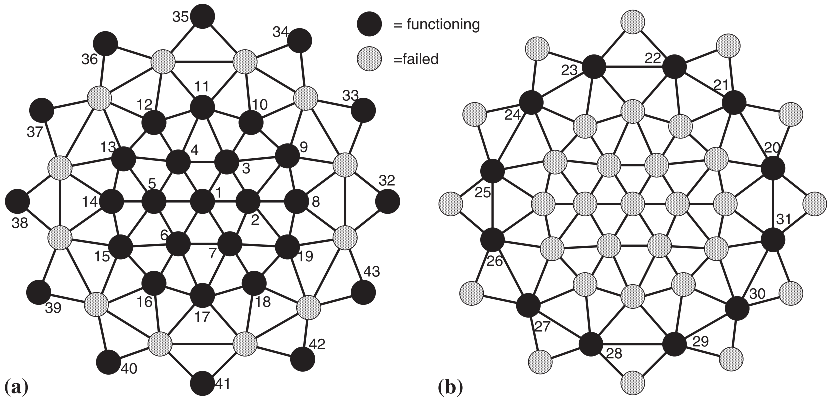

Fig. 5. Two examples of D_6-symmetric failures. (a) Uncontrollable because of D_6-symmetric failures. (b) Controllable in spite of D_6-symmetric failures.

Two examples of the D_6-symmetric failures are shown in Fig. 5. For the failure pattern shown in Fig. 5(a) with $M = \{1, 2, \ldots, 19, 32, 33, \ldots, 43\}$, the representations S_M and S_N of D_6 are decomposed as

$$\begin{aligned} S_M &= 6A_1 \oplus 3B_1 \oplus 2B_2 \oplus 5E_1 \oplus 5E_2, \\ S_N &= A_1 \oplus A_2 \oplus B_1 \oplus B_2 \oplus 2E_1 \oplus 2E_2. \end{aligned} \tag{56}$$

Therefore, condition (20) holds for $\mu = \nu = A_2 \in R(D_6)$. Consequently, Theorem 4 reveals that the system is uncontrollable.

The system shown in Fig. 5(b) has the D_6-symmetric failure pattern described by $M = \{20, 21, \ldots, 31\}$. The representations S_M and S_N are decomposed as

$$\begin{aligned} S_M &= A_1 \oplus A_2 \oplus B_1 \oplus B_2 \oplus 2E_1 \oplus 2E_2, \\ S_N &= 6A_1 \oplus 3B_1 \oplus 2B_2 \oplus 5E_1 \oplus 5E_2. \end{aligned} \tag{57}$$

Therefore, condition (20) does not hold for any $\mu, \nu \in R(D_6)$. The system is revealed to be controllable by Theorem 12 in Section 4.

3.2. Cubic network

Consider a cubic system as shown in Fig. 6(a). The system is symmetric with respect to the group O_h of order 48, the symmetry group of the cube. The group is obtained as the direct product of the octahedral group O of order 24 and the group of reflection $D_1 = \{e, \sigma\}$, where $\sigma^2 = e$. The 24 elements of O are classified into five conjugacy classes:

$$\mathscr{E} = \{e\},$$

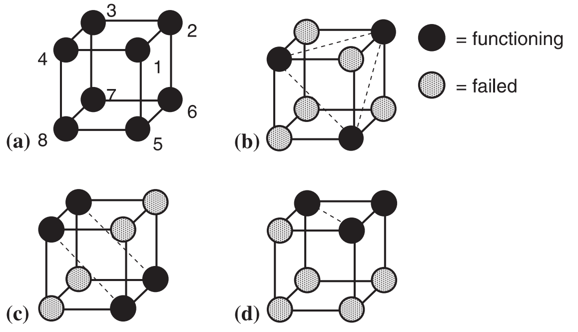

Fig. 6. (a) An example of cubic systems. (b) The system retains its controllability despite the D_3-symmetric failures. (c) The system becomes uncontrollable because of the D_{2h}-symmetric failures. (d) The system retains its controllability despite the D_1-symmetric failures.

$$\mathscr{C}_4^2 = \{\text{three rotations of } \pi \text{ about fourfold axes}\},$$
$$\mathscr{C}_2 = \{\text{six rotations of } \pi \text{ about twofold axes}\},$$
$$\mathscr{C}_4 = \{\text{three rotations of } \pi/2 \text{ and three rotations of } 3\pi/2 \text{ about fourfold axes}\},$$
$$\mathscr{C}_3 = \{\text{four rotations of } 2\pi/3 \text{ and four rotations of } 4\pi/3 \text{ about threefold axes}\},$$

according to the notation of [13]. The index set $R(\mathrm{O})$ of the irreducible representations of O is given by

$$R(\mathrm{O}) = \{A_1, A_2, E, T_1, T_2\},$$

where A_1 is the one-dimensional unit representation, A_2 is a nonunit one-dimensional representation, E is a two-dimensional irreducible representation, and T_1 and T_2 are three-dimensional ones. The products of elements in O with those of D_1 produces the 48 elements of O_h. The index set $R(O_h)$ of the irreducible representations is then given by

Table 1
The character table of the octahedral group O_h

	$\mathscr{E}$	$\mathscr{C}_4^2$	$\mathscr{C}_2$	$\mathscr{C}_4$	$\mathscr{C}_3$	$\mathscr{E}'$	$\mathscr{C}_4^{2\prime}$	$\mathscr{C}_2{}'$	$\mathscr{C}_4{}'$	$\mathscr{C}_3{}'$
A_1	1	1	1	1	1	1	1	1	1	1
A_2	1	1	−1	−1	1	1	1	−1	−1	1
E	2	2	0	0	−1	2	2	0	0	−1
T_1	3	−1	−1	1	0	3	−1	−1	1	0
T_2	3	−1	1	−1	0	3	−1	1	−1	0
A_1'	1	1	1	1	1	−1	−1	−1	−1	−1
A_2'	1	1	−1	−1	1	−1	−1	1	1	−1
E'	2	2	0	0	−1	−2	−2	0	0	1
T_1'	3	−1	−1	1	0	−3	1	1	−1	0
T_2'	3	−1	1	−1	0	−3	1	−1	1	0

$$R(\mathrm{O_h}) = \{A_1, A_2, E, T_1, T_2, A_1', A_2', E', T_1', T_2'\}.$$

The character table of the octahedral group $\mathrm{O_h}$ is given in Table 1.

The system matrices A and B for the cubic system (Fig. 6(a)) are given as

$$A = \left[\begin{array}{cccc|cccc} P & Q & O & Q & Q & O & O & O \\ Q & P & Q & O & O & Q & O & O \\ O & Q & P & Q & O & O & Q & O \\ Q & O & Q & P & O & O & O & Q \\ \hline Q & O & O & O & P & Q & O & Q \\ O & Q & O & O & Q & P & Q & O \\ O & O & Q & O & O & Q & P & Q \\ O & O & O & Q & Q & O & Q & P \end{array}\right],$$
$$B = \left[\begin{array}{cccc|cccc} K & O & O & O & O & O & O & O \\ O & K & O & O & O & O & O & O \\ O & O & K & O & O & O & O & O \\ O & O & O & K & O & O & O & O \\ \hline O & O & O & O & K & O & O & O \\ O & O & O & O & O & K & O & O \\ O & O & O & O & O & O & K & O \\ O & O & O & O & O & O & O & K \end{array}\right]. \tag{58}$$

The symmetry condition (4) holds for T and S defined naturally as representations of permutations. In this example, we assume that $n_0 = r_0 = 1$, hence $T = S$ follows. The irreducible representation decomposition of S of $\mathrm{O_h}$ is described as

$$S = A_1 \oplus T_2 \oplus A_1' \oplus T_2'. \tag{59}$$

We deal with three cases of failures as examples: a $\mathrm{D_3}$-symmetric failure, a $\mathrm{D_{2h}}$-symmetric one and a $\mathrm{D_1}$-symmetric one. The group D_m $(m = 1, 2, \ldots)$ is defined in (47) and the group $\mathrm{D_{2h}}$ of order 8 is defined as the direct product of $\mathrm{D_2}$ and $\mathrm{D_1}$. The index set $R(\mathrm{D_{2h}})$ of the irreducible representations is given by $R(\mathrm{D_{2h}}) = \{A, B_1, B_2, B_3, A', B_1', B_2', B_3'\}$, with the obvious meaning of primes.

An example of the first case with $\mathrm{D_3}$-symmetric failure is shown in Fig. 6(b) with $M = \{2, 4, 5\}$ and $N = \{1, 3, 6, 7, 8\}$. The restrictions of the irreducible representations in (59) to $\mathrm{D_3}$ are

$$A_1^{\mathrm{O_h}} \downarrow \mathrm{D_3} = A_1^{\mathrm{D_3}}, \quad A_1'^{\mathrm{O_h}} \downarrow \mathrm{D_3} = A_1^{\mathrm{D_3}},$$
$$T_2^{\mathrm{O_h}} \downarrow \mathrm{D_3} = A_1^{\mathrm{D_3}} + E_1^{\mathrm{D_3}}, \quad T_2'^{\mathrm{O_h}} \downarrow \mathrm{D_3} = A_1^{\mathrm{D_3}} + E_1^{\mathrm{D_3}},$$

where $\mu^{\mathrm{O_h}}$ denotes an index in $R(\mathrm{O_h})$ and $\nu^{\mathrm{D_3}}$ denotes an index in $R(\mathrm{D_3})$. The superscripts to μ and ν are omitted if there is no danger of confusion. The representations S_M and S_N of $\mathrm{D_3}$ are decomposed as

$$S_M = A_1 \oplus E_1, \quad S_N = 3A_1 \oplus E_1. \tag{60}$$

Since condition (20) is not satisfied by any pair of $\mu \in R(\mathrm{D}_6)$ and $\nu \in R(\mathrm{D}_3)$, Theorem 4 indicates that the system is likely to be controllable. In fact, Theorem 12, which will be shown in Section 4, reveals that the system is controllable. Thus, the system turns out to retain its controllability despite the D_3-symmetric failure with $M = \{2, 4, 5\}$.

The second case is where the failure pattern is symmetric with respect to $\mathrm{D}_{2\mathrm{h}}$. An example of the $\mathrm{D}_{2\mathrm{h}}$-symmetric failure patterns is shown in Fig. 6(c) with $M = \{3, 4, 5, 6\}$ and $N = \{1, 2, 7, 8\}$. The restrictions of the irreducible representations in (59) to $\mathrm{D}_{2\mathrm{h}}$ are

$$A_1^{\mathrm{O_h}} \downarrow \mathrm{D}_{2\mathrm{h}} = A^{\mathrm{D}_{2\mathrm{h}}}, \quad A_1'^{\mathrm{O_h}} \downarrow \mathrm{D}_{2\mathrm{h}} = B_3'^{\mathrm{D}_{2\mathrm{h}}},$$
$$T_2^{\mathrm{O_h}} \downarrow \mathrm{D}_{2\mathrm{h}} = A^{\mathrm{D}_{2\mathrm{h}}} + B_1^{\mathrm{D}_{2\mathrm{h}}} + B_2^{\mathrm{D}_{2\mathrm{h}}}, \quad T_2'^{\mathrm{O_h}} \downarrow \mathrm{D}_{2\mathrm{h}} = B_1'^{\mathrm{D}_{2\mathrm{h}}} + B_2'^{\mathrm{D}_{2\mathrm{h}}} + B_3'^{\mathrm{D}_{2\mathrm{h}}}.$$

The representations S_M and S_N of $\mathrm{D}_{2\mathrm{h}}$ are decomposed as

$$S_M = A \oplus B_1 \oplus B_2' \oplus B_3', \quad S_N = A \oplus B_2 \oplus B_1' \oplus B_3'. \tag{61}$$

Since condition (20) holds for $(\mu, \nu) = (T_2, B_2), (T_2', B_1')$, Theorem 4 reveals that the system is uncontrollable.

The third case is where the failure pattern is symmetric with respect to D_1. An example of the D_1-symmetric failure patterns is shown in Fig. 6(d) with $M = \{1, 2, 3\}$ and $N = \{4, 5, 6, 7, 8\}$. The restrictions of the irreducible representations in (59) to D_1 are

$$A_1^{\mathrm{O_h}} \downarrow \mathrm{D}_1 = A_1^{\mathrm{D}_1}, \quad A_1'^{\mathrm{O_h}} \downarrow \mathrm{D}_1 = A_1^{\mathrm{D}_1},$$
$$T_2^{\mathrm{O_h}} \downarrow \mathrm{D}_1 = 2A_1^{\mathrm{D}_1} + A_2^{\mathrm{D}_1}, \quad T_2'^{\mathrm{O_h}} \downarrow \mathrm{D}_1 = 2A_1^{\mathrm{D}_1} + A_2^{\mathrm{D}_1}.$$

The representations S_M and S_N of D_1 are decomposed as

$$S_M = 2A_1 \oplus A_2, \quad S_N = 4A_1 \oplus A_2. \tag{62}$$

Since condition (20) does not hold for any pair of $\mu \in R(\mathrm{O_h})$ and $\nu \in R(\mathrm{D}_1)$, the system is expected to be controllable. Actually, Theorem 12 in Section 4 confirms its controllability.

4. Sufficient condition for controllability

The main theorem of the present paper (Theorem 4) describes only a necessary condition for the controllability of a G-symmetric system after an H-symmetric failure. This section discusses sufficient conditions. The main result of this section, giving a sufficient condition for the controllability after some failures, is shown in Theorem 12, which has been applied to determine the controllability of the systems in Figs. 2(b), 4(a) and 5(b) in Section 3.1. We will then discuss the relationship between the sufficient condition in Theorem 12 and the necessary condition in Theorem

4, and clarify that the condition in Theorem 4 is necessary and sufficient if $H = G$, that is, if G-symmetric failures occur in G-symmetric systems as is the case in Fig. 5.

Our concern in this section is the controllability of the G-symmetric system after some failures. The symmetry of the system (A, B) has been defined in terms of Eq. (4) for the group G and its unitary representations T and S. We consider a generic system subject to this symmetry constraint. Since (A, B) satisfies this symmetry constraint if and only if A and B are decomposed as (28), the genericity with respect to the symmetry condition (4) is equivalent to the genericity of the matrices A^{μ} and B^{μ} in (28) in the sense that all of their entries are independent parameters. It should be emphasized that the following discussion is based on the genericity under the symmetry.

The controllability of the G-symmetric system (A, B) after a failure F is equivalent to that of system (34). Note that the matrices A^{μ} and $A^{\mu'}$ $(\mu \neq \mu')$ have no common eigenvalues by their genericity. The discussion above, together with the well-known lemma below, leads us to Lemma 10.

Lemma 9. *Suppose the matrices A and B for a system (A, B) are given as*

$$A = \begin{bmatrix} A_1 & O \\ O & A_2 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},$$

and square sub-matrices A_1 and A_2 have no common eigenvalues. Then the system (A, B) is controllable if and only if two subsystems (A_1, B_1) and (A_2, B_2) are controllable.

Lemma 10. *A generic G-symmetric system (A, B) remains to be controllable by a failure F if and only if $(\bigoplus_{k=1}^{N^{\mu}} A^{\mu}, (\bigoplus_{k=1}^{N^{\mu}} B^{\mu})(W^{\mu})^* F)$ is controllable for each μ satisfying $a^{\mu} \neq 0$.*

Accordingly, we are to investigate the sufficient condition for the controllability of the subsystem $(\bigoplus_{k=1}^{N^{\mu}} A^{\mu}, (\bigoplus_{k=1}^{N^{\mu}} B^{\mu})(W^{\mu})^* F)$ for μ with $a^{\mu} \neq 0$.

A sufficient condition for the controllability of a system $(\bigoplus_{k=1}^{N} A, (\bigoplus_{k=1}^{N} B)W^*)$ in general is given in the following, which serves as a key technical lemma of this section. The controllability of the system will be proved with reference to the rank of $\mathscr{C}_{\mathrm{m}}(A, B) = [A - \lambda I \mid B]$, $\lambda \in \mathbb{C}$, the modal controllability matrix of the system (A, B).

Lemma 11. *Consider a system $(\bar{A}, \bar{B})$ with the matrices $\bar{A}$ and $\bar{B}$ given as*

$$\bar{A} = \bigoplus_{k=1}^{N} A = \begin{bmatrix} A & & & \\ & A & & \\ & & \ddots & \\ & & & A \end{bmatrix}, \tag{63}$$

$$\bar{B} = \left(\bigoplus_{k=1}^{N} B\right) W^* = \begin{bmatrix} B(W^1)^* \\ B(W^2)^* \\ \vdots \\ B(W^N)^* \end{bmatrix},$$

where $A \in \mathbb{C}^{a\times a}$, $B \in \mathbb{C}^{a\times b}$, *and* $W = (W^k \mid 1 \leqslant k \leqslant N) \in \mathbb{C}^{f\times bN}$ *with* $W^k = (\boldsymbol{w}_j^k \mid 1 \leqslant j \leqslant b) \in \mathbb{C}^{f\times b}$. *Suppose that A and B are fully dense generic matrices. Then, the system* $(\bar{A}, \bar{B})$ *is controllable if there exists an integer* j $(1 \leqslant j \leqslant b)$ *such that*

$$\operatorname{rank} W_j^* = N, \tag{64}$$

where

$$W_j^* = \begin{bmatrix} (\boldsymbol{w}_j^1)^* \\ (\boldsymbol{w}_j^2)^* \\ \vdots \\ (\boldsymbol{w}_j^N)^* \end{bmatrix} \in \mathbb{C}^{N\times f}.$$

Proof. The modal controllability matrix of the system $(\bar{A}, \bar{B})$ is given as

$$\mathscr{C}_{\mathrm{m}}(\bar{A}, \bar{B}) = \left[\begin{array}{cccc|c} A-\lambda I & & & & B(W^1)^* \\ & A-\lambda I & & & B(W^2)^* \\ & & \ddots & & \vdots \\ & & & A-\lambda I & B(W^N)^* \end{array}\right], \quad \lambda \in \mathbb{C}. \tag{65}$$

If the equation

$$\begin{bmatrix} \boldsymbol{\xi}_1^* & \boldsymbol{\xi}_2^* & \cdots & \boldsymbol{\xi}_N^* \end{bmatrix} \mathscr{C}_{\mathrm{m}}(\bar{A}, \bar{B}) = O \tag{66}$$

holds only for $[\boldsymbol{\xi}_1^*\boldsymbol{\xi}_2^* \cdots \boldsymbol{\xi}_N^*] = O$, where $\boldsymbol{\xi}_k \in \mathbb{C}^a$ $(1 \leqslant k \leqslant N)$, the modal controllability matrix is of full row-rank, and thus the system $(\bar{A}, \bar{B})$ is controllable.

Eq. (66) is equivalent to

$$\boldsymbol{\xi}_k^*(A - \lambda I) = O \quad (1 \leqslant k \leqslant N), \tag{67}$$

$$\sum_{k=1}^{N} \boldsymbol{\xi}_k^* B(W^k)^* = O. \tag{68}$$

Eq. (67) shows that $\boldsymbol{\xi}_k^*$ $(1 \leqslant k \leqslant N)$ are left eigenvectors of A for the eigenvalue λ. By the assumed genericity of A, the eigenspace for an eigenvalue is one-dimensional. Therefore, the vectors $\boldsymbol{\xi}_k$ $(1 \leqslant k \leqslant N)$ are given as $\boldsymbol{\xi}_k = \alpha_k \boldsymbol{\xi}$ with $\alpha_k \in \mathbb{C}$ and a nonzero vector $\boldsymbol{\xi} \in \mathbb{C}^a$ satisfying $\boldsymbol{\xi}^* A = \lambda \boldsymbol{\xi}^*$. Then Eq. (68) is rewritten as

$$\sum_{k=1}^{N} \alpha_k \boldsymbol{\eta}^* (W^k)^* = O, \tag{69}$$

where $\boldsymbol{\eta} \in \mathbb{C}^b$ is given as

$$\boldsymbol{\eta}^* = \boldsymbol{\xi}^* B. \tag{70}$$

From condition (64), there exists an integer j_0 $(1 \leqslant j_0 \leqslant b)$ such that rank $W^*_{j_0} = N$. Therefore, the matrix $W^*_{j_0}$ contains N independent columns with the column indices $\{i_1, i_2, \ldots, i_N\}$, for which

$$\det W^*_{j_0}[i_1, i_2, \ldots, i_N] \neq 0, \tag{71}$$

$$\operatorname{rank} W^*[i_1, i_2, \ldots, i_N] = N, \tag{72}$$

where $W^*_{j_0}[i_1, i_2, \ldots, i_N]$ and $W^*[i_1, i_2, \ldots, i_N]$ denote the submatrices obtained by taking N columns indexed by $\{i_1, i_2, \ldots, i_N\}$ of $W^*_{j_0}$ and W^*, respectively. From Eq. (72), Eq. (69) contains N independent equations

$$\sum_{k=1}^{N} \alpha_k \boldsymbol{\eta}^* (W^k)^* [i_l] = O \quad (l = 1, 2, \ldots, N).$$

This is a system of N equations in N variables $\{\alpha_k \mid 1 \leqslant k \leqslant N\}$. If $(\alpha_1, \alpha_2, \ldots, \alpha_N) \neq (0, 0, \ldots, 0)$, the coefficient matrix must be singular, i.e.,

$$\det(\boldsymbol{\eta}^* (W^k)^* [i_l] \mid 1 \leqslant k, l \leqslant N) = 0 \tag{73}$$

holds. The left-hand side of Eq. (73) is a nontrivial polynomial in $\boldsymbol{\eta}^* = (\eta_1, \eta_2, \ldots, \eta_b)$, since the coefficient of the term $\eta_{j_0}^N$ in $\det(\boldsymbol{\eta}^* (W^k)^* [i_l])$ in (73) is equal to (71). Hence, (73) describes a nontrivial algebraic relation among the elements of A and B, since $\boldsymbol{\eta}^*$ defined in (70) is obtained by the eigenvector $\boldsymbol{\xi}$ of A and the matrix B. However, this is a contradiction to the genericity of A and B. Therefore, $\boldsymbol{\eta}^* = \mathbf{0}$ which leads to $\boldsymbol{\xi} = \mathbf{0}$ and thus the controllability of the system $(\bar{A}, \bar{B})$ is proved. □

Thus, the following theorem, giving a sufficient condition for the controllability, is obtained.

Theorem 12. *A generic G-symmetric system* (A, B) *retains its controllability in spite of a failure F, if, for each* $\mu \in R(G)$ *satisfying* $a^\mu \neq 0$, *there exists an integer* j_μ $(1 \leqslant j_\mu \leqslant b^\mu)$ *such that*

$$\operatorname{rank} \left(W^{\mu}_{j_\mu M} \right)^* = N^\mu. \tag{74}$$

Proof. Apply Lemma 11 to each subsystem $\left(\bigoplus_{k=1}^{N^\mu} A^\mu, \left(\bigoplus_{k=1}^{N^\mu} B^\mu\right)(W_M^\mu)^*\right)$ corresponding to μ. $\square$

Condition (74) means that the projection of the basis $W_{j_\mu}^\mu$ of an invariant subspace $\mathscr{U}_{j_\mu}^\mu$ in (22) onto the space of effective inputs $\mathbb{C}^f$ still serves as a basis.

In the examples in Section 3, such as Figs. 2(b), 4(a), and 5(b), the systems have been shown to be controllable by Theorem 12.

It is mentioned that the converse of the statement of Lemma 11 is not always true, as follows.

Example 13. Consider a system (A, B) given as

$$A = \alpha, \quad B = \begin{bmatrix}\beta_1 & \beta_2\end{bmatrix}, \quad N = 3, \quad W^* = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 1 \\ \hline 0 & 1 & 0 \\ 0 & 0 & 0 \\ \hline 1 & 0 & 0 \\ 0 & 0 & 2 \end{array}\right].$$

Since rank $W_j^* < N$ for all $j \in \{1, 2\}$, the condition is not satisfied. However, $\tilde{B}$ is calculated as

$$\tilde{B} = \begin{bmatrix} \beta_1 & 0 & \beta_2 \\ 0 & \beta_1 & 0 \\ \beta_1 & 0 & 2\beta_2 \end{bmatrix},$$

and thus the system is generically controllable.

In the case of $H = G$, the necessary condition in Theorem 4 turns out to be also sufficient for the controllability of the system.

Theorem 14. *A generic G-symmetric system (A, B) remains to be controllable in spite of a G-symmetric failure F if and only if there exist no absolutely irreducible representations μ of G such that*

$$a^\mu \neq 0, \quad b_M^\mu = 0, \tag{75}$$

where b_M^μ is the multiplicity of μ in the representations S_M of G.

Proof. The necessity has been shown in Theorem 4.

Since the system after the failure is also G-symmetric, the representations $S(g)$ of G are given as

$$S(g) = S_M(g) \oplus S_N(g), \quad g \in G, \tag{76}$$

where S_M and S_N are also representations of G (cf. (12)). The representations $S_M(g)$ and $S_N(g)$ are decomposed into a direct sum of irreducible representations as

$$\begin{aligned} W_M^* S_M(g) W_M &= \bigoplus_{\mu \in R(G)} \bigoplus_{j=1}^{b_M^\mu} D^\mu(g), \quad g \in G, \\ W_N^* S_N(g) W_N &= \bigoplus_{\mu \in R(G)} \bigoplus_{j=1}^{b_N^\mu} D^\mu(g), \quad g \in G, \end{aligned} \tag{77}$$

by unitary matrices W_M and W_N, respectively, where the nonnegative integers b_M^μ and b_N^μ are the multiplicities of μ in S_M and S_N, respectively and thus $b^\mu = b_M^\mu + b_N^\mu$ holds. Moreover, $b_M^\mu \neq 0$ holds for μ satisfying $a^\mu \neq 0$ since there exist no irreducible representations μ of G which satisfy (75). The controllability of the system (A, B_F) after the failure F is to be proved by the controllability of the subsystems $(\bigoplus_{k=1}^{N^\mu} A^\mu, (\bigoplus_{k=1}^{N^\mu} B^\mu)(W^\mu)^* F)$ for each μ satisfying $a^\mu \neq 0$ by Lemma 10.

Corresponding to the decomposition of $S(g)$ in (76), W in (23) is given here as $W = W_M \oplus W_N$. The matrix W^μ is then described as

$$W^\mu = \left[\begin{array}{c|c} W_M^\mu & O \\ \hline O & W_N^\mu \end{array}\right],$$

where $W_M = \left(W_M^\mu \mid \mu \in R(G)\right)$, $W_N = \left(W_N^\mu \mid \mu \in R(G)\right)$ with $W_M^\mu \in \mathbb{C}^{f \times b_M^\mu N^\mu}$ and $W_N^\mu \in \mathbb{C}^{(r-f) \times b_N^\mu N^\mu}$, and thus

$$(W^\mu)^* F = \left[\begin{array}{c|c} (W_M^\mu)^* & O \\ \hline O & O \end{array}\right].$$

Consequently, we are to prove the controllability of the subsystem

$$\left(\bigoplus_{k=1}^{N^\mu} A^\mu, \left(\bigoplus_{k=1}^{N^\mu} B^\mu \right) \begin{bmatrix} (W_M^\mu)^* \\ O \end{bmatrix} \right)$$

on the basis of Lemma 11. Corresponding to the block structure of $(\bigoplus_{k=1}^{N^\mu} B^\mu)$, the matrix W_M^μ is divided into

$$W_M^\mu = \left(\boldsymbol{w}_{Mkj}^\mu \mid 1 \leqslant k \leqslant N^\mu, 1 \leqslant j \leqslant b_M^\mu \right),$$

where $\boldsymbol{w}_{Mkj}^\mu \in \mathbb{C}^f$. Since the matrix W_M has been chosen to be unitary, the vectors $\{\boldsymbol{w}_{Mkj}^\mu \mid 1 \leqslant j \leqslant b_M^\mu, 1 \leqslant k \leqslant N^\mu\}$ are mutually independent. Note that $W_{Mj}^\mu \in \mathbb{C}^{f \times b_M^\mu}$ defined as

$$W_{Mj}^\mu = (\boldsymbol{w}_{Mkj}^\mu \mid 1 \leqslant k \leqslant N^\mu)$$

is the base of an invariant subspace for μ, and thus

$$\operatorname{rank} W_{Mj}^\mu = N^\mu$$

holds for all j $(1 \leqslant j \leqslant b_M^\mu)$, where $b_M^\mu \neq 0$ by the assumption. Consequently, the subsystem is controllable by Lemma 11. $\square$

5. Conclusion

This paper has discussed the fault-tolerance of symmetric systems with respect to controllability. We consider the controllability of a system as a characteristic that should be retained despite failures in some control channels, and have revealed the underlying mathematical mechanism of the loss of controllability for symmetric systems induced by failures. The main result has been clarified in the form of a necessary condition for symmetric systems to retain their controllability in spite of symmetric failures. In the discussion, the standard results in group representation theory have been properly applied. In particular, the irreducible representations have played an important role in order to decompose the symmetric system into subsystems. Moreover, a sufficient condition for the controllability despite the symmetric failures has been also discussed based on the genericity of the subsystems. The condition has been revealed to be necessary and sufficient if G-symmetric failures occur in G-symmetric systems. Further study on the controllability of symmetric systems can be found in [20,21].

Acknowledgement

This work is supported partly by a Grant-in-Aid of the Ministry of Education, Science, Sports and Culture of Japan.

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