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ESSAY REVIEW

Digging for Structure into the *Elements:* Euclid, Hilbert, and Mueller

Philosophy of Mathematics and Deductive Structure in Euclid's *Elements*. By Ian Mueller. Cambridge, Mass., London (MIT Press). 1981. xv + 378 pp. $45.00.

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. . . there is always a danger in historical interpretation of mistaking the obvious to us for the eternally obvious.—IAN MUELLER

Euclid's *Elements* (*Stoicheia*) was not the first book of its kind—an encyclopedic, postulatory–deductive treatment of elementary mathematics; Hippocrates of Chios, Leon, and Theudius of Magnesia wrote *Elements* before Euclid. These earlier writings were lost, while Euclid's synthesis survived to become almost synonymous with Greek elementary geometry and, at the same time, representative of Greek mathematics in general. Since its knowledge and methods are presupposed by the more advanced mathematical studies and their methods (conics, *neusis* constructions, invention of higher curves, arithmetico–algebraic investigations à la Diophantus, etc.), it is only fair (*pace* Fowler) [Fowler 1983] to see in the *Elements,* as Mueller does, a main source for the "understanding of the classical Greek conception of mathematics and its foundations and of the similarities and differences between that conception and our own" (p. viii).

To provide such an understanding is Mueller's main objective and, I believe, he succeeds remarkably well in achieving it. I do not believe, however, that his survey of the contents of the *Elements* is less "cumbersome . . . for someone neither familiar with the *Elements* nor willing to expend a great deal of labor to become familiar with them" (*sic;* p. viii) than Heath's. Perhaps the opposite is the case. Reading Mueller's book, I became more convinced than ever of the correctness of my views [Unguru 1975, 1979; Unguru & Rowe 1981, 1982], that ordinary and straightforward geometrical languages are the only historically acceptable metalanguages for explicating Euclid in particular and Greek mathematics in general. That algebra will not do is, by now, accepted by most historians (including Ian Mueller); that a symbolism à la Mueller will not do either, without too high a price to pay that makes it self-defeating, is at present also clear. (More on Mueller's symbolism below.)

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In order to achieve his aim, while emphasizing philosophical, foundational, and logical questions, Mueller investigates with great thoroughness and care the contents and the structure of the *Elements,* arriving, interestingly enough, at important conclusions notable precisely for their *historical* depth and accuracy. His account, then, though motivated by structural considerations, issues in judgments of high historical import and plausibility which reviewers so far have either glossed over or entirely overlooked.

The book contains seven chapters, four appendixes, a bibliography, and a (very inadequate) index. The material is arranged according to broad topics which are discussed in terms of some logically central propositions and their relations to the other propositions constituting the discussed topic. Mueller's methodological criterion, then, is the internal logic of the *Elements,* and this forces on him quite a bit of dislocation with respect to the order of the materials in the *Elements.* Thus he begins with a discussion of Book I (in which I.45 is the central proposition) and part of Book II, using Hilbert's *Grundlagen der Geometrie* as the modern counterpart to Euclid in order to emphasize the differences between the two, and he tackles from the very beginning the central issue of the relationship between geometry and algebra as relating to Book II. He next discusses Euclidean arithmetic (Books VII–IX), the theory of proportion (Book V) and its applications to plane rectilineal figures (Book VI), the circle and its relation to rectilineal figures (Books III and IV), in the context of which he also deals with Hippocrates of Chios and the method of exhaustion (XII.2). The last two chapters tackle, in turn, elementary solid geometry and the method of exhaustion (Books XI and XII) and the Platonic solids (Book XIII), in connection with which Mueller also investigates the classifications of irrational lines (Book X) and the remainder of Book II.

Ian Mueller's book is rich and varied in content, forcing reviewers to difficult selection criteria concerning those aspects of his thought that they find particularly important for emphasis. I deem fundamental Mueller's definitive and apodictic determination that Greek mathematics is unlike modern mathematics, with respect to the role of geometry in each, and his unambiguous rejection of the algebraic interpretation of the *Elements* as either historically justified or enlightening. I shall, therefore, concentrate my essay on this issue, the importance of which cannot be overemphasized. But before I do this, I would like to dispose of a problem that may trouble the unwary reader of the book. As Mueller repeatedly says, a basic difference between modern and ancient mathematics consists in "the dominant role of structure" in the former and "its virtual absence" in the latter (pp. ix–x). He goes on to point out that "Greek mathematics should not be interpreted in terms of structure" (p. 10). And yet his emphasis and major concern throughout the book is with deductive relationships, "deductive structure and foundations," logical and philosophical questions. Is there no contradiction here? Is not Mueller emphasizing the nonexistent? Not necessarily. Greek mathematics studies *objects* between which there exist logical relationships that the scholar can discern and analyze, while modern mathematics emphasizes and studies the logical relationships themselves that obtain between objects, such that the scholar looking at modern mathematics and dissecting it is twice removed from the mathematical objects. In other words, the objects of modern mathematics are themselves structures. Mueller's concern, then, is legitimate.

We come now to what I consider Mueller's main achievement, namely, his irrefutable demonstration that Euclid can be understood in his own right, as a geometer, and that appeal to algebraic interpretations is distorting, unnecessary, and illegitimate. This conclusion is reached with respect to any and all aspects of the *Elements* and gains in weight and impact as one advances in reading the book, additional arguments being continually added to it from various directions, fleshing out its far-reaching implications, and, ultimately, claiming the unmitigated assent of the attentive and not irredeemably biased reader. I shall mention only the main signposts on the road to this inescapable conclusion.

> It is quite clearly out of the question that one could establish the algebraic interpretation of much of Greek mathematics on the basis of vocabulary or explicit procedures. Rather, such an interpretation depends upon reading texts like book II as saying something other than what they appear to be saying. . . . the structural conception of mathematics, which is the core of algebra, is essentially foreign to Euclid. . . . the argument surely does provide *prima facie* grounds for doubting the viability of the algebraic interpretation. (p. 43)

Of course. And this *prima facie* evidence for the geometric interpretation is sufficiently plausible in the case of some of the most fundamental propositions of "geometric algebra" "to render the importation of algebra unnecessary" (p. 44). Moreover, when one performs the importation, different propositions in Euclid (II.1–3) become various forms (consequences) of the same proposition (II.1), which is not the way Euclid sees them, for whom each "states an independent geometric fact" (p. 46). This, by the way, is not an isolated case [1]. Furthermore, and in general, when one analyzes the central tenets of the algebraic interpretation, namely, that the lines and areas of "geometric algebra" express arbitrary quantities, that "geometric algebra" represents a geometric translation of Babylonian algebra, and that, as Van der Waerden put it, the line of thought in Greek mathematics is basically algebraic, one finds each and all of these tenets fundamentally bankrupt. It is clear then that "Euclid is approaching his subject by looking at the geometric properties of particular spatial configurations and not by considering abstract relations between quantities or formal relations between expressions" (p. 52).

This general assessment is buttressed by further arguments, each one representing one more nail in the coffin of the algebraic reading of Greek mathematics. Thus, showing that in the case of addition commutativity is taken for granted, while it is not so for multiplication (cf. VII.16), Mueller makes the very pertinent remark, supporting [Unguru & Rowe 1981] in their contention that rectangle formation is not multiplication [2]: "Of course commutativity would be obvious if multiplication were identified with forming a rectangle" (p. 73). Nor can one find the fundamental theorem of arithmetic in the *Elements*, VII.30–31 notwithstanding: "The sensible way to describe this situation would seem to be to say that although the *Elements* contains the materials for proving the fundamental theorem, it contains neither the theorem nor the equivalent of it" (p. 83). Moreover, compounding ratios and multiplication are not identical; this, needless to say, has devastating implications for the algebraic reading of the *Elements*. It follows immediately that neither does Euclid construe duplicating and triplicating as, respectively, squaring and cubing. Also, a careful examination of the arithmetic books makes it clear that for Euclid arithmetic and "geometric algebra" are separate endeavors, the former involving a combinatorial line of thought based on the notion of numbers as multitudes composed of units.

This brings us naturally to the Eudoxean theory of proportions contained in Book V. There is a world of difference between it and a theory of real numbers. The elements of such a theory cannot be uncovered in the *Elements* since ratios are not objects and there is no ordering relation of "being less than" between numerical ratios. Coupled with this is the fact that magnitudes in the *Elements* are abstractions from geometric objects (not including numbers), that Euclid's concerns in dealing with ratios are anything but calculational, and that Euclid failed to establish a correlation between his two treatments of proportionality (that for magnitudes and that for numbers) before attempting to combine them in Book X, a blemish in which Mueller sees "the greatest foundational flaw in the *Elements*" (p. 138). All this means that Book V, in spite of its "abstract" character, does not represent an attempt at complete axiomatization of proportion theory, but rather an attempt to abandon the realm of intuition for that of precise, formally correct definitions (5 and 7), enabling one to deal rigorously with equalities of ratios: "What is missing in Book V from a modern point of view is exactly the axiomatic foundation—the existential assumptions and combinatorial laws which underlie the whole book" (p. 148).

Book VI of the *Elements* has been typically considered, mainly because of VI.28–29, together with Book II, to be a mainstay of "geometric algebra." Mueller's thorough discussion of these propositions disposes effectively of their algebraic interpretation, showing that they are fully and satisfactorily graspable as geometric propositions [3] that function, moreover, as lemmas for other propositions (VI.28 for X.33–34 and implicitly for X.17–18 and many more in Book X, VI.29, which is never used in Book X, for VI.30). "Thus," to take VI.29 as an example, "it can be said that if the interpretation of II,4 as a geometric truth is accepted, the desire to prove VI.30 provides the basis for a satisfactory geometric explanation of the discovery and proof of VI.29" (p. 169). This, by the way, is not an isolated instance in the *Elements,* where a great many propositions comprising the so-called "geometric algebra" are proved and included not for their own sake but for their future geometric lemmatical use in other propositions.

Propositions VI.28–29 have been typically tied in with "Babylonian algebra" and a great deal has been made of their alleged identity and its implications for the contacts between the two mathematical cultures. Mueller rejects the Babylonian transfer myth (cf. pp. 170–172), though he chooses to say nothing about the historical legitimacy of the algebraic interpretation of Babylonian mathematics. For the sake of *his* arguments, he describes it lucidly, without taking an explicit and forceful stand on its real (historical) value. At times he even seems to embrace it, somewhat bashfully perhaps (cf. p. 176, n. 21). However, his intellectual honesty leaves him no choice but, in the end, to conclude:

> Nevertheless, I am not inclined to adopt the "Babylonian" explanation of Euclid's geometric algebra. Given the present state of our knowledge of the two bodies of mathematics and of the contact between the two cultures, the explanation raises as many problems as it solves. I shall therefore continue to try to establish that the "line of thought" in the *Elements* can be satisfactorily understood in terms of its surface geometric character. (p. 172)

In discussing elementary solid geometry and the method of exhaustion, Mueller makes the following pertinent points: (1) Elementary proportion theory is used in the *Elements* to avoid complex arguments concerning solid geometry but it is never used in an abstract computational manner as a replacement for geometric reasoning. (2) There are profound conceptual differences between Euclid's method of exhaustion, in which the "limit" (the geometrical object) is always given in advance, and the integral calculus in which one has to find the limit, if any, of a given sequence. Even though it is possible to see Euclid's technique as an "integration" technique, it is, nevertheless, always performed in innocence of a justificatory, theoretical law; its standard steps are always adapted to the particular geometrical issue at hand, precisely because of the absence of a theory of limits in Greek mathematics and the intuitive, unsatisfactory level of the concept of continuity which reigns in it. (3) Books XI and XII are exclusively geometric in character and one of their most striking features is the total elimination of the calculatory aspect.

Mueller concludes his analysis of the *Elements* with a connected discussion of Books X and XIII which also comprises a sprinkling of the pertinent propositions of Book II. What deserves mention here is that Mueller recognizes the purely classificatory and qualitative character of Book X while, at the same time, forcefully and cogently rejecting any attempt at algebraic explanations of any of the three Books. He argues, for example, that Euclid's motivation in proving propositions X.17–18 lies in the need to fit the edge of the regular icosahedron into a classificatory scheme. Indeed, ". . . the qualitative scheme evolved in book X is itself a sufficient explanation of the reasoning gone through to reach it" (p. 270). In the same connection, Mueller argues that X.91–102, in which the varius apotomes are discussed, are primary and fundamental because of their role in the determination of the edge value of the icosahedron; while the corresponding discussion of binomials and additive sides (X.54–65) is derivative, as it can easily be obtained from the former by simple combinatorial changes (it follows that the opposite is also true!), plays no role whatever in the *Elements*, and suffers from even more arbitrariness than the treatment of the apotomes. Mueller may be right. Still, his argumentation is largely *ad hoc*, the most telling point in its favor being the role played by the later propositions in obtaining the conclusions about the edge of the icosahedron. Finally, Mueller's discussions and drawings of the Platonic solids are much clearer than those one normally encounters in investigations of the topic and they make a very useful contribution toward an understanding of the matters dealt with by Euclid in Book XIII.

There is much, much more in Mueller's impressive book that deserves emphasis, but, for obvious reasons, what precedes should do [4]. Let me now mention, succinctly, my main criticisms of the book. Ian Mueller's logical symbolism is oppressive and opaque [5]. It seems to me that what one conceivably gains by its use in fathoming Euclid's deductive structure is achieved at the prohibitively high price of losing sight of Euclid's actual, commonsensical, convincing, elementary, syllogistico–geometric, rhetorical inference pattern (cf., for instance, pp. 70–71, 84–85, and the generalized discussions of Books X and XII). It is a symbolism consisting of a mixture of traditional and innovative notations that makes high demands on the reader and, at least in my case, forces him to go back very often to the Euclidean text for enlightenment and elucidation. This being the case with someone who is reasonably familiar with Euclid, it appears that Mueller's procedure is self-defeating and his goal of replacing Heath's *Euclid* unachievable by Mueller's means. Cases in point are the generalized discussion of proofs in Book XII and the abstract and largely diagramless, as well as disjointed, discussion (with respect to the Euclidean text) of the various irrational lines in Book X, in connection with the determination of the edge of the icosahedron. Both are exceedingly complicated, requiring constant recourse to good old Euclid and to various parts of Mueller's book. Thus, while Euclid's proof of XII.5 takes only slightly more than two pages in the *Great Books* edition of the *Elements*, Mueller's alternative proof—in which he also includes a very brief rendering of XII.3—is spread over almost four and a half dense pages of highly demanding and symbolically couched argument (cf. pp. 236–240) [6]. Concerning the treatment of Book X, I am not at all sure that Mueller's abstract, symbolic procedure has strong advantages over other, standard treatments of the book (Dijksterhuis', for example) and I can discern quite a few disadvantages.

It is clear that Mueller's book cannot replace the Euclidean text (or Heath's edition of it for that matter), because, among other reasons, Mueller plays havoc with the Euclidean arrangement and structure by rearranging and regrouping disparate things and by continually digging for structure. Additionally, both the enunciations and the proofs of Mueller are as a rule simplified, precisely because his guide is structural and, therefore, he will extract the "significant" and leave out the "superfluous," sometimes by contracting, other times by dilating (or separating) Euclidean enunciations, appealing to the converse proposition when Euclid deals with the direct, etc., very often needlessly complicating matters for the historically minded reader (cf., for instance, his discussion of III.10, on pp. 192–194). That Mueller is fully aware of his procedural dislocations is clear from some of his explicit statements:

> Euclid clearly thinks of subject matter as a more important organizing principle than deductive relevance—a fact which makes it difficult to read the *Elements* straight through. (p. 204)

> In general he [Euclid] does not seem to have a clear sense of the proportion–theoretic relations between various propositions so that cumbersome geometric argument is sometimes used when a simple proportionality argument would suffice. I have chosen not to go into the details of the geometric argument at such junctures in the belief that its complexity hinders more than it helps [!] in understanding Euclid's solid geometry. (p. 230)

Finally:

> This description involves some adaptation of the proofs [in which Euclid uses the method of exhaustion in Book XII] because Euclid does not follow a prescribed uniform procedure but adjusts his techniques to the problem at hand. (p. 231)

Ian Mueller is, as we saw, squarely against the algebraic interpretation of the *Elements*. The arguments advanced are presented cooly and in a deliberately balanced tone that is factual and nonjudgmental. This nonantagonistic, nonconfrontational tone is, clearly, meant to shun the ire of the "old guard" in the history of mathematics. It may even be successful. Sometimes, however, it leads to awkward situations as when, in the process of criticizing (again calmly and nonantagonistically) the central claims of the algebraic interpretation of Book II (cf. pp. 50–52), Mueller quotes H. G. Zeuthen (may he rest in peace) explicitly while referring (in the same footnote) only allusively and in a twice-removed fashion to Neugebauer (may he grace us with many more years of seminal contributions to the field). "Le style c'est l'homme." In the same connection, since he has shown conclusively that there is nothing algebraic about "geometric algebra," I find Mueller's use of the term without quotation marks unjustified and even misleading. Since a great many of his readers will naturally belong to the casual category (absorbing the important and deep "messages" of the book requires commitment to the topic and a lengthy time investment), there is a fair likelihood that they will be misled. That is a pity, since what is conceivably gained by assuaging the fury of the "geometrical algebraists" is more than counterbalanced by the damages stemming from faithfulness to the old, discredited usage. Thus, even some reviewers have failed to point out this crucial contribution of the book. This failure goes hand in hand, I think, with Mueller's own failure to discard the misleading term. *Cui prodest?* Illustrative of what I have in mind is the following passage:

> [Propositions II,] 4–7 are, of course, fundamental examples of geometric algebra and are used frequently in the *Elements*. As far as I am able to determine, there is nothing in the *Elements* themselves which makes the algebraic interpretation of the propositions more natural than the straightforward geometric one. (p. 301)

I would like to point out now a few more or less trivial mistakes that mar this otherwise excellent book. In two neighboring paragraphs (p. 289), Mueller says, concerning Euclid's classification of irrational lines, that "From a purely formalistic point of view one could hardly ask for more in a classification," a high praise, I assume; and, then, "Proposition 115 gives a clear demonstration of one of the limitations of the classificatory scheme developed in book X by showing that there are infinitely many kinds of straight lines which it fails to categorize," a serious criticism, presumably, of an important formal aspect of the classification. Obviously, then, one could ask Euclid to come up with a classificatory schema that would be exhaustive of the objects of his classification!

At times (p. 85), Mueller seems to think (wrongly, I believe) that it is possible to give non-Euclidean formulations to Euclidean propositions, while preserving their basically Euclidean character.

There appears to be a mistake on page 293, when Mueller claims that X.33–34 are never used subsequently; they are used, respectively in X.76 and X.77. On page 246 Archimedes' I.34, *Sphere and Cylinder,* is quoted mistakenly. On page 265 commensurability in length is mistakenly identified with commensurability in square. Finally, Hilbert's *Gesammelte Abhandlungen,* quoted on page 4, does not appear in the Bibliography.

In conclusion, Mueller's book represents a major contribution to Euclidean studies, a *vade mecum* for anyone interested in Greek mathematics that will take its honored place at the side of Heath's monumental edition, which it supplements and enriches. In his book Mueller demolishes, one after another, the shibboleths of the algebraic interpretation of the *Elements,* making a shambles of the central tenets of this interpretation, and he does this calmly, cooly and matter-of-factly, reaching the inescapable verdict about the geometric character of Euclid's masterpiece. It seems to me, therefore, that there is no longer a substantive excuse for anyone to go on speaking of "geometric algebra," "the Greek solution to quadratics," and the "Greek geometric translation of Babylonian algebra." Cleaving further to such an interpretation, then, is explainable in psycho–sociological rather than in substantive terms. It would be overly optimistic, however, to expect things to change quickly. But they must change and will change sooner or later, simply because inertia in scholarship, while both functional and valuable for its survival value, is not eternal. The need to rewrite the history of Greek mathematics has now been recognized and is, slowly but surely, gaining increasingly the acceptance of the scholarly community. Let it be.

As Ian Mueller put it, ". . . the significance of the *Elements* lies less in its final destination than in the regions travelled through to reach it. To a greater extent than perhaps any other major work in the history of mathematics, the *Elements* are [*sic*] a mathematical world" (p. 303), and this world, the Greek mathematical world, is *not* the world of algebra. This Mueller has shown beyond the shadow of a doubt and, although this may not have been his *telos,* I see in this his main achievement.

NOTES

1. For similar arguments, see [Unguru 1975].

2. It is gratifying to point out here that there is complete agreement between the arguments and conclusions of Mueller and those of [Unguru 1975; Unguru & Rowe 1981, 1982] with respect to the geometric, nonalgebraic character of the *Elements* and of much else in Greek mathematics.

3. See the lengthy discussion on the status of these propositions in [Unguru & Rowe 1982].

4. For instance, what Mueller says about Euclid's use of superposition and motion in geometry is both enlightening and convincing. It effectively disposes, I think, of the argument of Robert J. Wagner in "Euclid's Intended Interpretation of Superposition" (*Historia Mathematica* **10** (1983), 63–70). A prompt review of Mueller's book, then, might have conceivably saved the journal some space and caused the potential contributor to *Historia Mathematica* some second thoughts; all things considered, not too high a price to pay for speedy publication of reviews. Since a fair review means doing justice to an author, the known maxim applies: "Justice (review) delayed is justice denied."

5. For instance, Mueller's discussion of definition VII.21 complicates matters needlessly and is far from clear and convincing.

6. For fairness' sake, I should point out that Mueller's alternative proofs of propositions XII.10–12 are much shorter than Euclid's.

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