Bounds for determinants of meet matrices 
associated with incidence functions

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Abstract

We consider meet matrices on meet-semilattices as an abstract generalization of greatest common divisor (gcd) matrices. Some new bounds for the determinant of meet matrices and a formula for the inverse of meet matrices are given. © 2001 Elsevier Science Inc. All rights reserved.

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1. Introduction

Let \( S = \{x_1, x_2, \ldots, x_n\} \) be a set of \( n \) distinct positive integers. The \( n \times n \) matrix \((S)\) having the greatest common divisor \((x_i, x_j)\) of \( x_i \) and \( x_j \) as its \( i, j \)-entry is called the greatest common divisor (gcd) matrix of \( S \). The set \( S \) is said to be factor-closed if it contains every divisor of \( x \) for any \( x \in S \). The set \( S \) is said to be gcd-closed if \((x_i, x_j) \in S \) for all \( 1 \leq i, j \leq n \). Clearly, a factor-closed set is gcd-closed but not conversely.

Let \( f \) be an arithmetical function and let \((f(x_i, x_j))\) denote the \( n \times n \) matrix having \( f \) evaluated at the greatest common divisor \((x_i, x_j)\) of \( x_i \) and \( x_j \) as its \( i, j \)-entry. Let \( C_S \) denote the class of arithmetical functions defined as

\[
C_S = \{ f \mid (x \in S, d \mid x) \Rightarrow (f \ast \mu)(d) > 0 \},
\]

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where $\ast$ is the Dirichlet convolution and $\mu$ is the number-theoretic Möbius function. Hong [2] showed that if $f \in C_S$, then

$$ \det(f(x_i, x_j)) \geq \prod_{k=1}^{n} \sum_{d|x_k} (f \ast \mu)(d) $$

and the equality holds if and only if $S$ is gcd-closed. Hong [2, Theorem 3] also obtained an upper bound for $\det(f(x_i, x_j))$. (Note that this formula contains some errors.) In this paper we give abstract generalizations of these formulae considering bounds for determinants of meet matrices on meet-semilattices. Haukkanen [1] has previously studied meet matrices on meet-semilattices and this paper continues his work. Note that some notations differ from those used in [1].

2. Definitions

Let $(P, \leq)$ be a meet-semilattice such that the principal order ideal $\downarrow x = \{ y \in P \mid y \leq x \}$ is finite for all $x \in P$.

Let $S$ be a subset of $P$. We say that $S$ is lower-closed if for every $x, y \in P$ with $x \in S$ and $y \leq x$, we have $y \in S$. We say that $S$ is meet-closed if for every $x, y \in S$, we have $x \land y \in S$. Obviously the concepts “lower-closed” and “meet-closed” are generalizations of the concepts “factor-closed” and “gcd-closed”, respectively. It is also clear that a lower-closed set is always meet-closed but not conversely. The order ideal generated by $S$ is given as $\downarrow S = \{ y \in P \mid \exists x \in S : y \leq x \}$. Obviously $\downarrow S$ is the minimal lower-closed set containing $S$.

Let $f$ be a complex-valued function on $P \times P$ such that $f(x, y) = 0$ whenever $x \not\leq y$. Then we say that $f$ is an incidence function of $P$. If $f$ and $g$ are incidence functions of $P$, their sum $f + g$ is defined by $(f + g)(x, y) = f(x, y) + g(x, y)$ and their convolution $f \ast g$ is defined by $(f \ast g)(x, y) = \sum_{x \leq z \leq y} f(x, z)g(z, y)$. The set of all incidence functions of $P$ with addition and convolution forms a ring, where the identity $\delta$ is defined by $\delta(x, y) = 1$ if $x = y$, and $\delta(x, y) = 0$ otherwise. The incidence function $\zeta$ is defined by $\zeta(x, y) = 1$ if $x \leq y$, and $\zeta(x, y) = 0$ otherwise. The Möbius function $\mu$ of $P$ is the inverse of $\zeta$.

In what follows, let $P$ be a meet-semilattice such that all principal order ideals of $P$ are finite. Furthermore, let $S$ be a finite subset of $P$, and denote $S = \{ x_1, x_2, \ldots, x_n \}$ with $x_i < x_j \Rightarrow i < j$. For any incidence function $f$ of $P$ we denote $f(0, x) = f(x)$, where $0 = \min P$. For example, if $(P, \leq) = (\mathbb{Z}_+, |)$, then $\mu(1, n)$ is the usual number-theoretic Möbius function $\mu(n)$.

**Definition 2.1.** Let $C_S$ denote the class of incidence functions defined as

$$ C_S = \{ f \mid (x \in S, z \leq x) \Rightarrow (f \ast \mu)(z) > 0 \}. $$
Definition 2.2. If $f$ is an incidence function of $P$, then the $n \times n$ matrix $(S)_f = (s_{ij})$, where

$$s_{ij} = f(x_i \wedge x_j),$$

is called the meet matrix on $S$ with respect to $f$.

3. Structure theorem

Lemma 3.1. Let $f$ be an incidence function of $P$. Then

$$f(x, y) = \sum_{x \leq z \leq y} (f * \mu)(x, z)$$

for all $x, y \in P$.

Lemma 3.1 is a direct consequence of the formula $f = f * \delta = f * (\mu * \zeta) = (f * \mu) * \zeta$.

Lemma 3.2. Let $\downarrow S = \{y_1, y_2, \ldots, y_m\}$ with $y_i < y_j \Rightarrow i < j$, and let $f$ be an incidence function of $P$. Let $A$ denote the $n \times m$ matrix defined by

$$a_{ij} = \begin{cases} \sqrt{(f * \mu)(y_j)} & \text{if } y_j \leq x_i, \\ 0 & \text{otherwise} \end{cases}$$

Then $(S)_f = AA^T$.

Proof. Obviously $y_1 = \min P$. For $1 \leq i \leq n, 1 \leq j \leq m$ we have

$$(AA^T)_{ij} = \sum_{k=1}^{m} a_{ik}a_{jk} = \sum_{y_1 \leq y_k \leq x_i, y_1 \leq y_k \leq x_j} (f * \mu)(y_1, y_k) = \sum_{y_1 \leq y_k \leq x_i \wedge x_j} (f * \mu)(y_1, y_k).$$

Therefore it follows from Lemma 3.1 that

$$(AA^T)_{ij} = f(y_1, x_i \wedge x_j) = f(x_i \wedge x_j).$$

This completes the proof. □
4. Determinant of meet matrices

Haukkanen [1] has proved that if $S$ is meet-closed, then

$$\det(S)_f = \prod_{k=1}^{n} \sum_{\substack{z \leq x_k \\text{ for } k \leq t \leq k, \ z \neq x_t}} (f * \mu)(z).$$

(4.1)

Note that Haukkanen writes this formula without using convolution of incidence function. Also note that (4.1) is a generalization of Smith’s [4] famous formula: if $S = \{x_1, x_2, \ldots, x_n\}$ is a factor-closed set of positive integers and $f$ is an arithmetical function, then $\det(f(x_i, x_j)) = \prod_{i=1}^{n} (f * \mu)(x_i)$.

5. Lower bound for $\det(S)_f$

In this section we give a generalization of (1.1). The proof is adapted from that given by Hong [2].

**Theorem 5.1.** If $f \in C_S$, then

$$\det(S)_f \geq \prod_{k=1}^{n} \sum_{\substack{z \leq x_k \\text{ for } k \leq t \leq k, \ z \neq x_t}} (f * \mu)(z)$$

(5.1)

and the equality holds if and only if $S$ is meet-closed.

**Proof.** Define $S_k = \{z \in P \mid z \leq x_k, \ z \neq x_t, \ t < k\}, 1 \leq k \leq n$. Then for all $1 \leq i < j \leq n$ we have $S_i \cap S_j = \emptyset$. Otherwise, there exists $z \in S_k \cap S_t$, where $s < t$. Since $z \in S_i$ and $s < t$, we have $z \neq x_s$, and this contradicts $z \in S_k \cap S_t$. Obviously $S_1 \cup \cdots \cup S_n = \downarrow S$. To see this take $z \in \downarrow S$. Then for some $i$ we have $z \leq x_i$. From our assumptions we see that the interval $[z, x_i]$ is finite. We can therefore find the minimal $k$ such that $z \leq x_k$. Thus $z \neq x_t$ when $t < k$. This means that $z \in S_k$ and we have $z \in S_1 \cup \cdots \cup S_n$.

For all $1 \leq k \leq n$ let $S_k = \{y_{k,1}, y_{k,2}, \ldots, y_{k,p_k}\}$ with $y_{k,i} < y_{k,j} \Rightarrow i < j$. Obviously $y_{k,p_k} = x_k$, $1 \leq k \leq n$. Let $p_1 + \cdots + p_n = m$, and let

$$y_j = \begin{cases} y_{1,j} & \text{if } j = 1, 2, \ldots, p_1, \\ y_{k,t} & \text{if } j = p_1 + \cdots + p_{k-1} + t, \ 1 \leq t \leq p_k, \ k \geq 2, \end{cases}$$

for all $1 \leq j \leq m$. Then we have $\downarrow S = \{y_1, y_2, \ldots, y_m\}$ with $y_i < y_j \Rightarrow i < j$. To see the latter statement, let $y_i < y_j$. Then for $y_i$ there exist $d$ and $s$ such that $y_i = y_{d,s}$, where $i = p_1 + \cdots + p_{d-1} + s$, $1 \leq s \leq p_d$. In the same way, for $y_j$ there exist $e$ and $t$ such that $y_j = y_{e,t}$, where $j = p_1 + \cdots + p_{e-1} + t$, $1 \leq t \leq p_e$. Since
yd, s ⩽ ye, t, we have d ⩽ e. If d = e, we have trivially i < j. If d < e, then i ⩽ p1 + ⋅⋅⋅ + pd ⩽ p1 + ⋅⋅⋅ + pe−1 < j. The latter statement therefore holds.

Let A denote the n × m matrix defined by

\[ a_{ij} = \begin{cases} \sqrt{(f \ast \mu)(y_j)} & \text{if } y_j \leq x_i, \\ 0 & \text{otherwise.} \end{cases} \] (5.2)

By Lemma 3.2 we have

\[ \det(S)_f = \det(AA^T). \] (5.3)

Now let \{α1, α2, ..., αn\} denote the system of row vectors of A and let \{β1, β2, ..., βn\} denote the orthogonalization system obtained from \{α1, α2, ..., αn\} by using the Gram–Schmidt orthogonalization process

\[ \begin{align*}
\beta_1 &= \alpha_1, \\
\beta_k &= \alpha_k - \sum_{i=1}^{k-1} \frac{\langle \alpha_k, \beta_i \rangle}{\langle \beta_i, \beta_i \rangle} \beta_i, 
\end{align*} \] (5.4)

where 2 ⩽ k ⩽ n. Let finally B denote the n × m matrix having βi’s as its rows. From the orthogonalization algorithm we find that there exists an invertible matrix E, which is the product of elementary matrices, such that det E = 1 and EA = B. Thus

\[ \det(AA^T) = \det(E^{-1} BB^T (E^{-1})^T) = \det(BB^T). \] (5.5)

On the other hand, the set \{β1, β2, ..., βn\} is orthogonal. Thus

\[ BB^T = [\langle \beta_i, \beta_j \rangle] = \text{diag}(\langle \beta_1, \beta_1 \rangle, \langle \beta_2, \beta_2 \rangle, ..., \langle \beta_n, \beta_n \rangle) \] (5.6)

and

\[ \det(BB^T) = \prod_{k=1}^{n} \langle \beta_k, \beta_k \rangle. \] (5.7)

It follows from (5.3) and (5.5)–(5.7) that

\[ \det(S)_f = \prod_{k=1}^{n} \langle \beta_k, \beta_k \rangle. \] (5.8)

From the definition of the matrix A we see that

\[ \alpha_1 = \begin{pmatrix} \sqrt{(f \ast \mu)(y_{1,1})}, \ldots, \sqrt{(f \ast \mu)(y_{1,p_1})}, 0, \ldots, 0 \\ \vdots \\ \frac{1}{p_2 + \cdots + p_n} \end{pmatrix}, \]

\[ \alpha_k = \begin{pmatrix} \ast, \ldots, \ast, \sqrt{(f \ast \mu)(y_{k,1})}, \ldots, \sqrt{(f \ast \mu)(y_{k,p_k})}, 0, \ldots, 0 \\ \frac{1}{p_2 + \cdots + p_k - 1} \end{pmatrix}, \] (5.9)

where 2 ⩽ k ⩽ n. By orthogonalizing we have
\[
\beta_1 = \left( \sqrt{(f \ast \mu)(y_{1,1}), \ldots, \sqrt{(f \ast \mu)(y_{1,1}), 0, \ldots, 0}} \right)
\]
\[
\beta_k = \left( \ast, \ldots, \ast, \sqrt{(f \ast \mu)(y_{k,1}), \ldots, \sqrt{(f \ast \mu)(y_{k,1}), 0, \ldots, 0}} \right),
\]

(5.10)

where $2 \leq k \leq n$. We find that the Gram–Schmidt process changes only numbers marked by the asterisk. Since $f \in C_S$, we have

\[
\langle \beta_k, \beta_k \rangle \geq \sum_{i=1}^{p_k} (f \ast \mu)(y_{k,i}) = \sum_{z \leq x_k \atop z \not\in x_i} (f \ast \mu)(z) > 0
\]

and

\[
\det(S) f = \prod_{k=1}^{n} \langle \beta_k, \beta_k \rangle \geq \prod_{k=1}^{n} \sum_{z \leq x_k \atop z \not\in x_i} (f \ast \mu)(z).
\]

(5.11)

Therefore (5.1) holds.

Let $S$ be meet-closed. We prove by induction on $k$ that

\[
\beta_k = \left( 0, \ldots, 0, \sqrt{(f \ast \mu)(y_{k,1}), \ldots, \sqrt{(f \ast \mu)(y_{k,1}), 0, \ldots, 0}} \right),
\]

(5.13)

whenever $1 \leq k \leq n$. Since $\alpha_1 = \beta_1$, we have that (5.13) holds for $\beta_1$. Assume that (5.13) holds for $\beta_1, \beta_2, \ldots, \beta_{k-1}, 1 < k \leq n$. Now consider $\beta_k$ and let $i$ be an index such that $p_1 + \cdots + p_{e-1} < i \leq p_1 + \cdots + p_e$ and let $1 \leq e \leq k - 1$. By (5.10) we see that $\beta^{(i)}_1 = \cdots = \beta^{(i)}_{e-1} = 0$. By the induction assumption we have $\beta^{(i)}_{e+1} = \cdots = \beta^{(i)}_{k-1} = 0$. Thus

\[
\beta^{(i)}_k = \alpha^{(i)}_k - \frac{\langle \alpha_k, \beta_1 \rangle}{\langle \beta_1, \beta_1 \rangle} \beta^{(i)}_1 - \cdots - \frac{\langle \alpha_k, \beta_{k-1} \rangle}{\langle \beta_{k-1}, \beta_{k-1} \rangle} \beta^{(i)}_{k-1}
\]

\[
= \alpha^{(i)}_k - \frac{\langle \alpha_k, \beta_e \rangle}{\langle \beta_e, \beta_e \rangle} \beta^{(i)}_e.
\]

(5.14)

Since $x_i < x_j \Rightarrow i < j$ holds in $S$, we have either $x_e < x_k$ or $x_e \not\in x_k$. First assume that $x_e < x_k$. Now $y_{e,i} \leq x_e < x_k$; hence $\alpha^{(i)}_k = \beta^{(i)}_e = \sqrt{(f \ast \mu)(y_{e,i})}$. Since $i$ is chosen arbitrarily, we have $\langle \alpha_k, \beta_e \rangle = \langle \beta_e, \beta_e \rangle$. Now by (5.14) we have
\[ \beta_k^{(i)} = \alpha_k^{(i)} - \frac{(\alpha_k, \beta_e)}{(\beta_e, \beta_e)} \beta_e^{(i)} = \alpha_k^{(i)} - \beta_e^{(i)} = 0. \]

We now assume that \( x_e \not\leq x_k \). Then \( y_e, 1, y_e, 2, \ldots, y_e, p_e \not\leq x_k \). Otherwise, there exists \( s, 1 \leq s \leq p_e \), such that \( y_{e,s} \leq x_k \). Since \( S \) is meet-closed, there exists \( d < e \) such that \( x_e \land x_k = x_d \). Since \( y_{e,s} \leq x_k \) and \( x_e \not\leq x_k \), we have \( y_{e,s} \not\leq x_d \), which contradicts \( d < e \). So \( y_e, 1, y_e, 2, \ldots, y_e, p_e \not\leq x_k \) and \( \alpha_k^{(i)} = 0 \). Since \( i \) is chosen arbitrarily, we have \( \langle \alpha_k, \beta_e \rangle = 0 \). Now by (5.14) we have

\[ \beta_k^{(i)} = \alpha_k^{(i)} - \frac{(\alpha_k, \beta_e)}{(\beta_e, \beta_e)} \beta_e^{(i)} = 0. \]

Since \( \beta_k^{(i)} = 0 \) for \( p_1 + \cdots + p_{e-1} < i \leq p_1 + \cdots + p_e \) and \( 1 \leq e \leq k - 1 \), we see by (5.14) that (5.13) holds for \( \beta_k \). This completes the proof of (5.13). Now

\[ \langle \beta_k, \beta_k \rangle = \sum_{i=1}^{p_k} (f \ast \mu)(y_{k,i}) = \sum_{z \leq x_k, z \not\leq x_t, t < k} (f \ast \mu)(z) > 0 \]

and

\[ \det(S)_f = \prod_{k=1}^{n} \langle \beta_k, \beta_k \rangle = \prod_{k=1}^{n} \sum_{z \leq x_k, z \not\leq x_t, t < k} (f \ast \mu)(z). \]

Therefore, if \( S \) is meet-closed, then the equality holds in (5.1).

Now let \( S \) be a set such that the equality holds in (5.1). We show that \( S \) is meet-closed. On the contrary, assume that \( S \) is not meet-closed. Since \( \{x_1\} \) is meet-closed, there exists minimal \( a \geq 2 \) such that \( \{x_1, x_2, \ldots, x_{a-1}\} \) is meet-closed but \( \{x_1, x_2, \ldots, x_a\} \) is not meet-closed. Now (5.13) holds for \( \{x_1, x_2, \ldots, x_{a-1}\} \), that is,

\[ \beta_1 = \left( \sqrt{(f \ast \mu)(y_{1,1})}, \ldots, \sqrt{(f \ast \mu)(y_{1,p_1})}, 0, \ldots, 0 \right)_{p_2 + \cdots + p_n}, \]

\[ \vdots \]

\[ \beta_{a-1} = \left( 0, \ldots, 0, \sqrt{(f \ast \mu)(y_{a-1,1})}, \ldots, \sqrt{(f \ast \mu)(y_{a-1,p_{a-1}})}, 0, \ldots, 0 \right)_{p_{a-2} + \cdots + p_n}. \]

Let \( b \) be the minimal index such that \( 1 \leq b \leq a - 1 \) and \( x_a \land x_b \not\in S \). Clearly \( b \) exists. Since \( \downarrow S \) is lower-closed, it is meet-closed. Therefore \( x_a \land x_b = y_{d,c} \), where \( 1 \leq d \leq b \) and \( 1 \leq c < p_d \) (if \( c = p_d \), then \( y_{d,c} = x_d \), which leads to a contradiction). We show that \( d = b \). Otherwise, if \( d < b \), we have \( x_a \land x_d \not\in S \) by minimality of \( b \). Let \( x_a \land x_d = x_l \), \( l \leq d \). Since \( y_{d,c} \leq x_a \) and \( y_{d,c} \leq x_d \), we have \( y_{d,c} \leq x_l \).
Thus \( d \leq l \). Since \( l \leq d \), we have \( l = d \). Thus \( x_a \land x_d = x_d \) and \( x_d \leq x_a \). Since \( \{x_1, x_2, \ldots, x_{a-1}\} \) is meet-closed but \( \{x_1, x_2, \ldots, x_d\} \) is not meet-closed, where \( d < b < a \), we have \( x_b \land x_d \in S \). In the same way let \( x_b \land x_d = x_h, h \leq d \). Since \( y_{d,c} \leq x_b \) and \( y_{d,c} \leq x_d \), we have \( y_{d,c} \leq x_h \). Thus \( d \leq h \). Since \( h \leq d \), we have \( h = d \). Thus \( x_b \land x_d = x_d \) and \( x_d \leq x_b \). Now \( x_d \leq x_a \) and \( x_d \leq x_b \). We have therefore \( x_d \leq x_a \land x_b = y_{d,c} \). This leads to a contradiction, because \( y_{d,c} = x_d \in S \). Therefore \( d = b \) and \( x_a \land x_b = y_{b,c} \), where \( 1 \leq c < p_b \).

By (5.15) we have \( \beta_1^{(i)} = \cdots = \beta_{b-1}^{(i)} = \beta_{b+1}^{(i)} = \cdots = \beta_{a-1}^{(i)} = 0 \) and thus

\[
\beta_a^{(i)} = \alpha_a^{(i)} - \frac{\langle \alpha_a, \beta_1 \rangle}{\langle \beta_1, \beta_1 \rangle} \beta_1^{(i)} - \cdots - \frac{\langle \alpha_a, \beta_{a-1} \rangle}{\langle \beta_{a-1}, \beta_{a-1} \rangle} \beta_{a-1}^{(i)} = \alpha_a^{(i)} - \frac{\langle \alpha_a, \beta_b \rangle}{\langle \beta_b, \beta_b \rangle} \beta_b^{(i)}
\]

(5.16)

for all \( p_1 + \cdots + p_{b-1} < i \leq p_1 + \cdots + p_b \). First assume that \( \langle \alpha_a, \beta_b \rangle = 0 \) and let \( i = p_1 + \cdots + p_{b-1} + c \). Then by (5.16) we have \( \beta_a^{(i)} = \alpha_a^{(i)} = \sqrt{(f \ast \mu)(y_{b,c})} > 0 \) and

\[
\langle \beta_a, \beta_a \rangle \geq (f \ast \mu)(y_{b,c}) + \sum_{i=1}^{p_a} (f \ast \mu)(y_{a,i}) > \sum_{z \leq x_a}^{z \notin x_t} (f \ast \mu)(z) > 0.
\]

We now assume that \( \langle \alpha_a, \beta_b \rangle \neq 0 \) and let \( i = p_1 + \cdots + p_b \). Then

\[
\frac{\langle \alpha_a, \beta_b \rangle}{\langle \beta_b, \beta_b \rangle} \neq 0.
\]

Since \( x_b \nleq x_a \), we have \( \alpha_a^{(i)} = 0 \). Then by (5.16) we have

\[
\beta_a^{(i)} = -\frac{\langle \alpha_a, \beta_b \rangle}{\langle \beta_b, \beta_b \rangle} \sqrt{(f \ast \mu)(x_b)} \neq 0
\]

and

\[
\langle \beta_a, \beta_a \rangle \geq \left( \frac{\langle \alpha_a, \beta_b \rangle}{\langle \beta_b, \beta_b \rangle} \right)^2 (f \ast \mu)(x_b) + \sum_{i=1}^{p_a} (f \ast \mu)(y_{a,i}) > \sum_{z \leq x_a}^{z \notin x_t} (f \ast \mu)(z) > 0.
\]
In both cases we have
\[ \det(S)f = \prod_{k=1}^{n} \langle \beta_k, \beta_k \rangle > \prod_{k=1}^{n} \sum_{z \leq x_k} (f * \mu)(z) \]
and this is a contradiction. Therefore $S$ is meet-closed. This completes the proof of Theorem 5.1. □

6. Upper bound for $\det(S)f$

**Lemma 6.1.** If $f \in C_S$, then $(S)f$ is positive definite.

**Proof.** Let $f \in C_S$. Then $(f * \mu)(z) > 0$ whenever $z \leq x_i$ and $x_i \in S$. Define $S_i = \{x_1, x_2, \ldots, x_i\}$, $1 \leq i \leq n$. Then by Theorem 5.1 we have
\[ \det(S_i)f \geq \prod_{k=1}^{i} \sum_{z \leq x_k} (f * \mu)(z) > 0, \]
where $1 \leq i \leq n$. Thus the principal minors of $(S)f$ are positive. This completes the proof. □

**Lemma 6.2.** Suppose that
\[ A = \begin{bmatrix} B & D \\ D^* & C \end{bmatrix} \]
is a positive definite matrix that is partitioned so that $B$ and $C$ are square and non-empty, $D^*$ being the conjugate transpose of $D$. Then $\det A \leq (\det B)(\det C)$.

Lemma 6.2 is known as Fisher’s inequality and it can be found in [3]. Now we give an upper bound for $\det(S)f$. Haukkanen provided the same result in [1].

**Theorem 6.1.** If $f \in C_S$, then
\[ \det(S)f \leq f(x_1) \cdots f(x_n). \] (6.1)

We now provide a new upper bound for $\det(S)f$. The new upper bound (6.2) is sharper than (6.1) if we choose $m = 2$. To see this we need Lemma 6.3, which is also needed in the proof of the new upper bound.

**Lemma 6.3.** If $f \in C_S$, then $f(x_i) > 0$ for all $x_i \in S$. Furthermore, if $x < x_i$ and $x_i \in S$, then $f(x) < f(x_i)$. 
Proof. Let \( f \in C_S \) and \( x_i \in S \). Then \((f \ast \mu)(z) > 0\) for all \( z \leq x_i \). Thus by Lemma 3.1 we have \( f(x_i) = \sum_{z \leq x_i} (f \ast \mu)(z) > 0 \). Let \( x < x_i \). Then

\[
(f \ast \mu)(z) > 0 \text{ for all } z \leq x_i.
\]
Thus by Lemma 3.1 we have

\[
f(x_i) = \sum_{z \leq x_i} (f \ast \mu)(z) > 0.
\]
Let \( x < x_i \). Then

\[
f(x) < \sum_{z \leq x} (f \ast \mu)(z) \leq \sum_{z \leq x_i} (f \ast \mu)(z) = f(x_i).
\]
This completes the proof. \( \square \)

Theorem 6.2. If \( f \in C_S \), then

\[
\det(S)f \leq \frac{m!}{2} \left( 1 - \frac{f(x_{a_1} \wedge \cdots \wedge x_{a_m})^m}{f(x_{a_1}) \cdots f(x_{a_m})} \right) \prod_{k=1}^{n} f(x_k)
\]
whenever \( 1 \leq a_1 < \cdots < a_m \leq n \) and \( 2 \leq m \leq n \).

Proof. Let \( f \in C_S \). Define \( U = \{x_{a_1}, x_{a_2}, \ldots, x_{a_m}\} \), where \( 1 \leq a_1 < \cdots < a_m \leq n \) and \( 2 \leq m \leq n \). Let \( V = S \setminus U = \{x_{b_1}, x_{b_2}, \ldots, x_{b_{n-m}}\} \) with \( x_{b_i} < x_{b_j} \Rightarrow i < j \). Define

\[
A = \begin{bmatrix}
(U)_f & D \\
D^T & (V)_f
\end{bmatrix},
\]
where \( D = [f(x_{a_1} \wedge x_{b_1})] \). By Lemma 6.1, \((S)f\) is positive definite. Note that there is a permutation matrix \( Q \) such that \( A = Q^T(S)f Q \). Thus \( \det(S)f = \det A \) and \( A \) is positive definite. Thus, by Lemma 6.2 and Theorem 6.1, we have

\[
\det(S)f \leq (\det(U)_f)(\det(V)_f) \leq \det(U)_f \prod_{k=1}^{n-m} f(x_{b_k}).
\]
On the other hand,

\[
\det(U)_f = \sum_{\pi=(p_1, p_2, \ldots, p_m)} (\text{sgn } \pi) f(x_{a_1} \wedge x_{a_{p_1}}) \cdots f(x_{a_m} \wedge x_{a_{p_m}}),
\]
where \( \pi \) runs through the \( m! \) permutations of \((1, 2, \ldots, m)\) and where \( \text{sgn } \pi = 1 \) if \( \pi \) is even and \( \text{sgn } \pi = -1 \) if \( \pi \) is odd (see [3, p. 8]). Obviously there are \( m!/2 \) permutations of each type. Let \( \pi = (p_1, p_2, \ldots, p_m) \) be even. Since \( x_{a_1} \wedge x_{a_{p_1}} \leq x_{a_i} \) and \( f \in C_S \), by Lemma 6.3 we have \( 0 < f(x_{a_1} \wedge x_{a_{p_1}}) \leq f(x_{a_i}) \) for all \( 1 \leq i \leq m \). Thus

\[
f(x_{a_1} \wedge x_{a_{p_1}}) \cdots f(x_{a_m} \wedge x_{a_{p_m}}) \leq f(x_{a_1}) \cdots f(x_{a_m}).
\]
Let \( \pi = (p_1, p_2, \ldots, p_m) \) be odd. Since \( x_{a_1} \wedge \cdots \wedge x_{a_m} \leq x_{a_i} \wedge x_{a_{p_i}} \) and \( f \in C_S \), by Lemma 6.3 we have \( 0 < f(x_{a_1} \wedge \cdots \wedge x_{a_m}) \leq f(x_{a_i} \wedge x_{a_{p_i}}) \) for all \( 1 \leq i \leq m \). Thus
\[
\left[ f(x_{a_1} \land \cdots \land x_{a_m}) \right]^m \leq f(x_{a_1} \land x_{a_{p_1}}) \cdots f(x_{a_m} \land x_{a_{p_m}})
\]

and
\[
\det(U)_f = \sum_{\pi = (p_1, p_2, \ldots, p_m)} (\text{sgn } \pi) f(x_{a_1} \land x_{a_{p_1}}) \cdots f(x_{a_m} \land x_{a_{p_m}})
\]
\[
\leq \frac{m!}{2} \left( f(x_{a_1}) \cdots f(x_{a_m}) - \left[ f(x_{a_1} \land \cdots \land x_{a_m}) \right]^m \right).
\]

Therefore
\[
\det(S)_f \leq \left( \det(U)_f \right) \left( \det(V)_f \right)
\]
\[
\leq \frac{m!}{2} \left( f(x_{a_1}) \cdots f(x_{a_m}) - \left[ f(x_{a_1} \land \cdots \land x_{a_m}) \right]^m \right) \prod_{k=1}^{n-m} f(x_{b_k})
\]
\[
= \frac{m!}{2} \left( 1 - \frac{f(x_{a_1} \land \cdots \land x_{a_m})}{f(x_{a_1}) \cdots f(x_{a_m})} \right) \prod_{k=1}^{n} f(x_k).
\]

This completes the proof. \(\square\)

7. Inverse of \((S)_f\)

**Theorem 7.1.** Let \(S = \{x_1, x_2, \ldots, x_n\}\) be a meet-closed set and let \(f \in C_S\). Then \((S)_f\) is invertible and
\[
((S)^{-1})_{ij} = \sum_{x_i \leq x_k} \sum_{x_j \leq x_k} \frac{1}{\sum_{z \leq x_t \atop t < k} (f \ast \mu)(z)} \mu_S(x_i, x_k) \mu_S(x_j, x_k), \tag{7.1}
\]
where \(\mu_S = \zeta_S^{-1}\) and \(\zeta_S\) is the restriction of \(\zeta\) on \(S \times S\).

**Proof.** Define the order ideal as \(\downarrow S = \{y_1, y_2, \ldots, y_m\}\) and define the matrix \(A\) as in (5.2). Then \((S)_f = AA^T\). Let \(\{\alpha_1, \alpha_2, \ldots, \alpha_n\}\) denote the system of row vectors of \(A\) and let \(\{\beta_1, \beta_2, \ldots, \beta_n\}\) denote the orthogonalization system obtained from \(\{\alpha_1, \alpha_2, \ldots, \alpha_n\}\) by using the Gram–Schmidt orthogonalization process. Let \(B\) denote the \(n \times m\) matrix having \(\beta_i\)'s as its rows. Since \(S\) is meet-closed, we have
\[
BB^T = \text{diag} \left( \sum_{z \in x_1 \atop z \not\in x_2} (f \ast \mu)(z), \ldots, \sum_{z \in x_n \atop z \not\in x_t} (f \ast \mu)(z) \right).
\]
By (2.1) we see that \(BB^T\) is invertible and
\[(BB^T)^{-1} = \text{diag} \left( \frac{1}{\sum_{z \leq x_1} (f * \mu)(z)}, \ldots, \frac{1}{\sum_{z \leq x_n} (f * \mu)(z)} \right). \]

By algorithm (5.4) we have
\[\alpha_1 = \beta_1,\]
\[\alpha_k = \beta_k + \sum_{i=1}^{k-1} \frac{\langle \alpha_i, \beta_i \rangle}{\langle \beta_i, \beta_i \rangle} \beta_i,\]
where \(2 \leq k \leq n\). We note that \(EB = A\), where
\[E = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
\frac{\langle \alpha_2, \beta_1 \rangle}{\langle \beta_1, \beta_1 \rangle} & 1 & 0 & \cdots & 0 \\
\frac{\langle \alpha_3, \beta_1 \rangle}{\langle \beta_1, \beta_1 \rangle} & \frac{\langle \alpha_3, \beta_2 \rangle}{\langle \beta_2, \beta_2 \rangle} & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{\langle \alpha_n, \beta_1 \rangle}{\langle \beta_1, \beta_1 \rangle} & \frac{\langle \alpha_n, \beta_2 \rangle}{\langle \beta_2, \beta_2 \rangle} & \frac{\langle \alpha_n, \beta_3 \rangle}{\langle \beta_3, \beta_3 \rangle} & \cdots & 1
\end{bmatrix}.\]

Since \(S\) is meet-closed, we know by the proof of Theorem 5.1 (see the discussion after (5.14)) that \(\langle \alpha_i, \beta_j \rangle = \langle \beta_j, \beta_j \rangle\) if \(x_j < x_i\), and \(\langle \alpha_i, \beta_j \rangle = 0\) if \(x_j \not\leq x_i\). Thus \(E = [\xi_S(x_i, x_j)]^T\) and so \(E^{-1} = [\mu_S(x_i, x_j)]^T\). Then \((S)_f = AA^T = EBB^TE^T\) and \((S)^{-1}_f = (E^{-1})^T(BB^T)^{-1}E^{-1}\). Therefore
\[((S)^{-1}_f)_{ij} = \sum_{x_i \leq x_k} \sum_{x_j \leq x_k} \frac{1}{\sum_{z \leq x_i} (f * \mu)(z) \mu_S(x_i, x_k) \mu_S(x_j, x_k)}.\]

This completes the proof. \(\square\)

Haukkanen [1] has proved a similar formula for \((S)^{-1}_f\) when \(S\) is lower-closed. He also mentions the possibility of proving this more general result by using \(\mu_S\).

References