MATHEMATICS

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On varieties in multiple-projective spaces

Dedicated to N. G. de Bruijn on his sixtieth birthday

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SUMMARY

In this paper, ${}^mP$ will denote a projective space of dimension m, and ${}^{(m,n)}P$ will denote a doubly-projective space of dimension $m+n$, namely the space of all pairs of points $(x|y)$, where x varies in ${}^mP$ and y in ${}^nP$. Just so, ${}^{(m,n,s)}P$ will denote a triply-projective space of dimension $m+n+s$, and so on.

A variety V of dimension d in ${}^mP$ has just one degree g, namely the number of points of intersection of V with d generic linear hyperplanes $(ux)=0$, where (ux) means $\sum u_i x_i$. On the other hand, a variety V' of dimension d in ${}^{(m,n)}P$ has several degrees $g_{a,b}$ $(a+b=d)$, defined as follows: $g_{a,b}$ is the number of points of intersection of V' with a hyperplanes $(ux)=0$ and b hyperplanes $(vy)=0$.

Let $x_0, \ldots, x_m, y_0, \ldots, y_n$ be the homogeneous coordinates of a point in ${}^{m+n+1}P$. It sometimes happens that the equations of a variety V in ${}^{m+n+1}P$ are not only homogeneous in all variables x and y together, but even homogeneous in the x's and in the y's. In this case the same set of equations also defines a variety V' in the doubly-projective space ${}^{(m+n)}P$. If d is the dimension of V', the dimension of V is $d+1$, for to every point $(x|y)$ of V' corresponds a whole straight line of points $(x\alpha, y\beta)$ in V.

In some cases it is easier to determine the degrees $g_{a,b}$ of V' than to determine the degree g of V. For this reason, it is desirable to have a rule that enables us to calculate g from the $g_{a,b}$'s. Such a rule will be proved here. It says:

The degree g is the sum of all $g_{a,b}$ with $a+b=d$.

In the case of multiply-projective spaces ${}^{(m,n,\cdots)}P$ the same rule holds: g is the sum of the $g_{a,b,\ldots}$ with $a+b+\ldots=d$.

Examples of applications of this rule will be given at the end.

CYCLES AND THEIR INTERSECTIONS IN MULTIPLY-PROJECTIVE SPACES

For the sake of convenience, the ground field will be assumed to be the field of complex numbers. This assumption enables us to consider ${}^{(m,n)}P$ as a topological manifold and to apply the methods of homology theory to algebraic cycles in ${}^{(m,n)}P$. The algebraic intersection of two or more cycles is just the topological intersection cycle; topological intersection multiplicities of algebraic cycles are always positive and equal to the algebraic multiplicities [1]).

[1]) B. L. van der Waerden: Topologische Begründung des Kalküls der algebraischen Geometrie. Math. Ann. 102, 337–362 (1929).

The space ${}^{(m+n)}P$ is a topological product space ${}^{m}P \times {}^{n}P$. According to Künneth [2]), a homology basis for the product space can be obtained by multiplying the homology bases of the factor spaces. More precisely: if the cycles $A_1, \ldots, A_s$ form a homology basis for cycles of all dimensions in ${}^{m}P$, and $B_1, \ldots, B_t$ for ${}^{n}P$, then the cycles $A_i \times B_k$ form a basis for ${}^{(m,n)}P$. "Homology" is here meant as "Homology with division allowed" in the sense of Poincaré.

I shall use the very convenient notation of Schubert [3]). Let p denote any linear subspace $(ux)=0$, and q any linear subspace $(vy)=0$ in ${}^{(m,n)}P$. By p^2 Schubert denotes the intersection of any two subspaces $(ux)=0$ and $(u'x)=0$, by p^3 the intersection of any three such subspaces, provided the intersection has the right dimension $m+n-3$, which can always be achieved by shifting the subspaces. And so on.

Two cycles A and B of the same dimension d are called "equal", and Schubert writes $A=B$, if the intersections $A \cdot C$ and $B \cdot C$ with any algebraic cycle of complementary dimension $m+n-d$ consist of the same number of points. This is always the case if the cycles A and B are homologous.

It is well known that a homology basis for the cycles of all dimensions in ${}^{m}P$ is formed by the cycles

$$p^0, p^1, p^2, \ldots, p^m,$$

where p^0 is the whole space ${}^{m}P$. Just so, a basis of ${}^{n}P$ is formed by

$$q^0, q^1, \ldots, q^n.$$

Hence, according to Künneth, a basis for ${}^{(m,n)}P$ is formed by the cycles

$$p^a q^b \quad (a=0, 1, \ldots, m;\ b=0, 1, \ldots, n).$$

This implies: Every cycle Z of dimension d is equal (in the sense of Schubert) to a sum

$$Z = \sum_{a+b=d} p^{m-a} q^{n-b} g_{a,b}. \tag{1}$$

If one multiplies both sides of (1) by $p^a q^b$ one obtains

$$Z \cdot p^a q^b = p^m q^n g_{a,b}. \tag{2}$$

On the right, $p^m q^n$ is a cycle consisting of just one point of ${}^{(m,n)}P$. On the left, we have the intersection of Z with a hyperplanes $(ux)=0$ and b hyperplanes $(vy)=0$. So (2) means that the degrees of Z are just the coefficients $g_{a,b}$.

The easiest derivation of formula (1) seems to be the topological derivation just given. However, (1) is also true in the case of an arbitrary

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[2]) H. Künneth: Über die Bettischen Zahlen einer Produktmannigfaltigkeit. Math. Ann. 90, 65–85 (1923). See also Math. Ann. 91, 125 (1924).

[3]) H. Schubert: Kalkül der Abzählenden Geometrie.

ground field. This was proved by Grothendieck[4]). More generally, Grothendieck showed how to derive the intersection ring of cycles on any bundle of projective spaces over a smooth variety X from the intersection ring of cycles on X (modulo rational equivalence).

If one multiplies (1) by an arbitrary cycle W of dimension $m+n-d$, one obtains

$$Z \cdot W = p^m q^n \sum_{a+b=d} g_{a,b} h_{m-a,n-b}, \tag{3}$$

where $h_{e,f}$ are the degrees of W. Formula (3) is the generalization of Bézout's theorem to doubly-projective spaces.

It is easy to generalize (1), (2), (3) to multiply-projective spaces. For the space ${}^{(m,n,s)}P$ formula (1) reads

$$Z = \sum_{a+b+c=d} p^{m-a} q^{n-b} r^{s-c} g_{a,b,c}. \tag{4}$$

The proof is obvious: ${}^{(m,n,s)}P$ is just the topological product of ${}^{(m,n)}P$ and ${}^{s}P$, so Künneth's theorem can be applied once more.

THE VARIETIES V′ AND V

Let V' be an irreducible variety of dimension d in ${}^{(m,n)}P$. Every point $(x|y)$ of V' determines a straight line in ${}^{m+n+1}P$ formed by the points $(x\alpha, y\beta)$ having coordinates

$$(x_0\alpha, \ldots, x_m\alpha, y_0\beta, \ldots, y_n\beta).$$

The union of these lines is an irreducible variety V of dimension $d+1$ in ${}^{m+n+1}P$. A generic point of V can be obtained by starting with a generic point $(\xi|\eta)$ of V' and forming

$$(\xi_0 s, \ldots, \xi_m s, \eta_0 t, \ldots, \eta_n t)$$

with indeterminates s and t.

There is a rational mapping π which maps ${}^{m+n+1}P$ on to ${}^{(m,n)}P$, and V on to V'. The image of a point (x, y) is unique for all points (x, y) of ${}^{m+n+1}P$ except those for which $x=0$ or $y=0$. These exceptional points form two linear subspaces ${}^{n}P$ and ${}^{m}P$ in ${}^{m+n+1}P$. For all other points the mapping π is defined by

$$(x, y) \rightarrow (x|y).$$

If one wants to apply this mapping π to any variety U in ${}^{m+n+1}P$ which does not lie in ${}^{n}P$ or ${}^{m}P$, a good method is: Apply the mapping to a generic point Q of U, thus obtaining a generic point πQ of πU. All points of πU are spezializations of the generic point πQ. In this way,

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[4]) A. Grothendieck: Sur propriétés fondamentales en théorie des intersections. Séminaire Chevalley, deuxième année, Sécr. Math. Paris (1958). Mr. S. L. Kleiman, M.I.T. Cambridge (Mass.), has drawn my attention to this paper.

all difficulties arising from special points of U lying in ${}^nP$ or ${}^mP$ are avoided. If U has dimension d, the map πU may have dimension d or $d-1$.

Now let us return to our main problem: to find the degree of V, if the degrees $g_{a,b}$ of V' are given.

THE DEGREE OF V

To fix the ideas, let us suppose that V has codimension 2, i.e. that its dimension is $m+n-1$. In this case, the degree g is defined as the number of points of intersection with a generic plane U in ${}^{m+n+1}P$. The plane may be defined by three generic points A, B, C, whose coordinates we shall call

$$\begin{aligned} A&: (a_0, \ldots, a_m, d_0, \ldots, d_n) \text{ or just } (a, d)\\ B&: (b_0, \ldots, b_m, e_0, \ldots, e_n) \text{ or just } (b, e)\\ C&: (c_0, \ldots, c_m, f_0, \ldots, f_n) \text{ or just } (c, f). \end{aligned}$$

Now a generic point of the plane U may be obtained thus:

$$x = a\alpha + b\beta + c\gamma \tag{5}$$

$$y = d\alpha + e\beta + f\gamma \tag{6}$$

with indeterminates α, β, γ. The image of this generic point in the mapping π is given by exactly the same formulae (5) and (6). All points of the image $W = \pi U$ are obtained by giving special values to the indeterminates α, β, γ.

Our aim was: to determine the number of points of intersection of V and U. Since U is a generic plane, these points do not lie in ${}^nP$ or ${}^mP$, so we may apply the mapping π, thus obtaining points of intersection of V' and $\pi U = W$. Conversely, to every point of intersection of V' and W corresponds just one point of intersection of V and U. In such a point R, the tangential spaces to V and U have only the point R in common, because U is a generic plane in ${}^{m+n+1}P$. The mapping π diminishes the dimension of V by one unit, but it preserves the property of the tangential spaces to have only one point in common. Hence the point πR has multiplicity 1 in the intersection of V' and W, and this holds for all points of intersection. Hence the cycle $V' \cdot W$ consists of just g points, each counted once.

The rest is easy. According to formula (3), applied to the intersection $V' \cdot W$, we have

$$g = \sum_{a+b=d} g_{a,b} h_{m-a,n-b} \tag{7}$$

which means in our case $(d = m+n-2)$

$$g = g_{m-2,n} h_{2,0} + g_{m-1,n-1} h_{1,1} + g_{m,n-2} h_{0,2} \tag{8}$$

where the $h_{e,f}$ are the degrees of W. Now it turns out that these degrees are all equal to 1. Let us calculate, for instance, the degree $h_{1,1}$, i.e. the number of points of intersection of W with two linear hyperplanes

$$(ux)=0 \text{ and } (vy)=0.$$

For x and y we may substitute the parametric representations (5) and (6), from which we get

$$(ux)=(ua)\alpha+(ub)\beta+(uc)\gamma,$$
$$(vy)=(vd)\alpha+(ve)\beta+(vf)\gamma.$$

Thus we obtain two linear equations for α, β, γ, which have just one solution. Hence

$$h_{1,1}=1$$

and just so, if $m\geqslant 2$

$$h_{2,0}=1$$

and if $n\geqslant 2$

$$h_{0,2}=1.$$

Thus one obtains from (8)

$$g=g_{m-2,n}+g_{m-1,n-1}+g_{m,n-2}$$

and just so in the general case from (7)

$$g=\sum_{a+b=d} g_{a,b} \tag{9}$$

Of course, on the right side of (9), only terms $g_{a,b}$ satisfying the conditions

$$0\leqslant a\leqslant m$$
$$0\leqslant b\leqslant n$$

are to be retained.

Exactly the same proof holds in the still more general case of a multiply projective space ${}^{(m,n,\ldots)}P$. The degree of the variety V is always equal to the sum of those degrees $g_{a,b,\ldots}$, for which the sum $a+b+\ldots$ is d.

ANOTHER PROOF OF (9)

The proof of (9) just given was based on the theorem of Künneth. I shall now give an elementary proof by complete induction with respect to the dimension d of V'.

We may suppose V and V' to be absolutely irreducible. Hence, if d is zero, V' consists of just one point, and V of just one line. In this case the formula (9) is trivial:

$$g=g_{1,1}=1.$$

Now let us suppose that (9) is true for varieties U' of dimension $d-1$. We have to prove (9) for varieties V' of dimension d.

Let $v_0, \ldots, v_n$ be indeterminates, and let $F(v)$ be the field obtained from the ground field F by the adjunction of $v_0, \ldots, v_n$. The intersection of V (dimension $d+1$ in ${}^{m+n+1}P$) with the hyperplane

$$(vy) = v_0 y_0 + \ldots + v_n y_n = 0 \tag{10}$$

is a cycle Z of dimension d. The cycle Z consists of a variable part U, which is an irreducible variety over $F(v)$ counted only once [1]), and possibly a fixed part C not depending on the v_i and lying in the linear subspace ${}^mP$ defined by the equations

$$y_0 = 0, \ldots, y_n = 0. \tag{11}$$

So we have

$$Z = U + C. \tag{12}$$

Applying to U the mapping π, one obtains a variety U' of dimension $d-1$, the intersection of V' with the hyperplane $(vy)=0$ in ${}^{(m,n)}P$. This time, the intersection contains no fixed part, because the equations (11) have no solution in ${}^{(m,n)}P$. So the intersection cycle of V' with $(vy)=0$ is just U', counted only once.

To U' and U we may apply the induction hypothesis. It follows that the degree of U is the sum of the degrees $h_{a,b-1}$ of U'. Now we have, trivially,

$$h(a, b-1) = g(a, b)$$

for all $b > 0$. Hence:

The degree of U is equal to the sum of the $g(a, b)$ with $a+b=d$ and $b>0$.

The degree of V is of course equal to the degree of its hyperplane section $Z = U + C$. It remains to determine the degree of the cycle C. I shall prove that this degree is just $g_{d,0}$.

Let us consider the intersection of V with the $d+1$ hyperplanes (13) and (14):

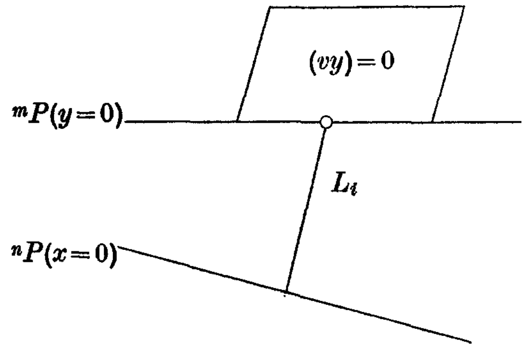
$$(u_1 x) = 0, \ldots, (u_d x) = 0 \tag{13}$$

$$(vy) = 0, \tag{14}$$

the coefficients u_i and v_k in (13) and (14) being independent indeterminates. We shall start with V and form intersections stepwise, taking into account

1) By Bertini's theorem. See e.g. my Einführung in die algebraische Geometrie, § 47.

one hyperplane after another, each time throwing away those parts of the intersection that lie in ${}^nP(x=0)$, because these are unimportant for our purpose. First we shall take the hyperplanes in the order $u_1, \ldots, u_d, v$ and next in the order $v, u_1, \ldots, u_d$. It is well known that the final multiplicities of the points of intersection are independent of the order of the steps.



Let us first consider the intersection of V with the hyperplanes (13). In ${}^{(m,n)}P$ the corresponding intersection is that of V' with the hyperplanes (13). It consists of $g_{d,0}$ points, each counted once. To these points correspond the same number of straight lines L_i, each counted once. The remainder of the intersection of V with (13) lies in the subspace ${}^nP$ and may be neglected.

Each of the $g_{d,0}$ straight lines L_i has just one point P_i in common with ${}^mP$ $(y=0)$. This point satisfies the equation $(vy)=0$, hence it is the point of intersection of the line L_i with the hyperplane (14). Hence that part of the intersection of V with (13) and (14) which does not lie in ${}^nP$ is the zero-dimensional cycle $\sum P_i$ consisting of $g_{d,0}$ points. Some of these points may coincide, but the degree of the cycle is exactly $g_{d,0}$. The total intersection of V with (13) and (14) consists of the points P_i in ${}^mP$ and a remainder in ${}^nP$.

Now we take the other order: first (14) and next (13). The intersection of V with (14) is the d-dimensional cycle $Z=U+C$. So we have to intersect U with (13) and C with (13) and to add the results. We have to consider only those points of intersection that lie in ${}^mP$. Now U does not lie in ${}^mP$, hence the meet of U and ${}^mP$ has at most dimension $d-1$, and the hyperplanes (13) do not contain any point of this meet. So we are left with the intersection of the cycle C in ${}^mP$ with the d hyperplanes (13). The degree of this intersection is just the degree of C.

Comparing the two results, we obtain the desired equation

$$\deg(C)=g_{d,0}.$$

Thus we have

$$\begin{aligned} g &= \deg(U) + \deg(C) \\ &= \sum_{b>0} g_{a,b} + g_{d,0} \\ &= \sum_{a+b=d} g_{a,b}, \end{aligned}$$

which is what we wanted to prove.

APPLICATIONS

If the matrix elements of an $m \times n$ matrix are interpreted either as coordinates in an affine space ${}^{mn}A$ or as homogeneous coordinates in a projective space ${}^{mn-1}P$, one may ask [5]:

What is the degree of the variety V formed by the $m \times n$ matrices of rank r at most?

To fix the ideas, let us consider the case $m=n=4$, $r=2$. The four columns of a 4×4 matrix may be interpreted as homogeneous coordinates of four points in ${}^3P$. If the rank is 2 at most there are at most 2 linearly independent points among the four, in other words, the four points are in a straight line. Hence the variety V' in ${}^{(3,3,3,3)}P$ consists of all those sequences of four points p, q, r, s in ${}^3P$ that are collinear. The dimension of this variety V' is $3+3+1+1=8$, since two of the four points may be chosen arbitrarily, while the remaining two have one degree of freedom each.

In order to determine the degree g of V, we first have to determine the degrees $g_{a,b,c,d}$ of V' $(a+b+c+d=8)$ and next to form their sum. We have 3 types of degrees $g_{a,b,c,d}$, namely

$$\begin{aligned} &g_{3,3,1,1} \text{ and permutations,} \\ &g_{3,2,2,1} \text{ and permutations,} \\ &g_{2,2,2,2}. \end{aligned}$$

The calculation of $g_{3,3,1,1}$ is easy. We have 3 linear conditions $(up)=0$, which determine the first point p completely, and 3 conditions $(vq)=0$ which determine q completely. Now the line pq is known, and one condition for each of the points r and s suffices to determine these two points. Hence $g_{3,3,1,1}=1$.

Just so, one sees that $g_{3,2,2,1}=1$. Three conditions determine p, and for q and r we have two conditions each, which means that q and r must lie on two given straight lines in generic positions. There is just one straight line through p that meets the two given lines. One linear condition for r suffices to determine the fourth point r.

To find $g_{2,2,2,2}$ we have to find the number of straight lines that meet

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[5]) This question was raised by V. Strassen during a walk in 1976.

four given straight lines. It is well known that this number is 2, provided the four lines are in generic positions.

So we have

$g_{3,3,1,1}=1$, six permutations, sum 6
$g_{3,2,2,1}=1$, twelve permutations, sum 12
$g_{2,2,2,2}=2$, only one permutation, sum 2
Total $g=6+12+2=20$.

This solves our problem. In the more general case of $4\times n$ matrices of rank 2 at most one finds by the same method

$$g_{3,3,1,1,\dots}=1,\quad \binom{n}{2} \text{ permutations, sum } \binom{n}{2}$$

$$g_{3,2,2,1,\dots}=1,\quad 3\binom{n}{3} \text{ permutations, sum } 3\binom{n}{3}$$

$$g_{2,2,2,1,\dots}=2,\quad \binom{n}{4} \text{ permutations, sum } 2\binom{n}{4}$$

$$\text{Total } g=\binom{n}{2}+3\binom{n}{3}+2\binom{n}{4}=\frac{n^2(n^2-1)}{12}$$

Other results, which may be obtained by the same method, are:

$m\times n$ matrices $(m\leqslant n)$, rank $m-1$ at most:

$$g=\binom{n}{m-1},$$

$m\times n$ matrices, rank 1 at most:

$$g=\binom{m+n-2}{m-1}.$$

The latter result can also be verified by a simpler method: An $m\times n$ matrix of rank 1 can be obtained as a product of a single column and a single row:

$$a_{ik}=b_ic_k.$$

If the a_{ik} are considered as homogeneous coordinates in ${}^{mn-1}P$, one obtains a variety V of dimension $m+n-2$ in ${}^{mn-1}P$. If we impose $(m+n-2)$ linear conditions on the a_{ik}, we obtain $(m+n-2)$ bilinear conditions for the b_i and c_k. The pairs of points (b,c) form a doubly-projective space ${}^{(m-1,n-1)}P$. A bilinear condition for b and c determines a hypersurface B of degrees 1,1 in ${}^{(m-1,n-1)}P$. According to formula (1), we have the Schubert equality

$$B=p+q.$$

The intersection of $(m+n-2)$ such bilinear hypersurfaces is

$$B^{m+n-2} = (p+q)^{m+n-2} = \binom{m+n-2}{m-1} p^{m-1} q^{n-1}.$$

The number of points of this 0-dimensional cycle is

$$g = \binom{m+n-2}{m-1}$$

in accordance with our earlier result.

In the most general case, when m, n and r are arbitrary, the same method can be applied. The problem of determining the degrees $g_{a,b,\ldots}$ of V' is seen to be equivalent to a well-known problem of enumerative geometry:

How many subspaces ${}^{r-1}P$ *of* ${}^{m-1}P$ *meet a certain number of given subspaces in generic positions*

$${}^{m-1-a}P,\ {}^{m-1-b}P,\ \ldots ?$$

This problem can be solved in any particular case by the methods explained in Volume 2, page 309–367 (Chapter XIV) of Hodge and Pedoe: Methods of Algebraic Geometry (Cambridge Univ. Press 1952). The main tool is "Pieri's Formula" (4), p. 354, which was first proved by W. V. D. Hodge.