Comparing Fuzzy Numbers: The Proportion of the Optimum Method

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ABSTRACT

The proportion of the optimum fuzzy number ranking procedure measures the consonance of the fuzzy number under comparison with the fuzzy ideals of $\widetilde{\max}$ *and* $\widetilde{\min}$. *This is accomplished by using three successive levels of analysis, where the number of levels utilized is dependent upon the difficulty of the ranking problem. This method is then compared to eight existing fuzzy number comparison methods. When evaluating all nine methods (using five examples) in terms of the method attributes of robustness, accuracy, and ease of use, the Lee-Li and proportion of the optimum methods are recommended. If, however, the decision maker desires the most flexible model, due to spread-preference differences, then the proportion of the optimum method is recommended.*

KEYWORDS: ***fuzzy sets, fuzzy numbers, fuzzy number comparison***

INTRODUCTION

The task of comparing fuzzy numbers to yield a totally ordered set can be extremely difficult. In many instances, only a partially ordered set results, which may be acceptable to a mathematician but not to an analyst, who may need a discrete evaluation of smallest and largest.

Many people have studied the comparison of fuzzy numbers and have proposed methods that yield a totally ordered set or ranking (Adamo [1], Baas and Kwakernak [2], Baldwin and Guild [3], Bortolan and Degani [4], Chang [5], Chen [6], Dubois and Prade [7], Jain [8, 9], Kerre [10], Li and Lee [11],

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Murakami [12], Nakamura [13], Tsukamoto et al [14], Watson et al [15], and Yager [16–18]). A review and comparison of these existing methods can be found in Li and Lee [11]. These methods range from the trivial to the complex, from including one fuzzy number attribute (mode) to including many fuzzy number attributes (such as mode, spread, and closeness to a fuzzy ideal). Some have higher discrimination capabilities and can solve more types of comparison problems (ie, are more robust). Still others have higher modeling accuracy. All are "goal-specific," meaning that they can be used only for a single type of decision-maker's comparison goal set.

This paper introduces a new fuzzy number comparison method, the proportion of the optimum [19]. It incorporates the fuzzy number attributes of mode and spread to determine the degree of consonance with the $\widetilde{\max}$ and $\widetilde{\min}$.

To illustrate the proportion of the optimum method in comparison with the existing comparison methods, five examples (Section 4) will be solved using each approach, highlighting the robustness, flexibility, accuracy, and ease of use of each method. The eight existing methods used for comparison are Yager's F1, F3, and F4 indices [18], Chang's [5], Kerre's [10], Murakami et al's [12], Nakamura's [13], and Lee and Li's [20]. The results will then be compared, and a discussion of the appropriate uses of each method will follow.

REVIEW OF FUZZY NUMBER COMPARISON METHODS

Since the proportion of the optimum method is not based on possibility, only the current methods that are not based on possibility theory will be reviewed.

Yager's F1 Index

Yager's first ranking index [16–18] can be considered a weighted mean value of the fuzzy number. In a general form, the index is

$$\mathrm{F1}(\tilde{A}_i) = \frac{\displaystyle\int_0^1 g(x)\mu_{\tilde{A}_i}(x)\,dx}{\displaystyle\int_0^1 \mu_{\tilde{A}_i}(x)\,dx} \tag{1}$$

where $g(x)$ is a weight function measuring the importance of the value x. When $g(x) = x$, the index can be thought of as the geometric center of A_i as seen in Figure 1. Note that the support of the fuzzy number is [0, 1]. If the support sets of the fuzzy numbers under comparison are not [0, 1], then they can be rescaled by dividing each of the members of $\max[\sup S_{A_i}]$, where i denotes the ith fuzzy number. The use of this scaling procedure will yield an index of $1/\max[\sup S_{\tilde{A}_i}]$ times the index if scaling is not used. The integration limits

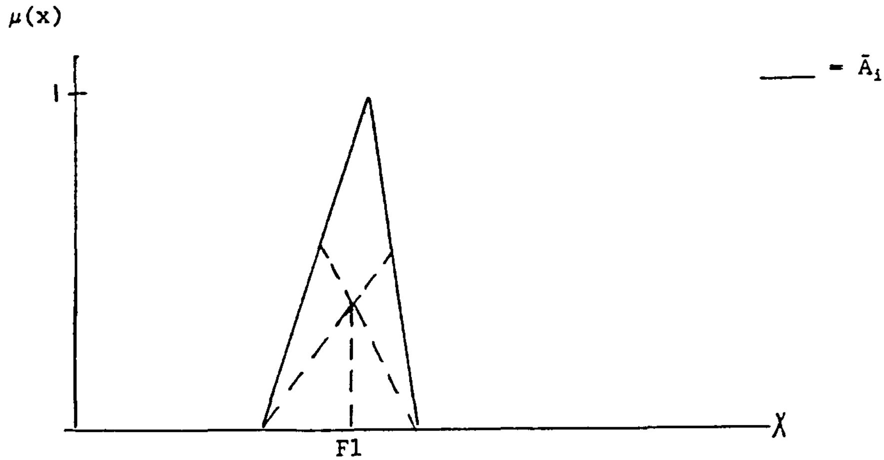

Figure 1. Yager's F1 index when $g(x) = x$.

are

$$\min_i[\inf S_{\tilde{A}_i}] \quad \text{and} \quad \max_i[\sup S_{\tilde{A}_i}]$$

If the exact indices are not desired, then scaling is not required. The decision logic is, the higher the index, the higher the ranking of the fuzzy number.

When $g(x) = x$ and the fuzzy numbers are triangular, Eq. (1) reduces to

$$F1(\tilde{A}_i) = \tfrac{1}{3}(a + b + c) \tag{2}$$

where $a = \inf S_{\tilde{A}_i}$, $\mu_{\tilde{A}_i}(b) = 1$, and $c = \sup S_{\tilde{A}_i}$.

The rankings of $\tilde{A}_1$ and $\tilde{A}_2$ in the five examples (Section 4) using this index, with $g(x) = x$, are listed in Table 1. Since Yager's F1 index is a measure of the mean value, it is subject to the pitfall of nondiscrimination when the modes of the fuzzy numbers are equivalent and the spreads are symmetric as seen in Example 2.

Yager's F3 Index

The F3 index [20] measures the area under the membership function from the mode back to the origin, as shown in Figure 2. The mathematical expression is

$$\mathrm{F3}(\tilde{A}_i) = \int_0^{\alpha_{max}} M(A_i^{\alpha})\, d\alpha \tag{3}$$

where $\alpha_{max} = \sup \mu_{\tilde{A}_i}(x)$ and $M(\tilde{A}_i^{\alpha})$ represents the average value of the elements having at least an α-grade of membership. Again, the higher the

Table 1. Rankings of $\tilde{A}_1$ and $\tilde{A}_2$ in Examples 1–5 by Method

Method	Example 1	2	3	4	5
Yager's F1 with $g(x)=x$	$\tilde{A}_1<\tilde{A}_2$	$\tilde{A}_1=\tilde{A}_2$	$\tilde{A}_1>\tilde{A}_2$	$\tilde{A}_1<\tilde{A}_2$	$\tilde{A}_1>\tilde{A}_2$
Yager's F3	$\tilde{A}_1<\tilde{A}_2$	$\tilde{A}_1=\tilde{A}_2$	$\tilde{A}_1>\tilde{A}_2$	$\tilde{A}_1<\tilde{A}_2$	$\tilde{A}_1<\tilde{A}_2$
Yager's F4	$\tilde{A}_1<\tilde{A}_2$	$\tilde{A}_1>\tilde{A}_2$	$\tilde{A}_1>\tilde{A}_2$	$\tilde{A}_1>\tilde{A}_2$	$\tilde{A}_1<\tilde{A}_2$
Chang's	$\tilde{A}_1<\tilde{A}_2$	$\tilde{A}_1>\tilde{A}_2$	$\tilde{A}_1>\tilde{A}_2$	$\tilde{A}_1>\tilde{A}_2$	$\tilde{A}_1<\tilde{A}_2$
Kerre's	$\tilde{A}_1<\tilde{A}_2$	$\tilde{A}_1=\tilde{A}_2$	$\tilde{A}_1>\tilde{A}_2$	$\tilde{A}_1<\tilde{A}_2$	$\tilde{A}_1>\tilde{A}_2$
Murakami's	$\tilde{A}_1<\tilde{A}_2$	$\tilde{A}_1=\tilde{A}_2$	$\tilde{A}_1>\tilde{A}_2$	$\tilde{A}_1<\tilde{A}_2$	$\tilde{A}_1>\tilde{A}_2$
Nakamura's with $\alpha=.5$	$\tilde{A}_1<\tilde{A}_2$	$\tilde{A}_1=\tilde{A}_2$	$\tilde{A}_1>\tilde{A}_2$	$\tilde{A}_1<\tilde{A}_2$	$\tilde{A}_1>\tilde{A}_2$
Lee and Li's (uniform densities)	$\tilde{A}_1<\tilde{A}_2$	$\tilde{A}_1<\tilde{A}_2$	$\tilde{A}_1>\tilde{A}_2$	$\tilde{A}_1<\tilde{A}_2$	$\tilde{A}_1>\tilde{A}_2$
Proportion of the optimum	$\tilde{A}_1<\tilde{A}_2$	$\tilde{A}_1<\tilde{A}_2$ or $\tilde{A}_1>\tilde{A}_2$	$\tilde{A}_1>\tilde{A}_2$	$\tilde{A}_1<\tilde{A}_2$	$\tilde{A}_1>\tilde{A}_2$

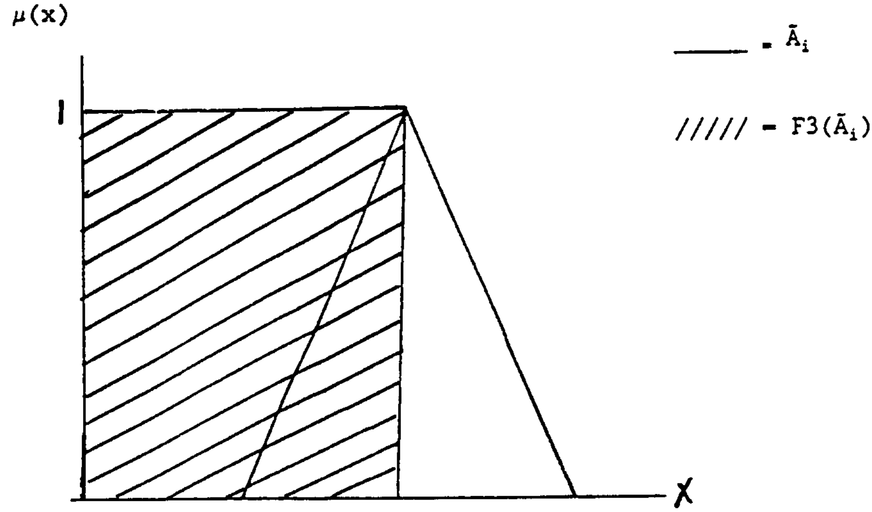

Figure 2. Yager's F3 index.

index value, the higher the ranking. Before $M(A_i^\alpha)$ can be calculated, however, the membership function must be expressed in terms of α (also known as the interval of confidence). The results for the five examples are listed in Table 1. Note that, like the F1 index, this index cannot discern fuzzy numbers with identical modes and symmetric spreads.

Yager's F4 Index

This index [20] measures the closeness, in terms of Hamming distance, of

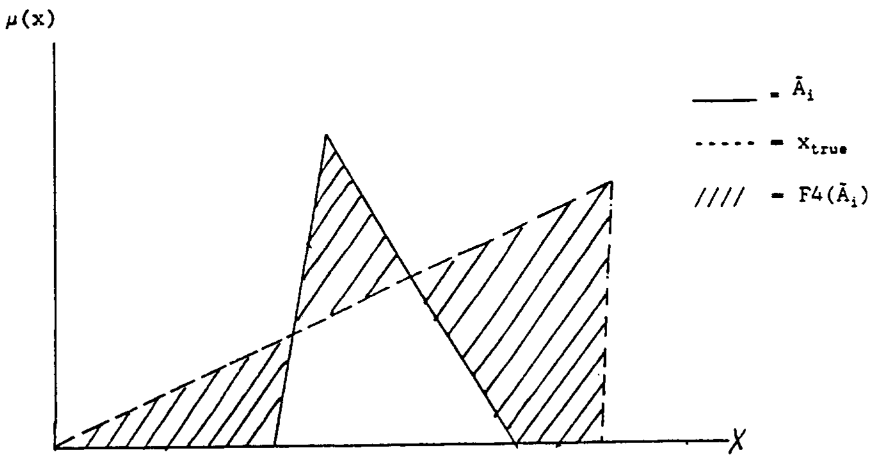

Figure 3. Calculation of Yager's F4 index.

the fuzzy number to a fuzzy "true" value defined as

$$\mu_{\text{true}}(x) = x \tag{4}$$

where the support of true is [0, 1]. If the supports of the fuzzy numbers in question are not [0, 1], the scaling method described for the F1 index can be used if desired.

The F4 index is exactly equal to the Hamming distance between $\mu_{\text{true}}(x)$ and $\mu_{\tilde{A}_i}(x)$. It is calculated by using

$$\text{F4}(\tilde{A}_i) = \int_0^1 \left| x - \mu_{\tilde{A}_i}(x) \right| dx \tag{5}$$

Figure 3 illustrates the $\text{F4}(\tilde{A}_i)$ calculation.

Note that in this case the *smaller* the F4 value, the *greater* the ranking of $\tilde{A}_i$. The rankings of $\tilde{A}_1$ and $\tilde{A}_2$ for Examples 1–5 are listed in Table 1. Unlike Yager's F1 and F3 indices, the F4 index *can* discriminate between fuzzy numbers with identical modes and symmetric spreads. Furthermore, this method ranks as higher those fuzzy numbers with larger spreads, while F1 and F3 rank fuzzy numbers with lower spreads as higher.

Chang's Method

Chang [5] proposes an index similar to Yager's F1 index. He defines his index as

$$C(\tilde{A}_i) = \int_{S_{\tilde{A}_i}} x\mu_{\tilde{A}_i}(x)\, dx \tag{6}$$

which reduces to

$$C(\tilde{A}_i) = \tfrac{1}{6}(b - a)(a + b + c) \tag{7}$$

for a triangular fuzzy number (a, b, c). Note that this is the numerator of Yager's F1 index with $g(x) = x$. Therefore, Chang's index is a nonnormalized version of Yager's F1 index with $g(x) = x$. Chang also has the restriction of the support of $\tilde{A}_i$ being [0, 1], so scaling must be employed if the support is not strictly [0, 1] and the exact indices are required. However, for comparison purposes only, the scaling need not be used, as the same decision will be reached with or without the scaling.

As with Yager's F1 index, the larger the index, the higher the ranking. The rankings for Examples 1–5 are also found in Table 1. Note that this method favors fuzzy numbers with larger spreads.

Kerre's Method

Like Yager, Kerre [10] applies the concept of minimizing the Hamming distance between the fuzzy number $\tilde{A}_i$ and some preidentified goal. Unlike

Yager, however, the goal Kerre chooses is the fuzzy maximum ($\widetilde{\max}$). The index is calculated by

$$K(\tilde{A}_i) = \int_{S_{\tilde{A}_i}} \left|\mu_{\tilde{A}_i}(x) - \mu_{\widetilde{\max}}(x)\right| dx \tag{8}$$

where $S_{\tilde{A}_i}$ is the support of $\tilde{A}_i$.

The smaller the index, the higher the ranking because of the smaller distance from the fuzzy maximum. Table 1 lists the rankings for Examples 1-5. This method behaves exactly like Yager's F1 with respect to nondiscrimination with identical modes and symmetric spreads.

The Murakami-Maeda-Imamura Methods

In 1983, Murakami, Maeda, and Imamura [12] introduced two ranking methods for triangular and trapezoidal fuzzy numbers: the α-cut method and the centroid method. The α-cut method is identical to Adamo's [1] method, which is based on possibility theory, so it will not be reviewed here.

The centroid method calculates the centroid (x_0, y_0) of each fuzzy number $\tilde{A}_i$, and the fuzzy number with the largest x_0 and y_0 values becomes the largest fuzzy number. The centroid values are defined as

$$x_0 = \frac{\int_0^1 x\mu_{\tilde{A}_i}(x)\,dx}{\int_0^1 \mu_{\tilde{A}_i}(x)\,dx} \tag{9}$$

and

$$y_0 = \frac{\int_0^1 x\mu_{\tilde{A}_i}(x)\tilde{A}_i(x)\,d\mu}{\int_0^1 x\mu_{\tilde{A}_i}(x)\,d\mu} \tag{10}$$

Note that these definitions implicitly restrict the support of the fuzzy numbers to $[0, 1]$, so scaling must be used if exact indices are desired. Also note that x_0 is exactly equal to Yager's F1 index when $g(x) = x$. Li and Lee [11] prove that y_0 also equals F1 when $g(x) = (1/2)\mu_{\tilde{A}}(x)$.

However, in special cases, the centroid values can lead to inconsistent conclusions. For example, let $\tilde{A}_i$ be the rectangular fuzzy number and $\tilde{A}_2$ the isosceles triangular fuzzy number illustrated in Figure 4. Note that when the x_0's are compared, the conclusion is $\tilde{A}_1 > \tilde{A}_2$. Recognizing this problem, Murakami et al suggest ranking the fuzzy numbers according to the importance the decision maker attaches to both x_0 and y_0. This means that the ordering is then

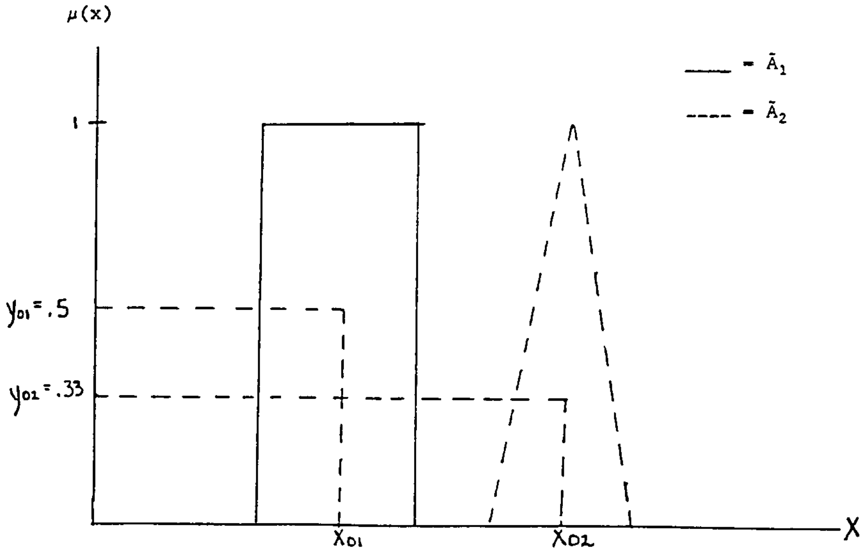

Figure 4. Illustration of special cases of the Murakami-Maeda-Imamura centroid method.

based on one coordinate of the centroid or the other. Bortolan and Degani [4] suggest that x_0 seems to be the only rational index, since most fuzzy numbers are normalized [sup $\mu_{\tilde{A}_i}(x) = 1$].

Using this method for the five examples yields the rankings listed in Table 1. When determining x_0 and y_0 for these triangular fuzzy numbers, $y_0 = 0.33$ for all of them (as for any triangular fuzzy number), so this method essentially defaults to Yager's F1 index with $g(x) = x$, with all its characteristics and qualities.

Nakamura's Method

Nakamura [13] builds upon Kerre's method and Tsukamoto et al's [14] method to form a comparison index $\mu_p(\tilde{A}_i > \tilde{A}_j)$ that combines the Hamming distance of both fuzzy numbers to the fuzzy minimum and the fuzzified best and worst states. Mathematically, $\mu_p(\tilde{A}_i > \tilde{A}_j)$ is defined as

$$\mu_p(\tilde{A}_i > \tilde{A}_j) = \begin{cases} \dfrac{1}{\Delta\alpha}[\alpha D(\tilde{A}_{Li}, \widetilde{\min}(\tilde{A}_{Li}, \tilde{A}_{Lj}) \\ \qquad + (1-\alpha) D(\tilde{A}_{Ri}, \widetilde{\min}(\tilde{A}_{Ri}, \tilde{A}_{Rj}))], & \Delta_\alpha \neq 0 \\ 1/2, & \Delta_\alpha = 0 \end{cases} \tag{11}$$

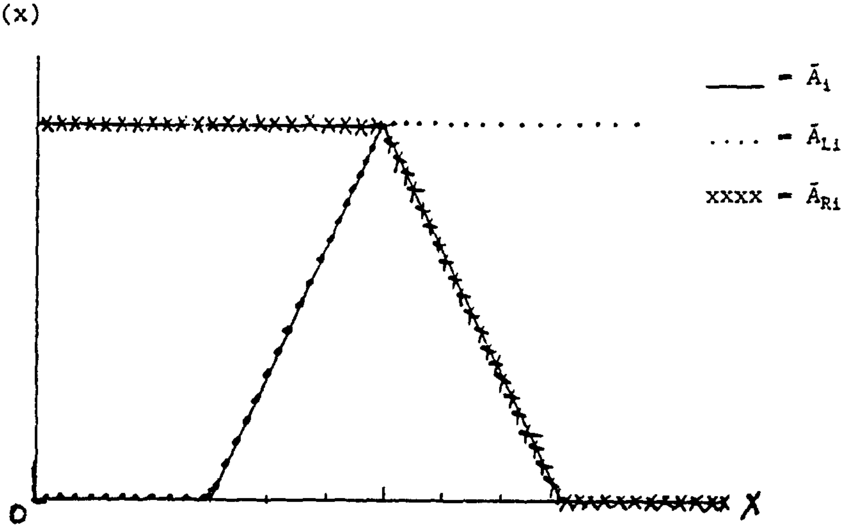

Figure 5. Illustration of A_{Li} and A_{Ri}.

where

$$\Delta_\alpha = \alpha[D(\tilde{A}_{Li}, \widetilde{\min}(\tilde{A}_{Li}, \tilde{A}_{Lj}) + D(\tilde{A}_{Li}, \widetilde{\min}(\tilde{A}_{Li}, \tilde{A}_{Lj}))] + (1-\alpha)[D(\tilde{A}_{Ri}, \widetilde{\min}(\tilde{A}_{Ri}, \tilde{A}_{Rj}) + D(\tilde{A}_{Rj}, \widetilde{\min}(\tilde{A}_{Ri}, \tilde{A}_{Rj}))] \quad (12)$$

$$D(\tilde{A}_i, \tilde{A}_j) = \int_S \left|\mu_{\tilde{A}_i}(x) - \mu_{\tilde{A}_j}(x)\right| dx \quad (13)$$

and

$$\mu_{\tilde{A}_{Li}}(r) = \sup_{r \geq x} \mu_{\tilde{A}_i}(x) \qquad \forall r \in R \quad (14)$$

and

$$\mu_{\tilde{A}_{Ri}}(r) = \sup_{r \leq x} \mu_{\tilde{A}_i}(x) \qquad \forall r \in R \quad (15)$$

$\tilde{A}_{Li}$ and $\tilde{A}_{Ri}$ are illustrated in Figure 5.

For clarification, the illustrations of $\widetilde{\min}(\tilde{A}_{Li}, \tilde{A}_{Lj})$ and $\widetilde{\min}(\tilde{A}_{Ri}, \tilde{A}_{Rj})$, which represent the fuzzified worst and best states, are shown in Figures 6 and 7, respectively. α reflects the decision-maker's preference placed on the left-hand side tail versus the right-hand side tail (ie, the desirability of erring lower versus erring higher). The index itself represents a weighted combination of the distances between a fuzzy number and the fuzzy best and worst

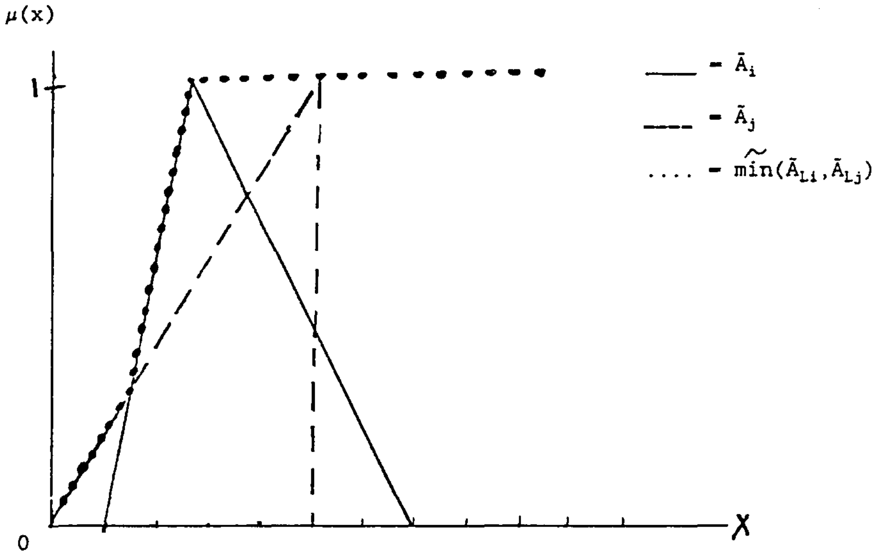

Figure 6. Illustration of $\min(A_{Li}, A_{Lj})$.

as a proportion of the sum of weighted combinations of the distances between both fuzzy numbers and the fuzzy best and worst. The logical use of the index is, when $\mu_p(\tilde{A}_i > \tilde{A}_j) \geq 0.5$ then $\tilde{A}_i > \tilde{A}_j$. Note that by definition $\mu_p(\tilde{A}_j > \tilde{A}_i) = 1 - \mu_p(\tilde{A}_i > \tilde{A}_j)$.

The solutions of Examples 1–5 when $\alpha = 0.5$ are listed in Table 1. In general, as α increases, fuzzy numbers with smaller spreads are judged to be greater. When $\alpha = 0.5$, Nakamura's method cannot distinguish fuzzy numbers with identical modes and symmetric spreads.

Lee and Li's Method

Very recently, Lee and Li [11] proposed a ranking method based on a generalized mean value and spread of the fuzzy numbers. In general, the generalized mean value is calculated as

$$m(\tilde{A}_i) = \frac{\displaystyle\int_{S_{\tilde{A}_i}} x\mu_{\tilde{A}_i}(x) f(\tilde{A}_i)\, dx}{\displaystyle\int_{S_{\tilde{A}_i}} \mu_{\tilde{A}_i}(x) f(\tilde{A}_i)\, dx} \tag{16}$$

where $f(\tilde{A}_i)$ is the probability density function of $\tilde{A}_i$. Note that this is a generalized version of Yager's F1 index function expressed in Eq. (1) with $g(x) = x$ and a uniform density $f(A_i) = 1/[A_i]$. However, the meanings are different.

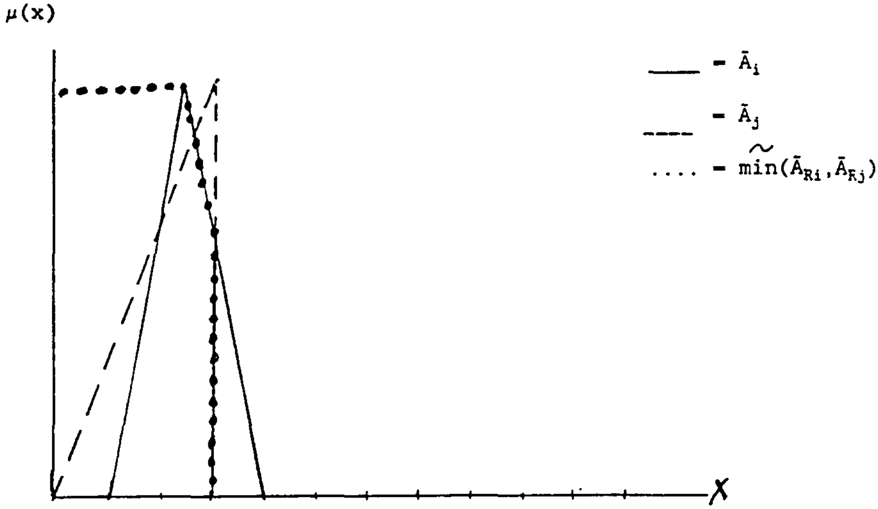

Figure 7. Illustration of $\min(A_{Ri}, A_{RJ})$.

Yager's F1 is a weighted mean value, whereas Lee and Li's is a probability measure of a fuzzy event. The spread is then calculated as

$$s(\tilde{A}_i) = \left[\frac{\int_{S_{\tilde{A}_i}} x^2 \mu_{\tilde{A}_i}(x) f(\tilde{A}_i)\, dx}{\int_{S_{\tilde{A}_i}} \mu_{\tilde{A}_i}(x) f(\tilde{A}_i)\, dx} - [m(\tilde{A}_i)] \right]^2 \tag{17}$$

When a uniform density function is assumed and the fuzzy numbers are triangular, Eqs. (16) and (17) reduce to

$$m(\tilde{A}_i) = (1/3)(a + b + c) \tag{18}$$

and

$$s(\tilde{A}_i) = (1/18)(a^2 + b^2 + c^2 - ab - ac - bc) \tag{19}$$

where $a = \inf S_{\tilde{A}_i}$, $\mu_{\tilde{a}_i}(b) = 1$, and $c = S_{\tilde{A}_i}$.

When a proportional density function, $f(\tilde{A}_i) = C\,\mu_{\tilde{A}}(x)$, is assumed, Eqs. (16) and (17) reduce to

$$m(\tilde{A}_i) = (1/4)(a + b + 2c) \tag{20}$$

$$s(\tilde{A}_i) = [(a - b)^2 + 2(a - c)^2 + 2(b - c)^2]/80 \tag{21}$$

Proofs of these reductions are given in Lee and Li [11]. Once the mean values and spreads are calculated for the fuzzy numbers, the ordering rule Li and Lee propose is

$$\text{If}\, m(\tilde{A}_i) > m(\tilde{A}_j)\, \text{or if}\, m(\tilde{A}_i) = m(\tilde{A}_i)\, \text{and}\, s(\tilde{A}_i) < s(\tilde{A}_j),\ \text{then}\, \tilde{A}_i > \tilde{A}_j.$$

Note that the spreads come into play only when the mean values are equal. No concurrent analysis of mean values and spreads is made.

When uniform probability distributions are assumed, the rankings for Examples 1–5 are those listed in Table 1. This method ranks as higher those fuzzy numbers with smaller spreads.

PROPORTION OF THE OPTIMUM METHOD

The proportion of the optimum method [19], like Kerre's and Nakamura's methods, compares the fuzzy numbers to specified fuzzy ideals. Recall that Kerre calculated the Hamming distance from the fuzzy number to the fuzzy maximum, and Nakamura calculated a ratio based on the Hamming distance from the fuzzy best and worst for each alternative to the global fuzzy best and worst. With the proportion of the optimum method, normalized fuzzy numbers will be compared on the basis of their consonance with the fuzzy max and fuzzy min. This will be accomplished using three levels or "cuts" of analysis:

1. "Rough cut"—the determination of the percentage of the fuzzy number used to form the fuzzy $\widetilde{\max}$ and fuzzy $\widetilde{\min}$, [$\widetilde{\max}$ (MP) and $\widetilde{\min}$ (mp) proportions].
2. "Fine cut"—the determination of the degree of propensity toward the $\widetilde{\max}$ or $\widetilde{\min}$ [composite $\widetilde{\max}$ (CMP) and composite $\widetilde{\min}$ (cmp) proportions].
3. "Final choice"—selection based on the decision-maker's preference of amount of spread or amount of vagueness desired (denominators of CMPs).

Not all comparison situations will require all three levels. In general, as the difficulty of the comparison increases, so does the number of levels used.

To perform the "rough cut," the $\widetilde{\max}$ (MP) and $\widetilde{\min}$ (mp) proportions are calculated using

$$\mathrm{MP}(\tilde{A}_i) = \frac{\displaystyle\int_{S_{\tilde{A}_i}} [\mu_{\widetilde{\max}(\tilde{A}_i \tilde{A}_j)}(x) \wedge \mu_{\tilde{A}_i}(x)]\, dx}{\displaystyle\int_{S_{\tilde{A}_i}} \mu_{\tilde{A}_i}(x)\, dx} \tag{22}$$

and

$$\mathrm{mp}(\tilde{A}_i) = \frac{\int_{S_{\tilde{A}_i}} [\mu_{\widetilde{\min}(\tilde{A}_i\tilde{A}_j)}(x) \wedge \mu_{\tilde{A}_i}(x)]\, dx}{\int_{S_{\tilde{A}_i}} \mu_{\tilde{A}_i}(x)\, dx} \tag{23}$$

After $\mathrm{MP}(\tilde{A}_i)$ and $\mathrm{mp}(\tilde{A}_i)$ are calculated for each alternative $\tilde{A}_i$, they are ranked according to the logic presented in Table 2. Note that if after the comparisons, $\mathrm{MP}(\tilde{A}_i)$ and $\mathrm{mp}(\tilde{A}_i)$ prove indeterminate, which may happen if a fuzzy number with small area is compared to a fuzzy number with large area, then the "fine cut" is employed using the composite $\widetilde{\max}$(CMP) and composite $\widetilde{\min}$(cmp) proportions:

$$\mathrm{CMP}(\tilde{A}_i) = \frac{\mathrm{MP}(\tilde{A}_i)}{\mathrm{MP}(\tilde{A}_i) + \mathrm{mp}(\tilde{A}_i)} = 1 - \mathrm{cmp}(\tilde{A}_i) \tag{24}$$

$$\mathrm{cmp}(\tilde{A}_i) = \frac{\mathrm{mp}(\tilde{A}_i)}{\mathrm{MP}(\tilde{A}_i) + \mathrm{mp}(\tilde{A}_i)} = 1 - \mathrm{cmp}(\tilde{A}_i) \tag{25}$$

These composite $\widetilde{\max}$ and $\widetilde{\min}$ proportions are compared using the logic presented in Table 3. [Note that because of the structure of $\mathrm{CMP}(\tilde{A}_i)+\mathrm{cmp}(\tilde{A}_i) = 1$, only the three instances listed in Table 3 are mathematically possible.] If the rankings are still indeterminate after this level, the "final choice" comparison factors, which are simply the denominators of the CMPs, are calculated. Specifically,

$$d[\mathrm{CMP}(\tilde{A}_i)] = \mathrm{MP}(\tilde{A}_i) + \mathrm{mp}(\tilde{A}_i) \tag{26}$$

The use of the denominators of the CMPs is dependent upon the decision-maker's preference of larger versus smaller spreads. (Note the last two translation "THEN" columns in Table 3.) In general, the third level will be invoked only if the fuzzy numbers under comparison are fairly indistinguishable (ie, same mode and/or symmetric spreads). Such a case is illustrated in Example 2. Out of the five examples presented, only Example 2 needs the third level for solution. The ranking for these five examples can also be found in Table 1.

If more than two fuzzy numbers are to be compared, then the $\mathrm{CMP}(\tilde{A}_i)$ and $\mathrm{cmp}(\tilde{A}_i)$ are based on a $\widetilde{\max}$ and $\widetilde{\min}$ formed by more than two fuzzy numbers. Two group rankings are then developed: one based on the $\mathrm{CMP}(\tilde{A}_i)$ and one based on $\mathrm{cmp}(\tilde{A}_i)$. Recall that, for the first ranking, the fuzzy number with the largest CMP is ranked largest and so on down. For the second ranking, the fuzzy number with the largest cmp is ranked smallest, and so on up. Often, the

Table 2. Ranking Logic for Proportion of the Optimum Method (MP and MP Proportions)

IF	AND	THEN
$\mathrm{MP}(\tilde{A}_i) > \mathrm{MP}(\tilde{A}_j)$	$\mathrm{mp}(\tilde{A}_i) \le \mathrm{mp}(\tilde{A}_j)$	$\tilde{A}_i > \tilde{A}_j$
	$\mathrm{mp}(\tilde{A}_i) > \mathrm{mp}(\tilde{A}_j)$	Calculate the composite proportions and compare, using Table 3.
$\mathrm{MP}(\tilde{A}_i) < \mathrm{MP}(\tilde{A}_j)$	$\mathrm{mp}(\tilde{A}_i) \ge \mathrm{mp}(\tilde{A}_j)$	$\tilde{A}_i < \tilde{A}_j$
	$\mathrm{mp}(\tilde{A}_i) < \mathrm{mp}(\tilde{A}_j)$	Calculate the composite proportions and compare, using Table 3.
$\mathrm{MP}(\tilde{A}_i) = \mathrm{MP}(\tilde{A}_j)$	$\mathrm{mp}(\tilde{A}_i) < \mathrm{mp}(\tilde{A}_j)$	$\tilde{A}_i > \tilde{A}_j$
	$\mathrm{mp}(\tilde{A}_i) = \mathrm{mp}(\tilde{A}_j)$	$\tilde{A}_i = \tilde{A}_j$
	$\mathrm{mp}(\tilde{A}_i) > \mathrm{mp}(\tilde{A}_j)$	$\tilde{A}_i < \tilde{A}_j$

orders will conflict in some way. Those fuzzy number relationships that conflict given initial group ranking are then further investigated on a pairwise basis to resolve the conflict, using the logic in Tables 2 and 3.

FIVE EXAMPLES

Example 1. Different Mode, Same Spreads

This example, illustrated in Figure 8, tests each method's capability of discriminating between means when the spreads of the fuzzy numbers are equal. The triangular membership functions of $\tilde{A}_1$ and $\tilde{A}_2$ are

$$\mu_{\tilde{A}_1}(x) = \begin{cases} x/3 - 1, & 3 \le x \le 6 \\ -x/3 + 3, & 6 < x \le 9 \\ 0, & \text{elsewhere} \end{cases} \tag{27}$$

and

$$\mu_{\tilde{A}_2}(x) = \begin{cases} x/3 - 5/3, & 5 \le x \le 8 \\ -x/3 + 11/3, & 8 < x \le 11 \\ 0, & \text{elsewhere} \end{cases} \tag{28}$$

Table 3. Ranking Logic for Proportions of the Optimum (CMP and CMP Proportion)

IF	AND	THEN	THEN (if smaller spreads preferred)	THEN (if larger spreads preferred)	THEN (if indifferent to type of spread preferred)
$\mathrm{CMP}(\tilde{A}_i) > \mathrm{CMP}(\tilde{A}_j)$	$\mathrm{cmp}(\tilde{A}_i) < \mathrm{cmp}(\tilde{A}_j)$	$\tilde{A}_i > \tilde{A}_j$			
$\mathrm{CMP}(\tilde{A}_i) < \mathrm{CMP}(\tilde{A}_j)$	$\mathrm{cmp}(\tilde{A}_i) > \mathrm{cmp}(\tilde{A}_j)$	$\tilde{A}_i < \tilde{A}_j$			
$\mathrm{CMP}(\tilde{A}_i) = \mathrm{CMP}(\tilde{A}_j)$	$\mathrm{cmp}(\tilde{A}_i) = \mathrm{cmp}(\tilde{A}_j)$	If $d[\mathrm{CMP}(\tilde{A}_i)] > d[\mathrm{CMP}(\tilde{A}_j)]$[a]	$\tilde{A}_i > \tilde{A}_j$	$\tilde{A}_i < \tilde{A}_j$	$\tilde{A}_i = \tilde{A}_j$
		If $d[\mathrm{CMP}(\tilde{A}_i)] < d[\mathrm{CMP}(\tilde{A}_j)]$	$\tilde{A}_i < \tilde{A}_j$	$\tilde{A}_i > \tilde{A}_j$	$\tilde{A}_i = \tilde{A}_j$
		If $d[\mathrm{CMP}(\tilde{A}_i)] = d[\mathrm{CMP}(\tilde{A}_j)]$	$\tilde{A}_i = \tilde{A}_j$	$\tilde{A}_i = \tilde{A}_j$	$\tilde{A}_i = \tilde{A}_j$

[a] Where $d[\mathrm{CMP}(\tilde{A}_i)]$ stands for the denominator of the composite proportion $\mathrm{CMP}(\tilde{A}_i)$, ie, $\mathrm{MP}(\tilde{A}_i) + \mathrm{mp}(\tilde{A}_i)$.

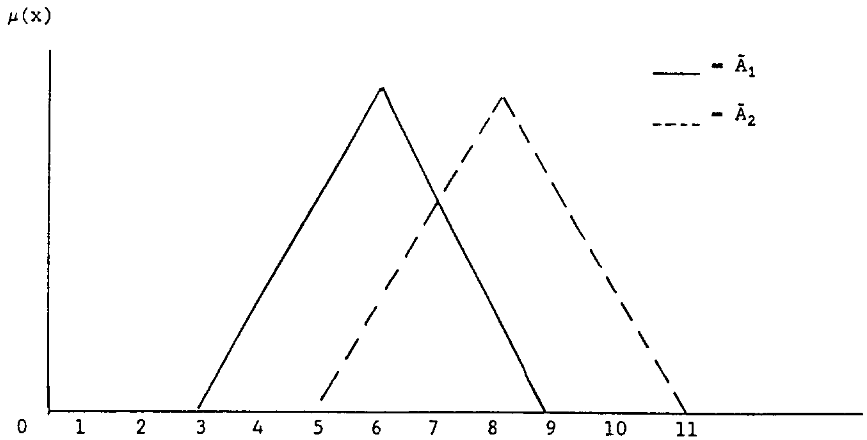

Figure 8. Example 1. Different mode, same spread.

Example 2. Same Mode, Different and Symmetric Spreads

This example tests each method's capability of discriminating between spreads when the modes of the fuzzy numbers are the same. Figure 9 illustrates this example.

$$\mu_{\tilde{A}_1}(x) = \begin{cases} x/3 - 1, & 3 \le x \le 6 \\ -x/3 + 3, & 6 < x \le 9 \\ 0, & \text{elsewhere} \end{cases} \tag{29}$$

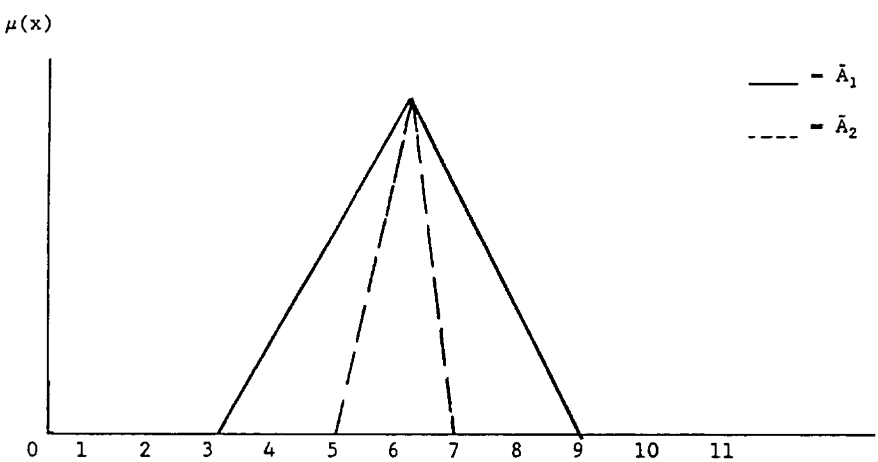

Figure 9. Example 2. Same mode, different and symmetric spreads.

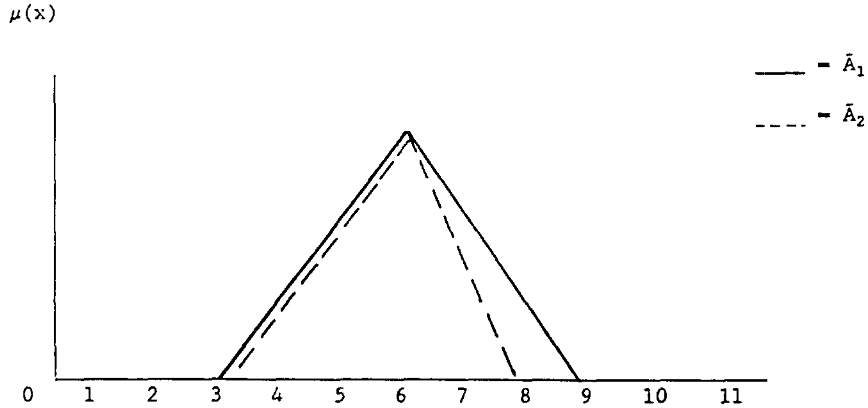

Figure 10. Example 3. Same LHS function.

and

$$\mu_{\tilde{A}_2}(x) = \begin{cases} x/3 - 5, & 5 \leq x \leq 6 \\ -x/3 + 7, & 6 < x \leq 7 \\ 0, & \text{elsewhere} \end{cases} \tag{30}$$

Example 3. Same Left-Hand-Side (LHS) Function

This example notes how the methods perform when the two fuzzy numbers being compared have a left-hand-side function in common. Figure 10 illustrates this example. The membership functions are

$$\mu_{\tilde{A}_1}(x) = \begin{cases} x/3 - 1, & 3 \leq x \leq 6 \\ -x/3 + 1, & 6 \leq x \leq 9 \\ 0, & \text{elsewhere} \end{cases} \tag{31}$$

and

$$\mu_{\tilde{A}_2}(x) = \begin{cases} x/3 - 1, & 3 \leq x \leq 6 \\ -x/2 + 4, & 6 < x \leq 8 \\ 0, & \text{elsewhere} \end{cases} \tag{32}$$

Example 4. Same Right-Hand-Side (RHS) Function

This example notes how the methods perform when the two fuzzy numbers being compared have a right-hand-side function in common. Figure 11 illus-

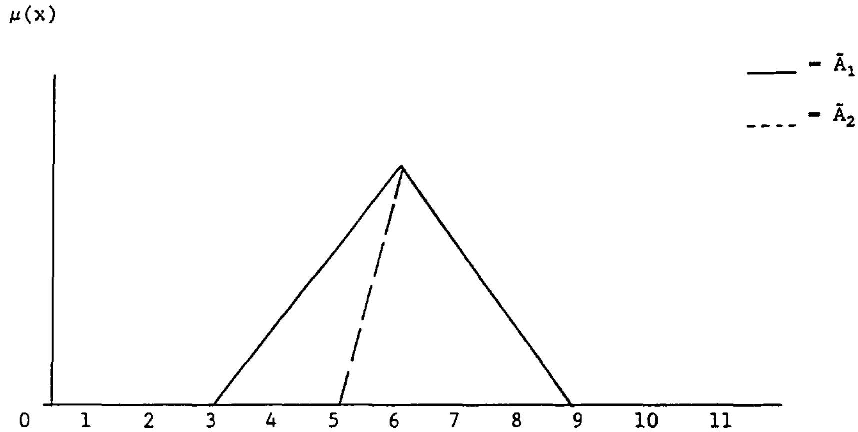

Figure 11. Example 4. Same RHS function.

trates this example. The membership functions are

$$\mu_{\tilde{A}_1}(x) = \begin{cases} x/3 - 1, & 3 \leq x \leq 6 \\ -x/3 + 3, & 6 < x \leq 9 \\ 0, & \text{elsewhere} \end{cases} \tag{33}$$

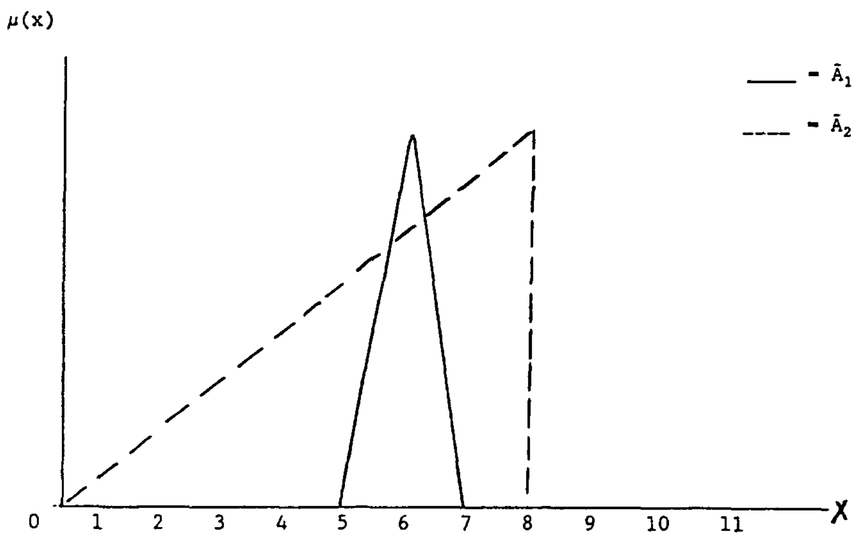

Figure 12. Example 5. Low mode, small spread versus high mode, large spread.

and

$$\mu_{\tilde{A}_2}(x) = \begin{cases} x-5, & 5 \le x \le 6 \\ -x/3+3, & 6 < x \le 9 \\ 0, & \text{elsewhere} \end{cases} \tag{34}$$

Example 5. Low Mode, Small Spread Versus High Mode, Large Spread

This example illustrates how the method "trades off" mode values versus spread values. This example is illustrated in Figure 12, and the membership functions are noted below.

$$\mu_{\tilde{A}_1}(x) = \begin{cases} x-5, & 5 \le x \le 6 \\ -x+7, & 6 < x \le 7 \\ 0, & \text{elsewhere} \end{cases} \tag{35}$$

$$\mu_{\tilde{A}_2}(x) = \begin{cases} x/7-7, & 0 \le x \le 7 \\ -x+8, & 7 < x \le 8 \\ 0, & \text{elsewhere} \end{cases} \tag{36}$$

COMPARISON OF PROPORTION OF THE OPTIMUM WITH THE OTHER METHODS

The nine ranking methods explored in this paper will be compared with one another using four criteria: robustness, accuracy, flexibility, and ease of use.

Robustness refers to the ability of the method to distinguish the larger from the smaller fuzzy number, that is, to come up with a discrete ranking without an equality. Therefore, the method that has the most discrete rankings (least number of equalities) would be judged the most robust.

Accuracy is the degree to which the method matches intuition. Without empirical data, intuition must suffice as the "real-world" base.

Flexibility is the ability of the method to incorporate different goal sets. For example, one decision maker may prefer smaller-spread fuzzy numbers; another may prefer larger-spread numbers. Those methods that can adapt to different goal sets are judged to be more flexible than those methods that cannot.

Ease of use incorporates the length of solution procedure with the ease of numerical computations required to solve a given fuzzy number comparison problem. The methods that require simpler numerical computations and a shorter solution algorithm would be judged easier to use.

Using intuition, the rankings for the five examples are listed below, and the behavior of the ranking methods is discussed.

EXAMPLE 1. DIFFERENT MODE, SAME SPREAD. The ranking should be $\tilde{A}_1 < \tilde{A}_2$, because the spreads are exactly the same and the mode of $\tilde{A}_2$ is larger than the mode of $\tilde{A}_1$.

All of the nine ranking methods matched intuition.

EXAMPLE 2. SAME MODE, DIFFERENT AND SYMMETRIC SPREADS. Here, the ranking could be $\tilde{A}_1 < \tilde{A}_2$ if the decision maker preferred less vagueness (smaller spread), or $\tilde{A}_1 > \tilde{A}_2$ if the decision maker preferred a greater amount of vagueness to ensure inclusion of all the extremes.

For this example, over half the models were unable to discriminate between the fuzzy numbers (Yager's F1, Yager's F3, Kerre's, Murakami's centroid, and Nakamura's with $\alpha = .5$). Yager's F4 and Chang's models implicitly favor the larger-spread numbers, and Lee and Li's model favors smaller-spread numbers. Only the proportion of the optimum method allows the decision maker to explicitly favor the smaller or larger spread numbers.

EXAMPLE 3. SAME LEFT-HAND-SIDE (LHS) FUNCTION. Here, the ranking should be $\tilde{A}_1 > \tilde{A}_2$, because the modes and LHS functions are identical and $\tilde{A}_1$ has a greater number of higher support values. Even if a decision maker is spread-averse, this is still the logical ranking due to the identical modes.

All of the nine ranking methods matched intuition.

EXAMPLE 4. SAME RIGHT-HAND-SIDE (RHS) FUNCTION. The ranking should be $\tilde{A}_1 < \tilde{A}_2$, because modes and LHS functions are identical and $\tilde{A}_1$ has a greater number of higher support values. Again, this should be the logical ranking even if the decision maker prefers larger spreads, because the additional vagueness is on the lower end.

The majority of the models, including the proportion of the optimum method, yielded this ranking. However, Yager's F4 and Chang's methods had the reverse ranking. These two methods seem to favor the larger spread numbers regardless of the direction of the spreads.

EXAMPLE 5. LOW MODE, SMALL SPREAD VERSUS HIGH MODE, LARGE SPREAD. Upon introspection, the ranking should be $\tilde{A}_1 > \tilde{A}_2$. If a decision maker is spread-averse, the trade-off of a higher-mean number with a large spread for a slightly lower-mean number with a substantially smaller spread would seem acceptable. Even if a decision maker is spread-seeking, the ranking should still be $\tilde{A}_1 > \tilde{A}_2$, because the bulk of the spread is with the left-hand or lower portion of the fuzzy number.

With the exception of Yager's F4 and Chang's methods, all the methods followed this intuition.

When evaluating all nine models in terms of robustness, Yager's F1 [with $g(x) = x$] and F3, Kerre's, Murakami's centroid, and Nakamura's (with $\alpha = 0.5$) methods cannot discriminate between fuzzy numbers with identical modes and symmetric spreads (as stated before, for Example 2). Therefore, Yager's F4, Chang's, Lee and Li's, and the proportion of the optimum methods should be considered more robust. In terms of flexibility, only the proportion of the optimum method allows the decision maker to incorporate a preference for larger or smaller spreads, assuming all other comparison attributes are equal, without a loss of accuracy. It is therefore *not* "goal-specific" and is more flexible than the other eight methods.

It is acknowledged that Nakamura's method allows for some flexibility in this regard, by the ability to change α. As α increases, smaller-spread fuzzy numbers are ranked higher, given relatively equal modes. Conversely, as α decreases to zero, larger-spread fuzzy numbers are ranked higher, given relatively equal modes, and the method would yield results similar to Yager's F4 and Chang's methods.

The most accurate models are Lee and Li's and the proportion of the optimum methods, for they match intuition exactly for all five examples, with no discrimination problems.

All of the methods presented in this paper require integration to calculate the ranking indices. A computer-coded numerical integration program could be easily implemented for each of them. Therefore, in terms of ease of use, all the methods would have a very similar ease-of-use level. However, recall that Yager's F1 and F4 and Chang's methods require a support set of [0, 1]. Scaling would have to be used for these approaches if the exact indices were desired before the integration could take place, reducing the ease of use of these three methods. Of the remaining six methods, only Lee and Li's and the proportion of the optimum provide a tiered solution structure based on the level of difficulty of the problem. Such a structure is attractive, for it does not force a lengthy solution algorithm if the problem is straightforward.

CONCLUSIONS

Overall, when comparing these nine ranking methods in terms of robustness, flexibility, accuracy, and ease of use, the Lee-Li and proportion of the optimum methods are superior to the remaining seven. When comparing these two methods with each other, however, note that while they are equivalent in terms of robustness and accuracy, the Lee-Li method *can* be easier to use without computerized calculations *if* the fuzzy numbers are triangular, while the proportion of the optimum method is more flexible in terms of the ability to explicitly incorporate the decision-maker's spread preferences. It is suggested, therefore, that if the decision maker wants to use the most flexible model possible, due to possible spread preference differences, then the proportion of the optimum method should be used.

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