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NOTE

SOME GENERALIZATIONS OF A COMBINATORIAL IDENTITY OF L. VIETORIS

H.M. SRIVASTAVA
Department of Mathematics, University of Victoria, Victoria, BC, Canada V8W 2Y2

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A remarkably simple proof is presented for an interesting generalization of a combinatorial identity given recently by L. Vietoris [*Monatsh. Math.* 97 (1984) 157–160]. It is also shown how this general result can be extended further to hold true for basic (or q-) series.

1. Introduction

For real or complex λ, let

$$\binom{\lambda}{0}=1, \qquad \binom{\lambda}{n}=\frac{\lambda(\lambda-1)\cdots(\lambda-n+1)}{n!}, \qquad n=1,2,3,\ldots, \tag{1}$$

so that

$$\binom{\lambda+n-1}{n}=(-1)^n\binom{-\lambda}{n}, \tag{2}$$

and, for integers m and n,

$$\binom{m}{n}=\frac{m!}{n!\,(m-n)!}=\binom{m}{m-n}, \quad 0\leq n\leq m. \tag{3}$$

Recently, Vietoris [3] proved the combinatorial identity:

$$(m+n)!=\frac{m!}{k!\,(m-k-1)!}\sum_{i=0}^{n}\binom{n}{i}(k+i)!\,(m+n-k-i-1)!, \tag{4}$$

where m, n, k are integers, with

$$0\leq k\leq m-1 \quad \text{and} \quad n\geq 0. \tag{5}$$

Making use of the definition (3), Vietoris's identity (4) can readily be rewritten in its *equivalent* form:

$$\binom{m+n}{n}=\sum_{i=0}^{n}\binom{k+i}{i}\binom{m+n-k-i-1}{n-i}, \tag{6}$$

where, as before, m, n, k are integers constrained by (5).

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A closer examination of the combinatorial identity (6) would suggest the existence of an interesting generalization of Vietoris's result (4) in the form:

$$\binom{\mu+n}{n}=\sum_{i=0}^{n}\binom{\lambda+i}{i}\binom{\mu-\lambda+n-i-1}{n-i}, \tag{7}$$

where λ and μ are arbitrary complex numbers, and $n=0, 1, 2, \ldots.$

Formula (7) can indeed be rewritten in a form analogous to (4) by using Gamma functions.

2. Derivation of Identity (7)

In view of the elementary relationship (2), we have

$$\binom{\lambda+i}{i}=(-1)^i\binom{-\lambda-1}{i}, \quad i=0, 1, 2, \ldots \tag{8}$$

and

$$\binom{\mu-\lambda+n-i-1}{n-i}=(-1)^{n-i}\binom{\lambda-\mu}{n-i}, \quad 0\leqslant i\leqslant n, \tag{9}$$

so that (7) evidently becomes

$$(-1)^n\binom{\mu+n}{n}=\sum_{i=0}^{n}\binom{-\lambda-1}{i}\binom{\lambda-\mu}{n-i}, \tag{10}$$

or, equivalently,

$$\binom{-\mu-1}{n}=\sum_{i=0}^{n}\binom{-\lambda-1}{i}\binom{\lambda-\mu}{n-i}, \tag{11}$$

where we have applied the relationship (2) once again.

Formula (11) is an immediate consequence of the well-known Vandermonde convolution [1]:

$$\binom{\lambda+\mu}{n}=\sum_{i=0}^{n}\binom{\lambda}{i}\binom{\mu}{n-i}, \tag{12}$$

which holds true for all (real or complex) values of λ and μ, and for integer $n\geqslant 0$.

Obviously, the above derivation of the general formula (7) is much simpler than Vietoris's proof of the special case of (7) when $\lambda=k$ and $\mu=m$, where k and m are integers constrained as in (5),

In order to present an *alternative* proof of (7), let $F(a, b; c; x)$ denote the Gaussian hypergeometric series defined by

$$F(a, b; c; x)=\sum_{i=0}^{\infty}\frac{(a)_i(b)_i}{(c)_i}\frac{x^i}{i!}, \tag{13}$$

where, for convenience,

$$(a)_0 = 1, \qquad (a)_i = a(a+1)\cdots(a+i-1), \quad i = 1, 2, 3, \ldots. \tag{14}$$

Since

$$\binom{\lambda+n-1}{n} = \frac{(\lambda)_n}{n!}, \quad n = 0, 1, 2, \ldots, \tag{15}$$

the right-hand side of the general identity (7) equals

$$(\mu-\lambda)_n F(-n, \lambda+1; \lambda-\mu-n+1; 1), \tag{16}$$

where the hypergeometric series is finite because n is a nonnegative integer.

Now apply a special case of Gauss's summation theorem [2, p. 19] in the form:

$$F(-n, b; c; 1) = \frac{(c-b)_n}{(c)_n}, \quad n = 0, 1, 2, \ldots, \tag{17}$$

and note from (14) that

$$\frac{(-\mu-n)_n}{(\lambda-\mu-n+1)_n} = \frac{(\mu+1)_n}{(\mu-\lambda)_n}, \quad n = 0, 1, 2, \ldots, \quad \lambda-\mu+1 \neq 1, 2, 3, \ldots, \tag{18}$$

and (16) immediately yields the left-hand side of the general identity (7) under the (easily removable) constraint that $\lambda-\mu+1$ is not a positive integer.

3. A basic (or q-) extension of Identity (7)

In terms of the basic (or q-) number $[\lambda]$ and basic (or q-) factorial $[n]!$ defined by

$$[\lambda] = \frac{1-q^\lambda}{1-q}; \qquad [n]! = [1][2][3]\cdots[n], \qquad [0]! = 1, \tag{19}$$

let the (Gaussian) basic (or q-) binomial coefficient be given by [cf. eq. (1) et seq.]

$$\begin{bmatrix}\lambda\\0\end{bmatrix} = 1, \qquad \begin{bmatrix}\lambda\\n\end{bmatrix} = \frac{[\lambda][\lambda-1]\cdots[\lambda-n+1]}{[n]!}, \quad n = 1, 2, 3, \ldots, \tag{20}$$

for arbitrary λ, so that

$$\begin{bmatrix}\lambda+n-1\\n\end{bmatrix} = (-1)^n q^{\frac{1}{2}n(2\lambda+n-1)} \begin{bmatrix}-\lambda\\n\end{bmatrix}, \quad n = 0, 1, 2, \ldots, \tag{21}$$

and, for integers m and n,

$$\begin{bmatrix}m\\n\end{bmatrix} = \frac{[m]!}{[n]!\,[m-n]!} = \begin{bmatrix}m\\m-n\end{bmatrix}, \quad 0 \leqslant n \leqslant m. \tag{22}$$

From the definitions (1) and (20) it is easily verified that

$$\lim_{q\to 1}\begin{bmatrix}\lambda\\ n\end{bmatrix}=\binom{\lambda}{n},\quad n=0,1,2,\ldots, \tag{23}$$

λ being a complex number.

By appealing to known q-extensions of familiar results like (12) and (17) (cf., e.g. [2, p. 348]), it is not difficult to prove the following q-extension of the general combinatorial identity (7):

$$\begin{bmatrix}\mu+n\\ n\end{bmatrix}=\sum_{i=0}^{n}\begin{bmatrix}\lambda+i\\ i\end{bmatrix}\begin{bmatrix}\mu-\lambda+n-i-1\\ n-i\end{bmatrix}q^{(\mu-\lambda)i}, \tag{24}$$

where λ and μ are arbitrary complex numbers, and n is a nonnegative integer.

Finally, in view of the limit relationship (23), the general combinatorial identity (7) would follow from (24) upon letting $q\to 1$.

Acknowledgments

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