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On functions with small differences

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SUMMARY

An Abel-Tauber theorem is proved and applied to multiplicative arithmetic functions.

1. INTRODUCTION

We prove an Abel-Tauber theorem that complements a well known result. It is shown that this provides simple proofs for some results by de Bruijn en van Lint [2] on multiplicative arithmetic functions. The theorem also provides an easy proof of an earlier result by Wirsing [12] on multiplicative arithmetic functions.

2. ABEL-TAUBER THEOREM

THEOREM 1. Suppose L is a measurable function, slowly varying at ∞, and U is such that its Laplace-Stieltjes transform $\hat{U}(\lambda)=\lambda\int_0^\infty e^{-\lambda x}U(x)\,dx$ exists for all $\lambda>0$. We also require that $U(0+)=0$ and that U is locally bounded.

a. If for all $x>0$

$$(1)\qquad \lim_{t\to\infty}\frac{U(tx)-U(t)}{L(t)}=0$$

then

$$(2)\qquad \lim_{t\to\infty}\frac{\hat{U}(1/t)-U(t)}{L(t)}=0.$$

b. If U is non-decreasing and for all $\lambda > 0$

$$(3)\qquad \lim_{t\downarrow 0} \frac{\hat{U}(\lambda t) - \hat{U}(t)}{L(1/t)} = 0,$$

then (2) holds.

REMARK. It is clear that (1) & (2) imply (3) and that (2) & (3) imply (1).

REMARK. Theorem 1 is an analogue of the main theorem of [7]. It is a generalisation of theorem 2 of Feller [4] which in turn contains theorem 3b, Ch. 5 in Widder [10] as a special case (note that the condition of slow variation of U in Feller's article is unnecessary). We shall make the connection with Feller's theorem after the proof.

For the proof of theorem 1 we need two lemmas.

LEMMA 1. Suppose L is a slowly varying function and $V: \mathbb{R}_+ \to \mathbb{R}_+$ is locally bounded. If for some $\beta > 0$

$$\lim_{t\to\infty} \frac{V(t)}{t^\beta L(t)} = 0,$$

then locally uniformly in $x \in [0, \infty)$

$$(4)\qquad \lim_{t\to\infty} \frac{V(tx)}{t^\beta L(t)} = 0.$$

PROOF. Take any two sequences $x_n \to x \in [0, \infty)$ and $t_n \to \infty$. If $\{t_n x_n\}$ is bounded the result holds since $t_n^\beta L(t_n) \to \infty$. If $\{t_n x_n\}$ is unbounded we take w.l.o.g. a subsequence with $t_n x_n \to \infty$. Then $V(t_n x_n) t_n^{-\beta} L^{-1}(t_n) = [V(t_n x_n)\{(t_n x_n)^\beta L(t_n x_n)\}^{-1}]$ $\{x_n^\beta L(t_n x_n)/L(t_n)\} \to 0$ since the first factor tends to zero and the other one is bounded.

LEMMA 2. Suppose L and V are as in Lemma 1 and moreover that the Laplace-Stieltjes transform $\hat{V}(\lambda) = \lambda \int_0^\infty e^{-\lambda x} V(x)\, dx$ exists for all $\lambda > 0$. Let α be a non-negative parameter.

a. If

$$(5)\qquad \lim_{t\to\infty} \frac{V(t)}{t^\alpha L(t)} = 0$$

then

$$(6)\qquad \lim_{t\to\infty} \frac{\hat{V}(1/t)}{t^\alpha L(t)} = 0.$$

b. Conversely, if V is non-decreasing, $V(0+) = 0$ and (6) holds then (5) is true.

PROOF. a) For $0<\varepsilon<1$

$$\frac{\hat{V}(1/t)}{t^\alpha L(t)}=\int_0^1 \frac{e^{-u}}{u^\varepsilon}\frac{(ut)^\varepsilon V(ut)}{t^{\alpha+\varepsilon}L(t)}\,du+\int_1^\infty e^{-u}\frac{V(ut)}{(ut)^\alpha L(ut)}u^\alpha\frac{L(ut)}{L(t)}\,du.$$

The first term tends to zero by Lemma 1 and for sufficiently large t the second term is bounded by

$$\varepsilon\int_1^\infty e^{-u}u^\alpha\frac{L(ut)}{L(t)}\,du$$

which by e.g. the representation of L tends to $\varepsilon\int_1^\infty e^{-u}u^\alpha\,du$ as $t\to\infty$.

b) Define

$$W_t(x):=\frac{V(tx)}{t^\alpha L(t)}.$$

According to (6) $\lim_{t\to\infty}\hat{W}_t(\lambda)=0$ for $\lambda>0$. It follows from the extended continuity theorem for Laplace transforms (Feller [5]) that $\lim_{t\to\infty}W_t(x)=0$ for $x>0$ i.e. (5) holds.

PROOF OF THEOREM 1. It is well known (cf. e.g. [3]) that (1) is equivalent to

$$\lim_{x\to\infty}\frac{U(x)-x^{-1}\int_0^x U(t)\,dt}{L(x)}=0. \tag{7}$$

Define $V(x)=xU(x)-\int_0^x U(t)\,dt$. Note that $x^{-1}V(x)$ is locally bounded on $x>0$ and that conversely $U(x)=x^{-1}V(x)+\int_0^x t^{-2}V(t)\,dt$ (cf. de Haan [8]).

a. Writing U in terms of V as above we get

$$\frac{U(t)-\hat{U}(t^{-1})}{L(t)}=\frac{V(t)}{t\,L(t)}-\int_0^\infty e^{-s}\frac{V(ts)}{ts\,L(t)}\,ds+\int_0^1\frac{1-e^{-s}}{s}\frac{V(ts)}{ts\,L(t)}\,ds+$$
$$-\int_1^\infty\frac{e^{-s}}{s}\frac{V(ts)}{ts\,L(t)}\,ds.$$

The second term tends to zero by Lemma 2 (for $\alpha=0$) and the remaining terms by similar arguments.

b. Now V is positive and non-decreasing, so $\hat{V}$ is non-increasing and

$$\frac{\hat{U}(2^{-1}t^{-1})-\hat{U}(t^{-1})}{L(t)}=\int_{2^{-1}}^1\frac{\hat{V}(st^{-1})}{t\,L(t)}\,ds\geq\frac{2^{-1}\hat{V}(t^{-1})}{t\,L(t)}.$$

Hence (3) implies (6); this in turn implies (5) with $\alpha=1$, which is (7) and we have already seen that (7) is equivalent to (1).

REMARK. Feller's [4] result is obtained if we take L constant and use the fact that (1) and (7) are equivalent.

Comparing theorem 1 of [7] and the present theorem 1 we see that the latter holds with the righthand sides of (1), (2) and (3) replaced by (respectively)

$c \log x$ $(c \geq 0)$, $-c\gamma$ (Euler's constant) and $-c \log \lambda$. It is clear from (7) that (1) implies

$$\text{(8)} \qquad \lim_{t\to\infty} \frac{Z(tx)-Z(t)}{L(t)}=0$$

with $Z(x)=x^{-1}\int_0^x U(t)\,dt$. This suggests that under certain conditions in the above we may replace U by Z.

LEMMA 3. a. If Z satisfies (8), then

$$\text{(9)} \qquad \lim_{t\to\infty} \frac{\hat{U}(t^{-1})-Z(t)}{L(t)}=0$$

b. If $\hat{U}$ satisfies (3) and Z is monotone, then (8) holds.

PROOF. Analogous to part of the proof of theorem 1 in Geluk [6]. Z satisfies (8) if and only if the function

$$H(x):=\int_0^x t\,dZ(t)=\int_0^x U(t)\,dt-\int_0^x\int_0^t U(s)\,ds\,\frac{dt}{t}$$

is $o(x\,L(x))$ for $x\to\infty$. Now

$$\hat{H}(1/t)=t\hat{U}(1/t)-t\,\hat{Z}(1/t).$$

If Z satisfies (8), then by lemma 2 $\hat{H}(1/t)=o(t\,L(t))$; by theorem 1 then $\hat{Z}(1/t)-Z(t)=o(L(t))$ hence (9) holds.

Conversely suppose Z is monotone and $\hat{U}$ satisfies (3). From (3) it follows that

$$t\,\hat{U}(1/t)-\int_0^t \hat{U}(1/s)\,ds=o(t\,L(t)) \qquad (t\to\infty)$$

by using the result from [3] as in (7).

Now $\int_0^t \hat{U}(1/s)\,ds=t\,\hat{Z}(1/t)$ hence $\hat{H}(1/t)=o(t\,L(t))$ and by Lemma 2 $H(x)=o(x\,L(x))$ i.e. Z satisfies (8).

We connect the results of Lemma 3 and theorem 1 in the following Lemma.

LEMMA 4. If (8) holds and U is non-decreasing, then (1) is true.

PROOF. It is easily verified that

$$x^{-1}\int_0^x U(t)\,dt=\int_0^x t^{-2}V(t)\,dt \quad \text{(where } V(x)=\int_0^x t\,dU(t) \text{ as before)}$$

so that for $x>1$

$$\frac{(tx)^{-1}\int_0^{tx}U(s)\,ds-t^{-1}\int_0^t U(s)\,ds}{L(t)}=\int_1^x \frac{V(ts)}{t\,L(t)}\frac{ds}{s^2}\geq\frac{V(t)}{t\,L(t)}\left(1-\frac{1}{x}\right).$$

Hence (7) is true which is equivalent to (1).

REMARK. A result like that of lemma 4 can be proved in a similar way in the situation mentioned above where the limits are non zero.

3. ARITHMETIC FUNCTIONS

Next we show how the above results can be used to get results on the asymptotic behaviour of multiplicative arithmetic functions. The first part of the following result (with slightly different conditions and a different proof) is well known (see de Bruijn and van Lint [2]).

THEOREM 2. Suppose λ is a real-valued multiplicative arithmetic function (i.e. $\lambda(mn)=\lambda(m)\lambda(n)$ for $(m,n)=1$) with $\lambda(n)\geq 0$ for $n\geq 1$,

(10) $$\sum_p \lambda(p)^2<\infty$$

where p runs through the primes, and

(11) $$\sum_{p,k\geq 2} \lambda(p^k)<\infty.$$

If for $x>0$

(12) $$\sum_{e^t<p\leq e^{tx}} \lambda(p)=(b+o(1))\log x \qquad (t\to\infty)$$

with $b\geq 0$, then

(13) $$\sum_{n\leq x} \lambda(n)=\left(\frac{e^{-b\gamma}}{\Gamma(b+1)}+o(1)\right)\left[\prod_p e^{-\lambda(p)}(1+\lambda(p)+\lambda(p^2)+\ldots)\right] e^{\sum_{p\leq x}\lambda(p)}$$

$$(x\to\infty)$$

where γ is Euler's constant.

Moreover

(14) $$\prod_{p\leq x} (1+\lambda(p)+\lambda(p^2)+\ldots)\sim e^{b\gamma}\Gamma(b+1) \sum_{n\leq x} \lambda(n). \qquad (x\to\infty)$$

If for $x>0$

(15) $$\sum_{e^t<p\leq e^{tx}} \lambda(p)\sim L(t)\log x \qquad (t\to\infty)$$

with $L(t)\to\infty$, then

(16) $$\log \sum_{n\leq e^x} \lambda(n)=\sum_{p\leq e^{1/s}} \lambda(p)+(1-\gamma)L(1/s)+o(L(1/s))$$

where $x\to\infty$, $s\downarrow 0$ and $xs\sim L(1/s)$.

PROOF. First we prove (13).

Since $\sum_{e^t<p\leq e^{tx}} \lambda(p)=(b+o(1))\log x$ $(t\to\infty)$ we get

(17) $$\sum_{p\leq e^x} \lambda(p)-\sum_p \frac{\lambda(p)}{p^{1/x}}=\gamma b+o(1) \qquad (x\to\infty).$$

For $b>0$ this result is well known (see [6]); in case $b=0$ it follows by application of theorem 1 with $L\equiv 1$.

Next we define the function $g(p,s)=\sum_{k=1}^{\infty}(\lambda(p^k)/p^{ks})$ where $s\geq 0$ and p denotes a prime.

Then we have for $s>0$

$$\log \sum_n \frac{\lambda(n)}{n^s}=\log \prod_p \left(1+\frac{\lambda(p)}{p}+\frac{\lambda(p^2)}{p^{2s}}+\dots\right)=\sum_p \log(1+g(p,s))$$

the convergence of the last series being implied by (11) and (12) as follows. First we have $g(p,s)\to 0$ $(p\to\infty)$ for $s>0$ since

$$\left|\sum_{k\geq 2}\frac{\lambda(p^k)}{p^{ks}}\right|\leq\left|\sum_{k\geq 2}\lambda(p^k)\right|$$

and this tends to zero if $p\to\infty$ by (11). Moreover we have $\lambda(p)/p^s\to 0$ $(p\to\infty)$ since $\lambda(p)$ is bounded. Now $\log(1+g(p,s))\sim g(p,s)$ $(p\to\infty)$ and

$$\sum_p g(p,s)=\sum_p \frac{\lambda(p)}{p^s}+\sum_{p,k\geq 2}\frac{\lambda(p^k)}{p^{ks}}<\infty$$

by (11) and (12). It now follows that

$$\text{(18)}\qquad \log \sum_n \frac{\lambda(n)}{n^s}=\sum_p g(p,s)-\sum_p \int_0^1 \frac{g(p,s)^2 t}{1+t\,g(p,s)}\,dt$$

$$=\sum_p \frac{\lambda(p)}{p^s}+c+o(1)\qquad (s\downarrow 0)$$

where

$$c=\sum_{p,k\geq 2}\lambda(p^k)-\sum_p g(p,0)^2\int_0^1\frac{t}{1+t\,g(p,0)}\,dt$$

$$=\sum_p\left[\log\left\{1+\sum_{k\geq 1}\lambda(p^k)\right\}-\lambda(p)\right].$$

Now (18) implies regular variation at zero with exponent $-b$ of $\sum_n(\lambda(n)/n^s)$ since $\exp\sum_p(\lambda(p)/p^s)$ is regularly varying at zero with exponent $-b$ by (12) and (17). Application of a well known theorem of Karamata (see [9] theorem 2.3) now yields

$$\text{(19)}\qquad \log\sum_{n\leq e^x}\lambda(n)-\log\sum_n\frac{\lambda(n)}{n^{1/x}}\to -\log\Gamma(b+1)\qquad (x\to\infty).$$

Combination of the results (17), (18) with $s=1/x$ and (19) gives expression (13). Similarly we find

$$\log\prod_{p\leq x}(1+\lambda(p)+\lambda(p^2)+\dots)=\sum_{p\leq x}\log(1+g(p,0))=$$

$$=\sum_{p\leq x}\lambda(p)+c+o(1)\quad (x\to\infty)$$

and (14) follows.

In order to prove (16) we proceed similarly: (17) is replaced by

$$\sum_{p\le e^x}\lambda(p)-\sum_p \frac{\lambda(p)}{p^{1/x}}=(\gamma+o(1))L(x)\qquad (x\to\infty). \tag{20}$$

The proof of (18) is unchanged.

In this case the regular variation of $\hat{U}(s):=\sum_n(\lambda(n)/n^s)$ at zero is replaced by

$$\left(\frac{\hat{U}(ys)}{\hat{U}(s)}\right)^{-1/L(1/s)}\to y\qquad (s\downarrow 0)\quad \text{for } y>0 \tag{21}$$

which is a consequence of (15), (18) and (20).

Now application of theorem 1 in [1] gives

$$\frac{\log \sum_{n\le e^t}\lambda(n)-\log\sum_n \frac{\lambda(n)}{n^s}}{L(1/s)}\to 1 \tag{22}$$

where $s\downarrow 0$ and $t\to\infty$ are related by $st\sim L(1/s)$.

Combination of (18), (20) and (22) now gives (16).

REMARK. The function L in (15) is necessarily slowly varying. See [8], th. 1.4.1. Next we apply the above results to prove the following theorem (compare Wirsing [11] and [12]).

THEOREM 3. If $f(n)\ge 0$ $(n=1,2,\dots)$ is a multiplicative arithmetic function satisfying

$$\sum_{p\le x}\frac{f(p)\log p}{p}=(\tau+o(1))\log x\qquad (x\to\infty)\quad \text{with}\quad \tau\ge 0 \tag{23}$$

$$\sum_{p,k\ge 2}\frac{f(p^k)}{p^k}<\infty,\qquad \sum_p\frac{f(p)^2}{p^2}<\infty, \tag{24}$$

then

$$\sum_{n\le x}\frac{f(n)}{n}=\left(\frac{e^{-\gamma\tau}}{\Gamma(\tau+1)}+o(1)\right)\prod_p\left[e^{-f(p)/p}\left(1+\frac{f(p)}{p}+\frac{f(p^2)}{p^2}+\dots\right)\right]e^{\sum_{p\le x}f(p)/p} \tag{25}$$

where γ is Euler's constant.

PROOF. We define

$$A(x):=\sum_{p\le x}\frac{f(p)\log p}{p}=(\tau+o(1))\log x\qquad (x\to\infty).$$

Then

$$\sum_{p\le x}\frac{f(p)}{p}=\int_1^x\frac{1}{\log t}\,dA(t)=\tau+o(1)+\int_1^x\frac{t(u)}{u\log u}\,du\quad \text{where}\ \lim_{u\to\infty}t(u)=\tau.$$

This gives

$$\sum_{p \le e^x} \frac{f(p)}{p} = \int_0^x \frac{t(e^v)}{v}\, dv + \tau + o(1),$$

hence condition (12) in theorem 2 is satisfied (with $\lambda(p) = f(p)/p$).

It is possible to give the behaviour of $\sum_{n \le x} f(n)/n$ in case $\sum_{p \le x} (f(p) \log p/p)$ tends to infinity more quickly:

THEOREM 4. Under the assumptions of theorem 3 with (23) replaced by

$$\sum_{p \le x} \frac{f(p) \log p}{p} = L(\log x) \log x \qquad (x \to \infty) \tag{26}$$

with $L(x) \to \infty$ slowly varying we have

$$\log \sum_{n \le e^x} \frac{f(n)}{n} = \sum_{p \le e^{1/s}} \frac{f(p)}{p} + (1 - \gamma + o(1)) L(1/s) \tag{27}$$

with $x \to \infty$, $s \downarrow 0$ and $xs \sim L(1/s)$.

In order to get the asymptotic behaviour of $1/x \sum_{n \le x} f(n)$ in the theorems 3 and 4 one needs extra assumptions. It seems difficult to get the behaviour of $1/x \sum_{n \le x} f(n)$ using Tauberian arguments.

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