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Lax Pairs and Bäcklund Transformations for a Coupled Ramani Equation and its Related System

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Abstract—A coupled Ramani equation and its related system are proposed. By dependent variable transformation, they are transformed into bilinear equations. Lax pairs and Bäcklund transformations are presented for these two systems. Soliton solutions and rational solutions to the systems could be obtained.

Keywords—Solitons, Integrable systems, Lax pairs, Bäcklund transformations.

It is known that many integrable models have played an important role in applied sciences. No doubt, it is one of central topics in soliton theory to search for as many integrable systems as possible. An effective way of seeking new integrable systems is to find integrable extensions of the known integrable systems. For example, for the celebrated KdV equation

$$u_t + 6uu_x + u_{xxx} = 0,$$

several coupled KdV equations have been found (see, e.g., [1–7]).

In this letter, we will consider an integrable extension of the following less studied Ramani equation:

$$u_{xxxxxx} + 15u_{xx}u_{xxx} + 15u_x u_{xxxx} + 45u_x^2 u_{xx} - 5(u_{xxxt} + 3u_{xx}u_t + 3u_x u_{xt}) - 5u_{tt} = 0. \quad (1)$$

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Equation (1) was first given by Ramani in [8]. We now propose the following coupled Ramani equation:

$$\begin{aligned} & u_{xxxxxx} + 15u_{xx}u_{xxx} + 15u_x u_{xxxx} + 45u_x^2 u_{xx} \\ & -5(u_{xxxt} + 3u_{xx}u_t + 3u_x u_{xt}) - 5u_{tt} + 18w_x = 0, \end{aligned} \tag{2}$$

$$w_t - w_{xxx} - 3w_x u_x - 3w u_{xx} = 0. \tag{3}$$

We will show that system (2),(3) is integrable in the sense of having a Bäcklund transformation and Lax pair. For this purpose, we set

$$u = 2(\ln f)_x, \qquad w = \left(\frac{g}{f}\right)_x.$$

Then (2),(3) can be transformed into the following bilinear equations:

$$\left(D_x^6 - 5D_x^3 D_t - 5D_t^2\right) f \cdot f + 18 D_x g \cdot f = 0, \tag{4}$$

$$\left(D_t - D_x^3\right) g \cdot f = 0, \tag{5}$$

where the bilinear operators are defined as follows [9–12]:

$$D_z^m D_t^k \; a \cdot b \equiv \left(\frac{\partial}{\partial z} - \frac{\partial}{\partial z'}\right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^k a(z,t) b(z',t')|_{z'=z,t'=t}.$$

Furthermore, by introducing an auxiliary variable z and letting $g = f_z$, (4) and (5) become the following bilinear equations for a single field f:

$$\left(D_x^6 - 5D_x^3 D_t - 5D_t^2 + 9D_x D_z\right) f \cdot f = 0, \tag{6}$$

$$D_z \left(D_t - D_x^3\right) f \cdot f = 0. \tag{7}$$

It is remarked that the technique of introducing auxiliary variables and dependent variable transformations and then transforming the original systems into bilinear equations is typical in Hirota's bilinear formalism. For example, in [13] Satsuma and Hirota transformed the following Hirota-Satsuma system

$$\begin{aligned} u_t &= \frac{1}{4} u_{xxx} + 3uu_x + 3\left(-\phi^2 + \omega\right)_x, \\ \phi_t &= -\frac{1}{2}\phi_{xxx} - 3u\phi_x, \\ \omega_t &= -\frac{1}{2}\omega_{xxx} - 3u\omega_x, \end{aligned}$$

into the bilinear form

$$\begin{aligned} \left(D_x D_t - \frac{1}{4} D_x^4 - \frac{3}{4} D_y^2\right) f \cdot f = 0, \\ D_y \left(D_t + \frac{1}{2} D_x^3\right) f \cdot f = 0, \end{aligned}$$

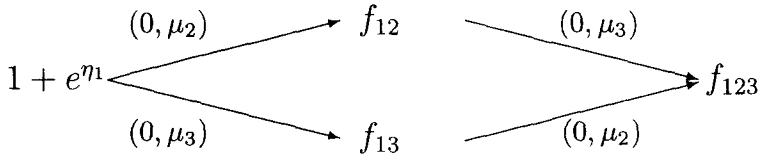
by the dependent variable transformation

$$u = (\ln f)_{xx}, \qquad \phi = \frac{1}{2}\frac{f_y}{f}, \qquad \omega = \frac{1}{2}\frac{f_{yy}}{f}.$$

By application of the exchange formalism, one can construct the following Bäcklund transformation for (6) and (7):

$$\left(D_x^3 - D_t + \lambda\right) f' \cdot f = 0, \tag{8}$$
$$\left(-D_x^5 + 5\lambda D_x^2 - 5D_x^2 D_t + 6D_z\right) f' \cdot f = 0, \tag{9}$$
$$D_x D_z f' \cdot f = \mu D_x f' \cdot f, \tag{10}$$

where λ and μ are arbitrary constants. By using BT (8)–(10), we can construct multiple soliton solutions and a series of rational solutions for (6) and (7). For example, we have



where

$$f_{1j} = \frac{p_1^5 - p_j^5}{p_1^5 + p_j^5} - e^{\eta_1} + e^{\eta_j} - \frac{p_1 - p_j}{p_1 + p_j} e^{\eta_1 + \eta_j}, \qquad j = 2, 3,$$

and

$$\begin{aligned} f_{123} = {} & \frac{p_3^5 - p_1^5}{p_1^5 + p_3^5} + \frac{p_1^5 - p_2^5}{p_1^5 + p_2^5} + \frac{p_2^5 - p_3^5}{p_2^5 + p_3^5} + \frac{p_2^5 - p_3^5}{p_2^5 + p_3^5} e^{\eta_1} \\ & + \frac{p_3^5 - p_1^5}{p_1^5 + p_3^5} e^{\eta_2} + \frac{p_1^5 - p_2^5}{p_1^5 + p_2^5} e^{\eta_3} - \frac{p_1 - p_2}{p_1 + p_2} e^{\eta_1 + \eta_2} - \frac{p_3 - p_1}{p_1 + p_3} e^{\eta_1 + \eta_3} \\ & - \frac{p_2 - p_3}{p_2 + p_3} e^{\eta_2 + \eta_3} + \frac{(p_3 - p_1)(p_1 - p_2)(p_2 - p_3)}{(p_1 + p_3)(p_1 + p_2)(p_2 + p_3)} e^{\eta_1 + \eta_2 + \eta_3} \end{aligned}$$

are 2-soliton and 3-soliton solutions of (6) and (7), respectively, with $\eta_i = p_i x + p_i^3 t + p_i^5 z + \eta_i^0$, $\mu_i = p_i^5$ and p_i, η_i^0 constants. We also have

$$\begin{aligned} & z + \frac{1}{120}x^5 + \frac{1}{2}x^2 t \xrightarrow{(0,0)} \frac{1}{3}z^3 + \frac{1}{2}z^2 t x^2 + \frac{1}{120}z^2 x^5 + \frac{1}{24}z t^2 x^4 + \frac{1}{3}z t^3 x + \frac{1}{2520} z t x^7 \\ & + \frac{1}{1814400} z x^{10} + \frac{2}{15} t^5 - \frac{1}{18} t^4 x^3 + \frac{11}{2160} t^3 x^6 + \frac{13}{181440} t^2 x^9 + \frac{1}{1360800} t x^{12} + \frac{1}{653184000} x^{15}. \end{aligned}$$

Here we have symbolically represented (8)–(10) by $f \xrightarrow{(\lambda,\mu)} f'$.

Next, we will derive a Lax pair for (2),(3). Set $f' = \psi f$, $u = 2(\ln f)_x$, $w = (f_z/f)_x$. Then from the bilinear BT (8)–(10) and by some calculations, we can obtain the following Lax pair for (2),(3):

$$\psi_t = \psi_{xxx} + 3u_x \psi_x + \lambda \psi, \tag{11}$$

$$\begin{aligned} & \psi_{xxxxxx} + 5u_x \psi_{xxxx} + 10u_{xx}\psi_{xxx} + \left(\frac{25}{3}u_{xxx} + \frac{5}{3}u_t + 5u_x^2\right)\psi_{xx} \\ & + \left(\frac{10}{3}u_{xxxx} + 10u_x u_{xx} + \frac{5}{3}u_{xt}\right)\psi_x + 2w\psi - \mu\psi_x = 0, \end{aligned} \tag{12}$$

which can be checked directly by using Mathematica [14].

Finally, we will derive another integrable system from the bilinear equations (6) and (7). We now exchange the roles played by t and z, i.e., we view t as an auxiliary variable and z a time variable; and set $u = 2(\ln f)_x$, $v = (f_t/f)_x$. In this case, we have from (6),(7) that

$$\begin{aligned} u_{zz} - u_{xxxxxz} - \frac{25}{3}u_{xxx}u_{xz} - 5u_x u_{xxxz} - 5u_x^2 u_{xz} - \frac{10}{3}u_{xxxx}u_z \\ -10u_{xx}u_{xxz} - 10u_x u_z u_{xx} - \frac{10}{3}v u_{xz} - \frac{10}{3}v_x u_z = 0, \end{aligned} \tag{13}$$

$$v_z = \frac{1}{2}(u_{xxxz} + 3u_{xx}u_z + 3u_x u_{xz}). \tag{14}$$

Since (13),(14) share the same bilinear form (6),(7) with (2),(3), we can also construct the soliton solutions and rational solutions for system (13),(14). In the following, we will give a Lax pair for (13),(14). In fact, set $f' = \psi f$, $u = 2(\ln f)_x$, $v = (f_t/f)_x$. Then from the bilinear BT (8)–(10) and by some calculations, we can obtain the following Lax pair for (13),(14):

$$\begin{aligned} \psi_{xxxxxx} + 5u_x\psi_{xxxx} + 10u_{xx}\psi_{xxx} + \left(\frac{25}{3}u_{xxx} + 5u_x^2 + \frac{10}{3}v\right)\psi_{xx} \\ + \left(\frac{10}{3}u_{xxxx} + 10u_x u_{xx} + \frac{10}{3}v_x\right)\psi_x + u_z\psi - \mu\psi_x = 0, \end{aligned} \tag{15}$$

$$\psi_z = \psi_{xxxxx} + 5u_x\psi_{xxx} + 5u_{xx}\psi_{xx} + \left(\frac{10}{3}u_{xxx} + 5u_x^2 + \frac{10}{3}v\right)\psi_x. \tag{16}$$

We have also checked that (15) and (16) do constitute a Lax pair for (13),(14) directly by using Mathematica.

To summarize, a coupled Ramani equation and a related system are proposed. By dependent variable transformation, their bilinear equations are given. Lax pairs and Bäcklund transformations are presented for them. Soliton solutions and rational solutions could be obtained.

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