Invariant and homogeneous bundles on $G/\Gamma$

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Abstract

Let $\Gamma$ be a cocompact lattice in a connected complex Lie group $G$. Given an invariant holomorphic vector bundle $E$ on $G/\Gamma$, we show that there is a trivial holomorphic subbundle $F \subset E$ such that any holomorphic section of $E$ factors through holomorphic sections of $F$. Given two homomorphisms $\gamma_1$ and $\gamma_2$ from $\Gamma$ to a complex linear algebraic Lie group $H$, with relatively compact image, we prove that any holomorphic isomorphism between the associated holomorphic principal $H$–bundles $E_H(\gamma_1)$ and $E_H(\gamma_2)$ is automatically $G$–equivariant.

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1. Introduction

Let $G$ be a connected complex Lie group, and let $\Gamma$ be a closed discrete subgroup such that the quotient $G/\Gamma$ is compact. It is known that the compact complex manifold $G/\Gamma$ is Kähler if and only if $G/\Gamma$ is a complex torus. Also, $G/\Gamma$ is Kähler if and only if $G$ is abelian. There are very large classes of examples of pairs $(G, \Gamma)$ of the above type such that $G$ is not abelian (see [8] for investigations of case $G = \text{SL}(2, \mathbb{C})$).

The image of any $g \in G$ in the quotient space $G/\Gamma$ will be denoted by $g\Gamma$. For any $g \in G$, let $\beta_g$ denote the holomorphic automorphism of $G/\Gamma$ defined by $x\Gamma \mapsto gx\Gamma$. A holomorphic vector bundle $E$ over $G/\Gamma$ is called invariant if for every $g \in G$, the pulled back vector bundle

$E|_{g\Gamma} = \beta_g^* E$.
\( \beta^*_g E \) is holomorphically isomorphic to \( E \). In [5] it was shown that that any invariant vector bundle is semistable. In [4,6] it was shown that any invariant vector bundle admits a holomorphic connection.

We prove the following (proved in Section 2.1):

**Theorem 1.1.** Let \( E \) be an invariant holomorphic vector bundle over \( G/\Gamma \). Then there is a unique holomorphically trivial subbundle \( F \subset E \) such that the natural homomorphism \( H^0(M, F) \hookrightarrow H^0(M, E) \) is surjective (equivalently, every holomorphic section of \( E \) factors through the subbundle \( F \)).

Let \( H \) be a complex linear algebraic Lie group. A homogeneous principal \( H \)-bundle on \( G/\Gamma \) is a pair of the form \((E_H, \rho)\), where \( E_H \) is a holomorphic principal \( H \)-bundle on \( G/\Gamma \), and \( \rho \) is a lift of the left-translation action of \( G \) on \( G/\Gamma \) to a holomorphic action of \( G \) on the total space of \( E_H \) which is compatible with the principal bundle structure of \( E_H \). A homogeneous principal \( H \)-bundle \((E_H, \rho)\) has a natural flat connection given by the action \( \rho \).

All homogeneous principal \( H \)-bundles are constructed using homomorphisms from \( \Gamma \) to \( H \). For any homomorphism \( \gamma : \Gamma \longrightarrow H \), the corresponding homogeneous principal \( H \)-bundle will be denoted by \( E^H(\gamma) \).

We prove the following (proved in Section 3.4):

**Theorem 1.2.** Let \( \gamma_i : \Gamma \longrightarrow H, i = 1, 2, \) be two homomorphisms such that the closures \( \gamma_1(\overline{\Gamma}) \) and \( \gamma_2(\overline{\Gamma}) \) are compact. If the two holomorphic homogeneous principal \( H \)-bundles \( E^H(\gamma_1) \) and \( E^H(\gamma_2) \) are isomorphic as holomorphic principal \( H \)-bundles, then they are isomorphic as holomorphic homogeneous principal \( H \)-bundles (meaning the holomorphic isomorphism can be chosen to be \( G \)-equivariant).

In fact, any holomorphic isomorphism \( E^H(\gamma_1) \longrightarrow E^H(\gamma_2) \) of principal \( H \)-bundles, where \( \gamma_1 \) and \( \gamma_2 \) are as in Theorem 1.2, is automatically \( G \)-equivariant (see Corollary 3.1). Theorem 1.2 is proved using Theorem 1.1 and the following proposition (proved in Section 3.2):

**Proposition 1.3.** Let \( \gamma : \Gamma \longrightarrow \text{GL}(n, \mathbb{C}) \) be a homomorphism such that the closure \( \overline{\gamma(\Gamma)} \subset \text{GL}(n, \mathbb{C}) \) is compact. Then any holomorphic section of the vector bundle \( E(\gamma) \) associated to \( \gamma \) is covariant constant with respect to the natural flat connection on \( E(\gamma) \).

**Corollary 1.4.** Let \( \gamma_i : \Gamma \longrightarrow H, i = 1, 2, \) be two homomorphisms such that the closures \( \overline{\gamma_1(\Gamma)} \) and \( \overline{\gamma_2(\Gamma)} \) are compact. If the two holomorphic principal \( H \)-bundles \( E^H(\gamma_1) \) and \( E^H(\gamma_2) \) are holomorphically isomorphic, then there is an element \( h \in H \) such that \( \gamma_2(g) = h^{-1} \gamma_1(g) h \) for all \( g \in \Gamma \).

The compact complex manifolds \( G/\Gamma \) of the above type with \( G \) non-commutative are the key examples of non-Kähler compact complex manifolds with trivial canonical bundle (non-Kähler analog of Calabi–Yau manifolds). These non-Kähler compact complex manifolds with trivial canonical bundle play an important role in string theory of theoretical physics (see [7,2,1,9] and references therein). They have also become an important topic of investigation in complex differential geometry (see [11,10]).
2. Sections of invariant vector bundles on $G/\Gamma$

Let $G$ be a connected complex Lie group. Let $\Gamma \subset G$ be a closed discrete subgroup such that the quotient $G/\Gamma$ is compact. This quotient $G/\Gamma$ will also be denoted by $M$. For each point $g \in G$, let

$$\beta_g : M := G/\Gamma \longrightarrow G/\Gamma$$

(2.1)

be the holomorphic automorphism defined by $x\Gamma \longmapsto gx\Gamma$.

A holomorphic vector bundle $E$ over $M$ is called invariant if the pulled back vector bundle $\beta_g^*E$ is holomorphically isomorphic to $E$ for every $g$.

2.1. Proof of Theorem 1.1

For any two holomorphic vector bundles $V$ and $W$ on $M$, the space of all holomorphic isomorphisms from $V$ to $W$ is a Zariski open subset of the affine space $H^0(M, W \otimes V^*)$. This can be seen as follows. If the ranks of $V$ and $W$ are different, then there is nothing to prove (there is no holomorphic isomorphism). So assume that $\text{rank}(V) = \text{rank}(W)$. Let $r$ be the common rank. Fix a point $x_0 \in M$. We have an algebraic morphism from the affine space

$$\delta : H^0(M, W \otimes V^*) \longrightarrow \text{Hom}\left(\bigwedge^r V, \bigwedge^r W\right)_{x_0} \cong \left(\bigwedge^r W_{x_0}\right) \otimes \left(\bigwedge^r V^*\right)$$

that sends any homomorphism

$$\psi : V \longrightarrow W$$

to the evaluation of $\bigwedge^r \psi : \bigwedge^r V \longrightarrow \bigwedge^r W$ at $x_0$. The space of holomorphic isomorphisms from $V$ to $W$ coincides with $\delta^{-1}\left((\bigwedge^r W_{x_0}) \otimes (\bigwedge^r V^*) \setminus \{0\}\right)$. Hence it is a Zariski open subset of the affine space $H^0(M, W \otimes V^*)$.

In particular, the space of holomorphic isomorphisms from $V$ to $W$ is connected (if it is nonempty).

Take $E$ as in the statement of the theorem. Let $A_E$ denote the space of all pairs of the form $(g, f_g)$, where $g \in G$ and $f_g$ is a holomorphic isomorphism $E \longrightarrow \beta_g^*E$ of vector bundles. As noted in [3], this $A_E$ is a complex Lie group (see [3, p. 609]). This group $A_E$ fits in a short exact sequence of complex Lie groups

$$e \longrightarrow \text{Aut}(E) \longrightarrow A_E \xrightarrow{p} G \longrightarrow e,$$

(2.2)

where $\text{Aut}(E)$ is the group of holomorphic automorphisms of the vector bundle $E$, and the projection $p$ is the forgetful map that sends any pair $(g, f_g)$ to $g$. The homomorphism $p$ is surjective because $E$ is an invariant vector bundle. Since both $\text{Aut}(E)$ and $G$ are connected, it follows that $A_E$ is connected. The group $A_E$ has a natural action on the total space of $E$. The action of any $(g, f_g)$ sends any $v \in E_x$ to $f_g(v) \in (\beta_g^*E)_x = E_{\beta_g(x)}$.

Let $E_0 := M \times H^0(M, E) \longrightarrow M$ be the trivial holomorphic vector bundle with fiber $H^0(M, E)$. Let

$$\phi : E_0 \longrightarrow E$$

(2.3)

be the evaluation map that sends any $(x, s) \in M \times H^0(M, E)$ to $s(x) \in E_x$. The action of $A_E$ on $E$ produces a linear action of $A_E$ on $H^0(M, E)$. The left-translation action of $G$ on $M$ produces an action of $A_E$ on $M$ through the projection $p$ in (2.2). Consider the diagonal action of $A_E$ on
Consider the restriction of the homomorphism \( \phi \)

\[
\phi_{e\Gamma'} : H^0(M, E) = (\mathcal{E}_0)_{e\Gamma'} \longrightarrow E_{e\Gamma'}
\]

to the point \( e\Gamma' \in M \). Let

\[
\mathcal{K} := \text{kernel}(\phi_{e\Gamma'}) \subset H^0(M, E)
\]

be the kernel of it. For the above action of \( A_E \) on \( H^0(M, E) \), the action of the subgroup \( p^{-1}(\Gamma') \subset A_E \) (see (2.2)) preserves the subspace \( \mathcal{K} \). Indeed, this follows immediately from the fact that the left-translation action of \( \Gamma' \) on \( M \) fixes the point \( e\Gamma' \). Our aim is to show that \( \mathcal{K} = 0 \).

The subgroup \( p^{-1}(\Gamma') \subset A_E \) is closed and is of finite co-volume with respect to the Haar measure, because

\[
A_E / p^{-1}(\Gamma') = G / \Gamma
\]

and \( G / \Gamma \) is compact. Since the action of \( p^{-1}(\Gamma') \) preserves the subspace \( \mathcal{K} \subset H^0(M, E) \), it follows that \( \mathcal{K} \) is preserved by the action of entire \( A_E \) [12, p. 864, Theorem 2] (it is a generalization of the Borel density theorem).

Take any \( \sigma \in \mathcal{K} \subset H^0(M, E) \), and also take a point \( g\Gamma' \in M \). To prove that \( \sigma(g\Gamma') = 0 \) (recall that we want to show that \( \mathcal{K} = 0 \)), choose an element

\[
\tilde{g} := (g^{-1}, f_{g^{-1}}) \in A_E;
\]

such an element \( \tilde{g} \) exists because \( E \) is invariant. Note that \( (\tilde{g} \cdot \sigma)(e\Gamma') = (f_{g^{-1}} \circ \sigma)(g\Gamma') \). On the other hand, \( (\tilde{g} \cdot \sigma)(e\Gamma') = 0 \), because \( \tilde{g} \cdot \sigma \in \mathcal{K} \) (we have shown above that \( \mathcal{K} \) is preserved by the action of \( A_E \)). Since

\[
(f_{g^{-1}} \circ \sigma)(g\Gamma') = (\tilde{g} \cdot \sigma)(e\Gamma') = 0,
\]

and \( f_{g^{-1}} \) is an isomorphism of vector bundles, we conclude that \( \sigma(g\Gamma') = 0 \). Consequently, the section \( \sigma \) vanishes identically, meaning \( \sigma = 0 \). Therefore, \( \mathcal{K} = 0 \).

This implies that the homomorphism \( \phi_{e\Gamma'} : H^0(M, E) \longrightarrow E_{e\Gamma'} \) that sends any section to its evaluation at \( e\Gamma' \) is injective.

Since

\begin{itemize}
  \item \( \phi \) in (2.3) is equivariant for the actions of \( A_E \),
  \item \( \phi \) is injective over the point \( e\Gamma' \) (as \( \mathcal{K} = 0 \)), and
  \item the action of \( A_E \) on \( G / \Gamma \) is transitive,
\end{itemize}

we conclude that \( \phi \) makes \( \mathcal{E}_0 \) a holomorphic subbundle of \( E \).

The vector bundle \( \mathcal{E}_0 = M \times H^0(M, E) \) is holomorphically trivial. Hence it is generated by its global holomorphic sections. Also, any holomorphic section of \( E \) is in the image of \( \phi \). More precisely, any holomorphic section \( \sigma \) of \( E \) is the image, under \( \phi \), of \( M \times \{ \sigma \} \).

Therefore, the subbundle \( \phi(\mathcal{E}_0) \subset E \) satisfies the conditions on \( F \) in Theorem 1.1. This completes the proof of Theorem 1.1. \( \square \)

**Remark 2.1.** The holomorphic vector bundle \( F \) in Theorem 1.1 is holomorphically trivial. Hence it is semistable; in fact, it is polystable. The vector bundle \( F \) is stable if and only if \( \dim H^0(M, E) = 1 \).
2.2. A lemma

The proof of Lemma 2.2 is due to S. G. Dani.

**Lemma 2.2.** Let \( \eta : G \to \text{GL}(n, \mathbb{C}) \) be a holomorphic homomorphism such that the closure \( \overline{\eta(\Gamma)} \) is compact. Then \( \eta \) is the trivial homomorphism.

**Proof.** Since \( G/\Gamma \) is compact, there exists a compact subset \( K \subset G \) such that \( G = K\Gamma \). Therefore, \( \overline{\eta(G)} = \overline{\eta(K\Gamma)} = \eta(K)\eta(\Gamma) \) is a compact subset of \( \text{GL}(n, \mathbb{C}) \).

Let \( \exp : \text{Lie}(G) \to G \) be the exponential map on the Lie algebra of \( G \). The composition \( \eta \circ \exp \) is a bounded entire function with values in \( \mathbb{C}^n \). Hence \( \eta \circ \exp \) is a constant function by Liouville’s theorem. From this it follows that \( \eta \) is the trivial homomorphism. \( \square \)

Lemma 2.2 will be needed in Section 3.

3. Homogeneous bundles on \( M \)

3.1. Flatness of sections

Let \( G \) and \( M \) be as before. A **homogeneous** vector bundle on \( M \) is a holomorphic vector bundle \( \alpha : E \to M \) together with a holomorphic left action on the total space \( \rho : G \times E \to E \) such that

- \( \alpha \circ \rho(g, v) = \beta_g(\alpha(v)) \) for all \( (g, v) \in G \times E \), where \( \beta_g \) is defined in (2.1), and
- for each point \( g \in G \), the map \( \rho_g : E \to E \) defined by \( v \mapsto \rho(g, v) \) is a holomorphic isomorphism of the holomorphic vector bundle \( E \) with the pulled back holomorphic vector bundle \( \beta_g^*E \).

Therefore, any homogeneous vector bundle is invariant. A homogeneous vector bundle \((E, \rho)\) has a natural holomorphic connection, which we will recall now.

For any point \( v \in E \), the horizontal tangent subspace of \( T_vE \) for this connection is the image of the Lie algebra \( \text{Lie}(G) \) in \( T_vE \) for the differential of the action \( \rho \). This holomorphic connection is clearly flat. In fact, the orbits give locally defined flat sections (an orbit need not be a global section).

The quotient map \( G \to G/\Gamma \) defines a principal \( \Gamma \)-bundle. Since \( \Gamma \) is discrete, this principal \( \Gamma \)-bundle has a flat connection; this flat connection will be denoted by \( \nabla_0 \). Given a homomorphism \( \gamma : \Gamma \to \text{GL}(n, \mathbb{C}) \), let \( E(\gamma) \to M \) be the vector bundle of rank \( n \) given by the principal \( \text{GL}(n, \mathbb{C}) \)-bundle associated to the principal \( \Gamma \)-bundle \( G \to G/\Gamma \) for \( \gamma \) and the standard representation of \( \text{GL}(n, \mathbb{C}) \). We recall that \( E(\gamma) \) is a quotient of \( G \times \mathbb{C}^n \) where any two elements \((g_1, v_1)\) and \((g_2, v_2)\) of \( G \times \mathbb{C}^n \) are identified if there is an element \( g \in \Gamma \) such that \( g_2 = g_1 g \) and \( v_2 = \gamma(g^{-1})(v_1) \). Note that the fiber \( E(\gamma)_e\Gamma \) is canonically identified with \( \mathbb{C}^n \) by sending any \( v \in \mathbb{C}^n \) to the equivalence class of \((e, v) \in G \times \mathbb{C}^n \). The above flat connection \( \nabla_0 \) induces a flat connection on \( E(\gamma) \); this induced flat connection on \( E(\gamma) \) will be denoted by \( \nabla^\gamma \). This flat connection \( \nabla^\gamma \) defines a holomorphic structure on \( E(\gamma) \). The left-translation action of \( G \) on
itself produces an action of $G$ on the associated bundle $E(\gamma)$. More precisely, the action of any $g \in G$ sends the equivalence class of any $(g', v') \in G \times \mathbb{C}^n$ to the equivalence class of $(gg', v')$. Therefore, $E(\gamma)$ is a homogeneous vector bundle. It is easy to see that all homogeneous vector bundles of rank $n$ on $M$ are given by homomorphisms from $\Gamma$ to $\text{GL}(n, \mathbb{C})$.

Note that the connection $\nabla^\gamma$ on $E(\gamma)$ (the one induced by $\nabla_0$) coincides with the connection on the homogeneous vector bundle $E(\gamma)$ constructed above using the differential of the action of $G$ on $E(\gamma)$.

### 3.2. Proof of Proposition 1.3

Consider the subbundle $F \subset E(\gamma)$ given by Theorem 1.1. Recall that $F$ is the trivial vector bundle on $M$ with fiber $H^0(M, E(\gamma))$ (the evaluation homomorphism $\phi$ in (2.3) was shown to be an isomorphism onto the image of $\phi$). Consider the fiber

$$H^0(M, E(\gamma)) = F_e \Gamma \subset E(\gamma)_e \Gamma = \mathbb{C}^n$$

(we noted earlier that $E(\gamma)_e \Gamma = \mathbb{C}^n$). The action of $G$ on $E(\gamma)$ produces a linear action of $G$ on $H^0(M, E(\gamma)) = F_e \Gamma$. The action of the subgroup $\Gamma' \subset G$ on $F_e \Gamma \subset \mathbb{C}^n$ clearly coincides with the one obtained by restricting the action $\Gamma'$ on $\mathbb{C}^n$ given by the homomorphism $\gamma$ through the standard action of $\text{GL}(n, \mathbb{C})$ on $\mathbb{C}^n$. Since $\gamma(\Gamma') \subset \text{GL}(n, \mathbb{C})$ is compact, it follows immediately that the image of $\gamma(\Gamma')$ in $\text{GL}(F_e \Gamma)$ is compact. Therefore, from Lemma 2.2 we conclude that the action of $G$ on $H^0(M, E(\gamma))$ is trivial. In other words, the action of $\Gamma$ on $\mathbb{C}^n$ (given by $\gamma$) fixes pointwise the subspace $H^0(M, E(\gamma)) \subset \mathbb{C}^n$. This implies that the holomorphic sections of $E(\gamma)$ are covariant constant with respect to the connection $\nabla^\gamma$. This completes the proof of Proposition 1.3. □

### 3.3. Homogeneous principal bundles

Let $H$ be a complex linear algebraic Lie group. Let $G$ and $M$ be as before. A homogeneous holomorphic principal $H$-bundle on $M$ is defined to be a pair of the form $(E_H, \rho)$, where

- $f : E_H \longrightarrow M$ is a holomorphic principal $H$-bundle, and
- $\rho : G \times E_H \longrightarrow E_H$ is a holomorphic left-action on the total space of $E_H$,

such that the following two conditions hold:

1. $(f \circ \rho)(g, z) = \beta_g(f(z))$ for all $(g, z) \in G \times E_H$, where $\beta_g$ is defined in (2.1), and
2. the actions of $G$ and $H$ on $E_H$ commute.

A homogeneous principal $H$-bundle $(E_H, \rho)$ has a natural flat holomorphic connection, which is described as follows. For any point $z \in E_H$, the horizontal tangent subspace of $T_z E_H$ for this connection is the image of the Lie algebra $\text{Lie}(G)$ in $T_z E_H$ by the differential of the action $\rho$.

Let $\gamma : \Gamma \longrightarrow H$ be a homomorphism. Let $E_H(\gamma)$ be the principal $H$-bundle over $M$ associated to the principal $\Gamma$-bundle $G \longrightarrow G/\Gamma$ for $\gamma$. The flat connection $\nabla_0$ on the principal $\Gamma$-bundle $G \longrightarrow G/\Gamma$ induces a flat connection on the associated principal $H$-bundle $E_H(\gamma)$. This flat connection, which we will denote by $\nabla^\gamma$, defines a holomorphic structure on $E_H(\gamma)$. The left-translation action of $G$ on itself produces a left action of $G$ on $E_H(\gamma)$. This action of $g \in G$ on $E_H(\gamma)$ sends the equivalence class of $(g', z) \in G \times H$ to the equivalence class of $(gg', z)$ (recall that $E_H(\gamma)$ is a quotient of $G \times H$).
The above holomorphic structure on $E_H(\gamma)$ and the action of $G$ on $E_H(\gamma)$ together make $E_H(\gamma)$ a homogeneous principal $H$-bundle.

All homogeneous principal $H$-bundles arise this way from homomorphisms from $\Gamma$ to $H$.

Just as for vector bundles, the connection $\nabla^{\gamma_1}$ on $E_H(\gamma_1)$ (respectively, $\nabla^{\gamma_2}$) be the flat connection on the principal $H$-bundle $E_H(\gamma_2)$ constructed earlier using the differential of the action of $G$ on $E_H(\gamma)$.

3.4. Proof of Theorem 1.2

Let $\nabla^{\gamma_1}$ (respectively, $\nabla^{\gamma_2}$) be the flat connection on the principal $H$-bundle $E_H(\gamma_1)$ (respectively, $E_H(\gamma_2)$) induced by the flat connection $\nabla_0$ on the principal $\Gamma$-bundle $G \to G/\Gamma$. Assume that the principal $H$-bundle $E_H(\gamma_1)$ is holomorphically isomorphic to the principal $H$-bundle $E_H(\gamma_2)$. To prove the theorem, it suffices to show that the two flat principal $H$-bundles $(E_H(\gamma_1), \nabla^{\gamma_1})$ and $(E_H(\gamma_2), \nabla^{\gamma_2})$ are isomorphic.

Fix a holomorphic isomorphism

$$\Phi : E_H(\gamma_1) \to E_H(\gamma_2)$$

(3.1)

of principal $H$-bundles. Our aim is to show that $\Phi$ takes the connection $\nabla^{\gamma_1}$ to the connection $\nabla^{\gamma_2}$.

Fix a holomorphic embedding

$$\iota : H \to \text{GL}(n, \mathbb{C})$$

(recall that $H$ is linear algebraic). Let $E(\iota \circ \gamma_1)$ and $E(\iota \circ \gamma_2)$ be the homogeneous vector bundles on $M$ associated to the homomorphisms $\iota \circ \gamma_1$ and $\iota \circ \gamma_2$ respectively. Note that $E(\iota \circ \gamma_1)$ is the vector bundle on $M$ obtained by extending the structure group of the principal $H$-bundle $E_H(\gamma_1)$ using the homomorphism $\iota$. Define the homogeneous vector bundle

$$\mathcal{W} := \text{Hom}(E(\iota \circ \gamma_1), E(\iota \circ \gamma_2)) = E(\iota \circ \gamma_2) \otimes E(\iota \circ \gamma_1)^*;$$

(3.2)

the actions of $G$ on $E(\iota \circ \gamma_1)$ and $E(\iota \circ \gamma_2)$ together define an action of $G$ on $\mathcal{W}$.

The group $\Gamma$ acts on $\text{End}_\mathbb{C}(\mathbb{C}^n)$ as follows: the action of any $g \in \Gamma$ sends any $T \in \text{End}_\mathbb{C}(\mathbb{C}^n)$ to the composition

$$\mathbb{C}^n \xrightarrow{\iota \gamma_1 (g^{-1}T)} \mathbb{C}^n \xrightarrow{T} \mathbb{C}^n \xrightarrow{\iota \gamma_2 (g)} \mathbb{C}^n.$$

Let $\tilde{\gamma} : \Gamma \to \text{GL}(\text{End}_\mathbb{C}(\mathbb{C}^n))$ be the homomorphism produced by this action. It is evident that the corresponding homogeneous vector bundle $E(\tilde{\gamma})$ is canonically identified with the vector bundle $\mathcal{W}$ defined in (3.2).

Let

$$\tilde{\Phi} : E(\iota \circ \gamma_1) \to E(\iota \circ \gamma_2)$$

(3.3)

be the holomorphic isomorphism of associated vector bundles produced by the isomorphism $\Phi$ in (3.1). So $\tilde{\Phi}$ gives a holomorphic section

$$\tilde{\Phi}' \in H^0(M, \mathcal{W}) = H^0(M, E(\tilde{\gamma})).$$

(3.4)

Let $\nabla^{\tilde{\gamma}}$ be the flat connection on $E(\tilde{\gamma})$ induced by the flat connection $\nabla_0$ on the principal $\Gamma$-bundle $G \to G/\Gamma$. From Lemma 2.2 we know that the holomorphic section $\tilde{\Phi}'$ in (3.4) is covariant constant with respect to the connection $\nabla^{\tilde{\gamma}}$ on $E(\tilde{\gamma})$. 
Since $\tilde{\Phi}'$ in (3.4) is covariant constant for the connection $\nabla^{\gamma}$, the isomorphism $\tilde{\Phi}$ in (3.3) takes the flat connection on $E(\iota \circ \gamma_1)$ (given by $\nabla_0$) to the flat connection on $E(\iota \circ \gamma_2)$ given by $\nabla_0$. It was noted earlier that $E(\iota \circ \gamma_1)$ is identified with the vector bundle obtained by extending the structure group of the principal $H$-bundle $E_H(\gamma_1)$ using the homomorphism $\iota$. This identification clearly takes the connection on $E(\iota \circ \gamma_1)$ given by $\nabla_0$ to the one given by $\nabla^{\gamma_1}$.

As the homomorphism of Lie groups $\iota$ is an embedding, the corresponding homomorphism of Lie algebras is injective. Therefore, from the fact that $\Phi$ in (3.3) takes the flat connection on $E(\iota \circ \gamma_1)$ to the flat connection on $E(\iota \circ \gamma_2)$ it follows immediately that the isomorphism $\Phi$ in (3.1) takes the connection $\nabla^{\gamma_1}$ to $\nabla^{\gamma_2}$. This completes the proof of Theorem 1.2. □

3.5. Some corollaries

Since $\Phi$ in (3.1) takes the connection $\nabla^{\gamma_1}$ to $\nabla^{\gamma_2}$, we have the following corollary:

**Corollary 3.1.** Let $\gamma_1$ and $\gamma_2$ be as in Theorem 1.2. Any holomorphic isomorphism between the holomorphic principal $H$-bundles $E_H(\gamma_1)$ and $E_H(\gamma_2)$ is $G$-equivariant.

Take two homomorphisms $\gamma_i : \Gamma \rightarrow H$, $i = 1, 2$. The two homogeneous principal $H$-bundles $E_H(\gamma_1)$ and $E_H(\gamma_2)$ are isomorphic (as homogeneous bundles) if and only if there is an element $h \in H$ such that $\gamma_2(g) = h^{-1}\gamma_1(g)h$ for all $g \in \Gamma$. Therefore, Theorem 1.2 has the following corollary:

**Corollary 3.2.** Let $\gamma_i : \Gamma \rightarrow H$, $i = 1, 2$, be two homomorphisms such that the closures $\gamma_1(\Gamma)$ and $\gamma_2(\Gamma)$ are compact. If the two holomorphic principal $H$-bundles $E_H(\gamma_1)$ and $E_H(\gamma_2)$ are holomorphically isomorphic, then there is an element $h \in H$ such that $\gamma_2(g) = h^{-1}\gamma_1(g)h$ for all $g \in \Gamma$.

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