Linear Algebra and its Applications 433 (2010) 72–79

Contents lists available at ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

Generalized competition index of a primitive digraph

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ARTICLE INFO

Article history:
Received 4 August 2009
Accepted 21 January 2010
Available online 21 February 2010

Submitted by R.A. Brualdi

AMS classification:
05C50
15A48
05C20

Keywords:
Competition index
m-Competition index
Scrambling index
Primitive digraph

ABSTRACT

For positive integers k and m, and a digraph D, the k-step m-competition graph $C_m^k(D)$ of D has the same set of vertices as D and an edge between vertices x and y if and only if there are distinct m vertices $v_1, v_2, \ldots, v_m$ in D such that there are directed walks of length k from x to v_i and from y to v_i for $1 \leqslant i \leqslant m$. In this paper, we present the definition of m-competition index for a primitive digraph. The m-competition index of a primitive digraph D is the smallest positive integer k such that $C_m^k(D)$ is a complete graph. We study m-competition indices of primitive digraphs and provide an upper bound for the m-competition index of a primitive digraph.

1. Introduction

For terminology and notation used here we follow [1,3,7]. Let $D = (V, E)$ denote a *digraph* (directed graph) with vertex set $V = V(D)$, arc set $E = E(D)$ and order n. Loops are permitted but multiple arcs are not. A $x \to y$ *walk* in a digraph D is a sequence of vertices $x, v_1, \ldots, v_t, y \in V(D)$ and a sequence of arcs $(x, v_1), (v_1, v_2), \ldots, (v_t, y) \in E(D)$, where the vertices and arcs are not necessarily distinct. A *closed walk* is a $x \to y$ walk where $x = y$. A *cycle* is a closed $x \to y$ walk with distinct vertices except for $x = y$.

The *length of a walk* W is the number of arcs in W. The notation $x \xrightarrow{k} y$ is used to indicate that there is a $x \to y$ walk of length k. The *distance* from vertex x to vertex y in D is the length of a shortest walk from x to y, and denote this by $d(x, y)$. An l-cycle is a cycle of length l, and denote this by C_l. If the

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doi:10.1016/j.laa.2010.01.033

digraph D has at least one cycle, the length of a shortest cycle in D is called the *girth* of D, and denote this by $s(D)$.

A digraph D is called *strongly connected* if for each pair of vertices x and y in $V(D)$ there is a walk from x to y. For a strongly connected digraph D, the *index of imprimitivity* of D is the greatest common divisor of the lengths of the cycles in D, and is denoted by $l(D)$. If D is a trivial digraph of order 1, $l(D)$ is undefined. For a strongly connected digraph D, D is *primitive* if $l(D) = 1$. If D is primitive, there exists some positive integer k such that there is a walk of length exactly k from each vertex x to each vertex y. The smallest such k is called the *exponent* of D, denoted by $\exp(D)$.

Proposition 1 [3]**.** *Let D be a primitive digraph with n vertices, and let s be the girth of D. Then we have*

$$\exp(D) \leqslant n + s(n-2).$$

Cohen [6] introduced the notion of competition graph in connection with a problem in ecology. The *competition graph* of a digraph D, denoted by $C(D)$, has the same set of vertices as D and an edge between vertices x and y if and only if there is a vertex z such that (x, z) and (y, z) are arcs of D. Since the notion of competition graphs was introduced, there has been numerous literature on competition graphs. For surveys of the literature of competition graphs, see [8]. In addition to ecology, their various applications include applications to channel assignments, coding, and modeling of complex economic and energy systems.

Cho et al. [5] generalized competition graph to m-step competition graph. Let D be a digraph with vertex set V and let k be a positive integer. A vertex z of D is a *k-step common prey* for x and y if $x \xrightarrow{k} z$ and $y \xrightarrow{k} z$. The *k-step competition graph of D*, denoted by $C^k(D)$, has the same vertex set as D and an edge between vertices x and y if and only if x and y have a k-step common prey in D and $x \neq y$. The k-step digraph of D, denoted by D^k, has the same vertex set as D and an arc (x, y) if and only if there is a $x \xrightarrow{k} y$ in D. The k-step competition graph of D is the competition graph of D^k, $C^k(D) = C(D^k)$, see [7]. The concept of k-step digraph and k-step graph are not new, and some asymptotic behavior of D^k is well known, see [3]. For all undefined graph terminology, see [4,5].

For a positive integer m, *m-competition graph* of a digraph D, denoted by $C_m(D)$, has the same vertex set as D and an edge x and y if and only if there are at least m distinct vertices $v_1, v_2, \ldots, v_m$ and arcs (x, v_i) and (y, v_i) for $1 \leqslant i \leqslant m$. That is an edge x and y if and only if there is at least m common preys for vertices x and y. The *k-step m-competition graph* $C_m^k(D)$ has the same vertex set as D and an edge x and y if and only if there are at least m distinct vertices $v_1, v_2, \ldots, v_m$ such that v_i $(1 \leqslant i \leqslant m)$ is k-step common prey for vertices x and y, i.e. there exist $x \xrightarrow{k} v_i$ and $y \xrightarrow{k} v_i$. By the definition of m-competition graph, we have the following.

Proposition 2. *For any digraph D and positive integers m and k, we have*

$$C_m^k(D) = C_m(D^k).$$

Lemma 3. *For a primitive digraph D of order n* $(\geqslant 3)$ *and each positive integer m where* $1 \leqslant m \leqslant n$*, there is a positive integer k such that* $C_m^k(D) = K_n$*, where* K_n *denotes a complete graph of order n. We also have* $C_m^{k+1}(D) = K_n$ *if* $C_m^k(D) = K_n$*.*

Proof. We have $C_m(D^{\exp(D)}) = K_n$ by the definition of $\exp(D)$ and m-competition graph. Let $k = \exp(D)$. Then by Proposition 2 we have $C_m^k(D) = K_n$.

Next suppose k be a positive integer such that $C_m^k(D) = K_n$. Each pair of vertices has at least m common preys and each vertex has at least m preys in D^k. Consider two vertices x and y in $V(D)$. There exist vertices u and v such that there are arcs (x, u) and (y, v). If $u = v$, we can find m preys of u in D^k. If $u \neq v$, we can find m common preys of u and v in D^k. In all cases, we can find m common preys of x and y in D^{k+1}. Therefore, we have $C_m^k(D) = C_m^{k+1}(D) = K_n$. This establishes the result. □

2. Competition index and scrambling index of a primitive digraph

In this section, we assume that D is a primitive digraph and m is a positive integer such that $1 \leqslant m \leqslant n$. The *m-competition index* of D is the smallest positive integer k such that for every pair of vertices x and y, there exist distinct vertices $v_1, v_2, \ldots, v_m$ such that $x \xrightarrow{k} v_i$ and $y \xrightarrow{k} v_i$ for $1 \leqslant i \leqslant m$ in D. That is, the m-competition index of D is the smallest positive integer k such that every pair of vertices x and y have at least m common preys in D^k. The m-competition index of D is denoted by $k_m(D)$. From Lemma 3, $k_m(D)$ is the smallest positive integer k such that $C_m^k(D) = C_m^{k+i}(D) = K_n$ for every positive integer i. An analogous definition can be given for nonnegative matrices. The *m-competition index* of a primitive matrix A, denoted by $k_m(A)$, is the smallest positive integer k such that any two rows of A^k have positive elements in at least m identical columns.

Akelbek and Kirkland [1] introduced the scrambling index of a primitive digraph D, denoted by $k(D)$, and in 2008, Kim [7] introduced the competition index of a digraph. In the case of primitive digraphs, the definitions of scrambling index and competition index are the same. Furthermore, these definitions are the same as our definition of the m-competition index of a primitive digraph when $m = 1$. The m-competition index is a generalization of the competition index and the exponent of a primitive digraph.

For a primitive digraph D and $x, y \in V(D)(x \neq y)$, we define the *local m-competition index* of vertices x and y as

$$k_m(D : x, y) = \min\{k : x \text{ and } y \text{ have } m \text{ common preys in } D^t \text{ where } t \geqslant k\}.$$

We may define

$$k_m(D : x, x) = \min\{k : x \text{ has at least } m \text{ preys in } D^t \text{ where } t \geqslant k\}.$$

Consider a vertex $x \in V(D)$. We define the *local m-competition index* of x as

$$k_m(D : x) = \max_{y \in V(D)} \{k_m(D : x, y)\}.$$

Then,

$$k_m(D) = \max_{x \in V(D)} k_m(D : x) = \max_{x,y \in V(D)} k_m(D : x, y).$$

From the definitions of $k_m(D)$, $k_m(D : x)$, and $k_m(D : x, y)$, we have

$$k_m(D : x, y) \leqslant k_m(D : x) \leqslant k_m(D).$$

By the definitions of the m-competition index and the exponent of D of order n, we have

$$k_m(D) \leqslant \exp(D),$$

where m is a positive integer with $1 \leqslant m \leqslant n$. Furthermore we have

$$k_n(D) = \exp(D).$$

Proposition 4. *For a primitive digraph D of order n and for positive integers i, j such that $1 \leqslant i < j \leqslant n$, we have*

$$k_i(D) \leqslant k_j(D).$$

Furthermore, we have $k(D) \leqslant k_m(D)$ for a positive integer m $(\leqslant n)$.

Proof. For each pair of vertices x and y, $k_i(D : x, y) \leqslant k_j(D : x, y)$. This establishes the result. □

From the Proposition 4, we have the following relation between the competition index (scrambling index) and the exponent of a primitive digraph D:

$$k(D) = k_1(D) \leqslant k_2(D) \leqslant \cdots \leqslant k_n(D) = \exp(D).$$

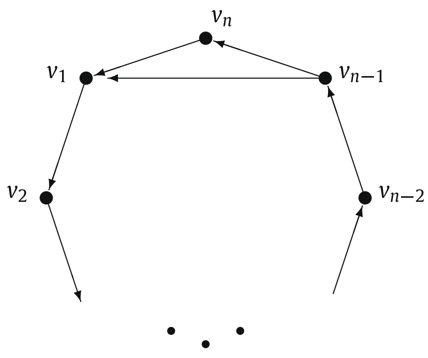

Fig. 1. The Wielandt graph W_n.

3. Upper bound for the *m*-competition index of a primitive digraph

In [1,2,7], there are some results related to $k(D) = k_1(D)$.

Proposition 5 [1]. *Let D be a primitive digraph with n vertices, and let s be the girth of D. Then, we have*

$$k(D) \leqslant n - s + \begin{cases} \binom{\frac{n-1}{2}} s, & \text{when } s \text{ is even,} \\ \binom{\frac{s-1}{2}} n, & \text{when } s \text{ is odd.} \end{cases}$$

Theorem 6 [1,7]. *Let D be a primitive digraph of order n, and let W_n be the Wielandt graph as shown in Fig. 1. Then,*

$$k(D) \leqslant k(W_n).$$

Equality holds if and only if $D = W_n$.

Next we determine the upper bound of $k_m(D)$ for a primitive digraph. First, we study the m-competition index of the Wielandt graph W_n.

Lemma 7. *For the Wielandt graph W_n and $1 \leqslant m \leqslant n$ $(n \geqslant 3)$, we have*

$$k_m(W_n) = 1 + \begin{cases} \binom{\frac{n+m-2}{2}} (n-1), & \text{when } n+m \text{ is even,} \\ \binom{\frac{n+m-3}{2}} n, & \text{when } n+m \text{ is odd.} \end{cases}$$

Proof. Let the vertex set of W_n be labeled as in Fig. 1.

Case 1. $n + m$ is even.

Consider two vertices x and y in $\{v_1, v_2, \ldots, v_{n-1}, v_n\}$. There are arcs (x, v_i) and (y, v_j) in W_n such that $1 \leqslant i, j \leqslant n - 1$. Consider the digraph $(W_n)^{n-1}$.

$$V((W_n)^{n-1}) = \{v_1, v_2, \ldots, v_n\},$$
$$E((W_n)^{n-1}) = \{(v_i, v_i) | i = 1, 2, \ldots, n-1\} \cup \{(v_i, v_{i-1}) | i = 2, 3, \ldots, n\} \cup \{(v_1, v_n)\}.$$

Since v_i and v_j are the loop vertex of $(W_n)^{n-1}$, the minimum number of vertices that can be reached from v_i at $\binom{\frac{n+m-2}{2}}$-step in $(W_n)^{n-1}$ is $\binom{\frac{n+m}{2}}$ and the minimum number of vertices that can be reached from v_j is the same. Therefore, v_i and v_j have at least m common preys at $\binom{\frac{n+m-2}{2}}$-step in $(W_n)^{n-1}$. That is, $k_m((W_n)^{n-1} : v_i, v_j) \leqslant \binom{\frac{n+m-2}{2}}$. Therefore, $k_m(W_n : x, y) \leqslant 1 + \binom{\frac{n+m-2}{2}} (n-1)$.

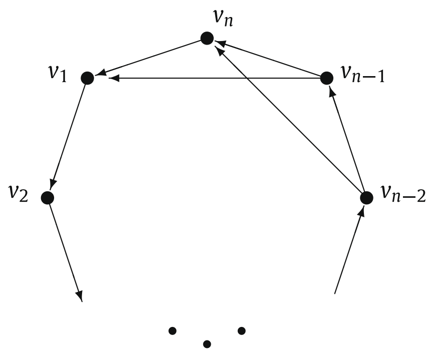

Fig. 2. W_n'.

Consider vertices v_n and $v_{\lfloor \frac{n}{2} \rfloor}$. In $(W_n)^{n-1}$, v_n and $v_{\lfloor \frac{n}{2} \rfloor}$ have no m common preys at $\left(\frac{n+m-2}{2}\right)$-step. Therefore, we have

$$k_m\left(W_n : v_n, v_{\lfloor \frac{n}{2} \rfloor}\right) > \left(\frac{n+m-2}{2}\right)(n-1),$$

which, in turn, leads to $k_m(W_n) = 1 + \left(\frac{n+m-2}{2}\right)(n-1)$.

Case 2. $n+m$ is odd.

Since $n+m$ is odd, we may suppose that $1 \leqslant m \leqslant n-1$. Consider two vertices x and y in $\{v_1, v_2, \ldots, v_{n-1}, v_n\}$. There are arcs (x, v_i) and (y, v_j) in W_n such that $1 \leqslant i, j \leqslant n-1$. Consider the digraph $(W_n)^n$. Each vertex has a loop, and $\{v_1, v_2, \ldots, v_{n-1}\}$ forms an $(n-1)$-cycle in $(W_n)^n$.

$$V((W_n)^n) = \{v_1, v_2, \ldots, v_n\},$$
$$E((W_n)^n) = \{(v_i, v_i) | i = 1, 2, \ldots, n\} \cup \{(v_i, v_{i+1}) | i = 1, 2, \cdots, n-1\} \cup \{(v_{n-1}, v_1), (v_n, v_1)\}.$$

Since v_i and v_j are the loop vertex of $(W_n)^n$, the minimum number of vertices that can be reached from v_i at $\left(\frac{n+m-3}{2}\right)$-step in $(W_n)^n$ is $\left(\frac{n-1+m}{2}\right)$ and the minimum number of vertices that can be reached from v_j is the same. Therefore, v_i and v_j have at least m common preys at $\left(\frac{n+m-3}{2}\right)$-step in $(W_n)^n$. That is, $k_m((W_n)^n : v_i, v_j) \leqslant \left(\frac{n+m-3}{2}\right)$. Therefore, $k_m(W_n : x, y) \leqslant 1 + \left(\frac{n+m-3}{2}\right) n$.

Consider vertices v_n and $v_{\frac{n-m+1}{2}}$. In $(W_n)^n$, v_n and $v_{\frac{n-m+1}{2}}$ have no m common preys at $\left(\frac{n+m-3}{2}\right)$-step. Therefore, we have

$$k_m\left(W_n : v_n, v_{\frac{n-m+1}{2}}\right) > \left(\frac{n+m-3}{2}\right) n,$$

and $k_m(W_n) = 1 + \left(\frac{n+m-3}{2}\right) n$. This establishes the result. □

By a similar argument, we can find the m-competition index of another digraph of order n and with girth $(n-1)$.

Example 8. For the digraph W_n' in Fig. 2 with girth $(n-1)$ and $1 \leqslant m \leqslant n$ $(n \geqslant 3)$, we have

$$k_m(W_n') = k_m(W_n) - 1.$$

Theorem 9. *Let D be a primitive digraph of order n $(\geqslant 3)$ and let s be the girth of D. For a positive integer m such that $1 \leqslant m \leqslant n$, we have*

$$k_m(D) \leqslant \begin{cases} n + \binom{\frac{n+m-4}{2}} s, & \text{when } n+m \text{ is even,} \\ n - 1 + \binom{\frac{n+m-3}{2}} s, & \text{when } n+m \text{ is odd.} \end{cases}$$

Proof. Let C be an s-cycle and consider two vertices x and y.

Case 1. $n+m$ is even.

We can find vertices u and v in $V(C)$ such that there exist $x \xrightarrow{n-s} u$ and $y \xrightarrow{n-s} v$. Since u and v are vertices in $V(C)$, there are loops, one containing u and the other containing v in D^s. D^s is primitive since D is primitive. For a positive integer l such that $1 \leqslant l \leqslant n$, the minimum number of vertices that can be reached from a vertex with a loop at l-step is $(l+1)$. Therefore, the minimum number of vertices that can be reached from u at $\binom{\frac{n+m-2}{2}}$-step is $\binom{\frac{n+m}{2}}$ in D^s. The minimum number of vertices that can be reached from v is the same. We hence have $k_m(D^s : u, v) \leqslant \binom{\frac{n+m-2}{2}}$, and $k_m(D : u, v) \leqslant \binom{\frac{n+m-2}{2}} s$. Therefore, we have

$$k_m(D : x, y) \leqslant n - s + \left(\frac{n+m-2}{2}\right) s.$$

Case 2. $n+m$ is odd.

There exists a vertex u in $V(C)$ such that there exists $x \xrightarrow{n-s-1} u$ or $y \xrightarrow{n-s-1} u$. Without loss of generality, we may assume that $x \xrightarrow{n-s-1} u(\in V(C))$. Then, we can find a vertex v in $V(C)$ such that there exists $y \xrightarrow{n-1} v$ since $n - s \leqslant n - 1$. The minimum number of vertices that can be reached from u at $\binom{\frac{n+m-1}{2}}$-step is $\binom{\frac{n+m+1}{2}}$ in D^s, and the minimum number of vertices that can be reached from v at $\binom{\frac{n+m-3}{2}}$-step is $\binom{\frac{n+m-1}{2}}$ in D^s. Therefore, we have

$$k_m(D : x, y) \leqslant n - s - 1 + \left(\frac{n+m-1}{2}\right) s.$$

This establishes the result. □

Denote

$$K(n, s, m) = \begin{cases} n + \binom{\frac{n+m-4}{2}} s, & \text{when } n+m \text{ is even,} \\ n - 1 + \binom{\frac{n+m-3}{2}} s, & \text{when } n+m \text{ is odd.} \end{cases}$$

The next example shows that Theorem 9 is sharp for a special case.

Example 10. Let D be a primitive digraph whose adjacency matrix A is given as

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The order of D is 5 and $s(D) = 3$. Thus, we can check

$$\begin{aligned} k_1(A) &= 7 \neq 8 = K(5,3,1), \\ k_2(A) &= 10 = K(5,3,2), \\ k_3(A) &= 11 = K(5,3,3), \\ k_4(A) &= 13 = K(5,3,4), \\ k_5(A) &= 14 = K(5,3,5). \end{aligned}$$

Remark. Let D be a primitive digraph of order n ($\geqslant 3$) and let s be the girth of D. From Theorem 9, we have

$$\exp(D) = k_n(D) \leqslant n + (n-2)s.$$

This is the same result of Proposition 1. We also have $k_{n-1}(D) \leqslant n - 1 + (n-2)s$ by Theorem 9, and equality holds when $D = W_n$ by Lemma 7.

Theorem 11. *Let D be a primitive digraph of order n ($\geqslant 3$). For a positive integer m such that $1 \leqslant m \leqslant n$, we have*

$$k_m(D) \leqslant k_m(W_n).$$

Equality holds if and only if $D = W_n$.

Proof. Let $s(D) = s$. By Theorem 6, this theorem holds when $m = 1$. Furthermore, by Lemma 7 and Example 8, this theorem holds when $s = n - 1$. Let us suppose $m \geqslant 2$ and $s \leqslant n - 2$. We will show that $k_m(D) < k_m(W_n)$.

Case 1. $n + m$ is even.

$$\begin{aligned}
k_m(D) &\leqslant n + \left(\frac{n+m-4}{2}\right)s \quad \text{(by Theorem 9)}\\
&\leqslant n + \left(\frac{n+m-4}{2}\right)(n-2) \quad \left(\frac{n+m-4}{2} \geqslant 0\right)\\
&= 3 - \frac{n+m}{2} + \left(\frac{n+m-2}{2}\right)(n-1)\\
&< 1 + \left(\frac{n+m-2}{2}\right)(n-1) \quad (n \geqslant 3,\ m \geqslant 2)\\
&= k_m(W_n).
\end{aligned}$$

Case 2. $n + m$ is odd.

$$\begin{aligned}
k_m(D) &\leqslant n - 1 + \left(\frac{n+m-3}{2}\right)s \quad \text{(by Theorem 9)}\\
&\leqslant n - 1 + \left(\frac{n+m-3}{2}\right)(n-2) \quad \left(\frac{n+m-3}{2} \geqslant 0\right)\\
&= 2 - m + \left(\frac{n+m-3}{2}\right)n\\
&< 1 + \left(\frac{n+m-3}{2}\right)n \quad (m \geqslant 2)\\
&= k_m(W_n).
\end{aligned}$$

If $m \geqslant 2$ and $s \leqslant n - 2$, then we have $k_m(D) < k_m(W_n)$. This establishes the result. □

4. Closing remark

Akelbek and Kirkland [1,2] obtained an upper bound for 1-competition index, and they also characterized all the primitive digraphs with this upper bound. In [9], it was carried out a research on $\exp(D)$ and its generalization for primitive matrices. These results of these studies can be generalized to m-competition indices. In the present study, we introduce $k_m(D)$ as another generalization of $\exp(D)$ and competition index of D.

Acknowledgements

The author would like to thank an anonymous referee for his or her outstanding job of suggesting changes. And the author would like to thank professor Han Hyuk Cho for his help.

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