Available online at www.sciencedirect.com
SCIENCE DIRECT
Discrete Mathematics 268 (2003) 287–291
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

Note

The existence of hyper-L triple-loop networks[☆]

Chiuyuan Chen[a,*], F.K. Hwang[a], J.S. Lee[b], S.J. Shih[a]

[a] *Department of Applied Mathematics, National Chiao Tung University, Hsinchu 300, Taiwan*
[b] *Department of Mathematics, National Kaohsiung Normal University, Kaohsiung 802, Taiwan*

Received 27 August 2001; received in revised form 15 October 2002; accepted 22 October 2002

Abstract

Aguiló et al. (Discrete Math. 167/168 (1997) 3–16) have presented some necessary conditions for the existence of hyper-L triple-loop networks. In this paper, we will give necessary and sufficient conditions for the existence of hyper-L triple-loop networks.

Keywords: Triple-loop network; Hyper-L tile; Diameter

1. Introduction

Multi-loop networks were first proposed by Wong and Coppersmith [4] for organizing multi-module memory services. A *multi-loop network* ML(N; s_1, $s_2,\ldots,$ s_d) has N nodes $0,1,2,\ldots,N-1$ and dN links, $i \to i+s_1$, $i \to i+s_2,\ldots,i \to i+s_d \pmod N$, $i=0,1,\ldots,N-1$. Multi-loop networks are now being widely studied because of their relevance to the design of some interconnection or communication networks. For details of multi-loop networks, see [1–3].

A triple-loop network TL($N;s_1,s_2,s_3$) has N nodes $0,1,2,\ldots,N-1$ and $3N$ links, $i \to i+s_1$, $i \to i+s_2$, $i \to i+s_3 \pmod N$, $i=0,1,\ldots,N-1$. It is an extension of the double-loop network DL($N;s_1,s_2$) with one more fixed step s_3 for each node. It is well known [4] that DL($N;s_1,s_2$) and TL($N;s_1,s_2,s_3$) are strongly connected if and only if $\gcd(N,s_1,s_2)=1$ and $\gcd(N,s_1,s_2,s_3)=1$, respectively. In this paper we are only concerned with strongly connected networks and if the parameters do not satisfy the above conditions, we simply say the corresponding network does not exist.

[☆] This research was partially supported by the National Science Council of the Republic of China under the Grants NSC90-2115-M-009-012, NSC90-2115-M-009-013, and NSC91-2115-M-009-015.

* Corresponding author.
E-mail addresses: cychen@mail.nctu.edu.tw (C. Chen), fhwang@math.nctu.edu.tw (F.K. Hwang).

0012-365X/03/$ - see front matter
doi:10.1016/S0012-365X(03)00030-X

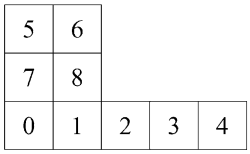

Fig. 1. An MDD for $N = 9$, $s_1 = 1$, $s_2 = 7$.

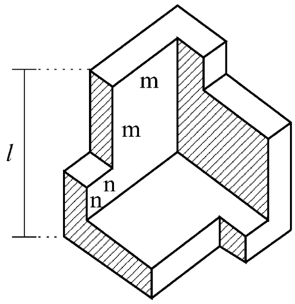

Fig. 2. A hyper-L tile.

A minimum distance diagram MDD(v) for DL($N; s_1, s_2$) is a two-dimensional array which gives the shortest paths from node v to every other node. Since DL($N; s_1, s_2$) is node-symmetric, we need only study MDD(0), or simply, MDD. Wong and Coppersmith [4] gave an O(N) time construction of MDD by sequentially adding nodes to the diagram which can be reached from node 0 in i steps for $i = 0, 1, \ldots$, until every node appears exactly once. Fig. 1 illustrates an MDD where each horizontal step signifies an s_1-step and each vertical an s_2-step.

It is well known that an MDD for a double-loop network is an L-shape which includes the degenerate form of a rectangle. This L-shape plays a crucial role in proving many desirable properties for DL($N; s_1, s_2$).

The MDD for a triple-loop network is a three-dimensional array with each step in the x_i-axis signifying an s_i-step. Unfortunately, the MDD does not have a uniform nice shape like the L-shape, and this fact has hampered the study of triple-loop networks. Aguiló et al. [1] overcame this difficulty by skipping the triple-loop network and going directly to a nice three-dimensional shape which they called *hyper-L tile*. A hyper-L tile is characterized by three parameters l, m, n, and is highly structured and symmetrical (see Fig. 2). Note that l, m, n are integers, $m \geqslant n \geqslant 0$ and $l > m + n$. They proved that the diameter of a hyper-L tile can approach $(27N/2)^{1/3}$ which is the best construction for a triple-loop network so far.

While the hyper-L tile seems to be a promising tool for studying the triple-loop network, we must be able to verify that those hyper-L tiles producing good results are indeed the MDDs of some triple-loop networks. While Aguiló et al. [1] have presented some necessary conditions, we will give necessary and sufficient conditions.

2. Necessary and sufficient conditions

Let HL(l, m, n) denote a hyper-L tile with parameters l, m, n. Define

$$M = \begin{pmatrix} l & -m & -n \\ -n & l & -m \\ -m & -n & l \end{pmatrix}.$$

Aguiló et al. [1] observed that HL(l,m,n) tessellates $\mathfrak{R}^3$. By studying the distribution of node 0 in $\mathfrak{R}^3$, they obtained

$$M \times \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \pmod N$$

or

$$M \times \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} N \quad \text{for some integers } \alpha, \beta, \gamma. \tag{1}$$

Also, $N = \det M = l^3 - m^3 - n^3 - 3lmn$. They said it was proven in their Ref. [8] that two necessary conditions for the existence of a triple-loop network whose MDD is HL(l,m,n) are:

(i) $\gcd(N,l,m,n) = 1$, and
(ii) $\gcd(2 \times 2$ minors of $M) = 1$.

We now show that condition (ii) implies condition (i). Suppose $\gcd(N,l,m,n) = d > 1$. Then $\gcd(2 \times 2$ minors of $M) = \gcd(l^2 - mn, m^2 + ln, n^2 + lm) \geqslant d > 1$, since d divides every term.

For convenience, we call a triple-loop network whose MDD is HL(l,m,n) an HL(l,m,n) triple-loop. We give a necessary and sufficient condition for the existence of an HL(l,m,n) triple-loop.

Theorem 1. *A necessary and sufficient condition for the existence of an* HL(l,m,n) *triple-loop is* $\gcd(l^2 - mn, m^2 + ln, n^2 + lm) = 1$.

Proof. Suppose an HL(l,m,n) triple-loop exists. From (1),

$$\begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = M^{-1} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} N = \begin{pmatrix} l^2 - mn & n^2 + lm & m^2 + ln \\ m^2 + ln & l^2 - mn & n^2 + lm \\ n^2 + lm & m^2 + ln & l^2 - mn \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}.$$

Setting $(\alpha, \beta, \gamma) = (1, 0, 0)$, we obtain the solution

$$(s_1, s_2, s_3) = (l^2 - mn, m^2 + ln, n^2 + lm).$$

So if $\gcd(l^2 - mn, m^2 + ln, n^2 + lm) = 1$, then clearly

$$\gcd(N, s_1, s_2, s_3) = 1$$

and TL$(N; s_1, s_2, s_3)$ exists.

On the other hand, if

$$\gcd(l^2 - mn, m^2 + ln, n^2 + lm) = d > 1,$$

then each s_i, $i = 1, 2, 3$, is a linear combination of terms divisible by d. Furthermore,

$$N = l^3 - m^3 - n^3 - 3lmn = l(l^2 - mn) - m(m^2 + ln) - n(n^2 + lm)$$

is also a linear combination of terms divisible by d. Hence

$$\gcd(N, s_1, s_2, s_3) \geqslant d > 1$$

and $\mathrm{TL}(N; s_1, s_2, s_3)$ does not exist. □

We next give a useful necessary condition.

Theorem 2. *A necessary condition for the existence of an* $\mathrm{HL}(l, m, n)$ *triple-loop is* $\gcd(l + m, l + n, m - n) = 1$.

Proof. We prove Theorem 2 through Theorem 1. Suppose $\gcd(l + m, l + n, m - n) = d > 1$. Then

$$l^2 - mn = l(l + m) - m(l + n),$$

$$m^2 + ln = m(l + m) - l(m - n),$$

$$n^2 + lm = n(l + n) + l(m - n).$$

So each of the three terms can be represented as a linear combination of terms divisible by d. It follows that

$$\gcd(l^2 - mn, m^2 + ln, n^2 + lm) \geqslant d > 1. \quad □$$

In application, the condition $\gcd(l+m, l+n, m-n) = 1$ involves only the first-order terms of l, m, n and may hence require less computation. It may also be easier to check as in the following case.

Corollary 1. *An* $\mathrm{HL}(l, m, m)$ *triple-loop does not exist.*

Proof. $\gcd(l + m, l + n, m - n) = \gcd(l + m, l + m, 0) = l + m > 1$. □

3. Applications

We use the conditions given in Section 2 to check the existence of some $\mathrm{HL}(l, m, n)$ triple-loop for those $\mathrm{HL}(l, m, n)$ proposed in [1].

$\mathrm{HL}(3x, x, x)$ was used in Theorem 1 of [1] to give an upper bound of N on a given diameter D. By Corollary 1, an $\mathrm{HL}(3x, x, x)$ triple-loop does not exist. So the upper bound

$$N = \tfrac{2}{27}(D + 3)^3$$

cannot be exactly reached.

Three other families of hyper-L tiles were proposed:

(i) $HL(3x, x+1, x-1)$ with $N = 16x^3 + 3x$ and $D = 6x - 2$,
(ii) $HL(3x+1, x+1, x)$ with $N = 16x^3 + 12x^2 + 3x$ and $D = 6x - 1$,
(iii) $HL(3x+2, x+1, x)$ with $N = 16x^3 + 36x^2 + 27x + 7$ and $D = 6x + 2$.

It can be verified that these families satisfy the conditions in Theorems 1 and 2. Furthermore, when x goes to infinitive, $(27/2)N/D^3$ approaches 1.

Acknowledgements

We thank the referees for many constructive comments that greatly improve the presentation of this paper.

References

[1] F. Aguiló, M.A. Fiol, C. Garcia, Triple-loop networks with small transmission delay, Discrete Math. 167/168 (1997) 3–16.
[2] J.-C. Bermond, F. Comellas, D.F. Hsu, Distributed loop computer networks: a survey, J. Parallel. Distribut. Comput. 24 (1995) 2–10.
[3] F.K. Hwang, A complementary survey of the double-loop networks, Theoret. Comput. Sci. A 263 (2001) 211–229.
[4] C.K. Wong, D. Coppersmith, A combinatorial problem related to multimodule organizations, J. Assoc. Comput. Mach. 21 (1974) 392–402.