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ELSEVIER
Journal of Computational and Applied Mathematics 160 (2003) 297–305
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
www.elsevier.com/locate/cam

# On the $q$-analogues of the Zassenhaus formula for disentangling exponential operators

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Received 15 April 2003

To our dear friend Srinivasa Rao with admiration, affection, and best wishes on his sixtieth birthday

**Abstract**

Katriel, Rasetti and Solomon introduced a $q$-analogue of the Zassenhaus formula written as $\mathrm{e}_q^{(A+B)} = \mathrm{e}_q^A \mathrm{e}_q^B \mathrm{e}_q^{c_2} \mathrm{e}_q^{c_3} \mathrm{e}_q^{c_4} \mathrm{e}_q^{c_5} \ldots$ , where $A$ and $B$ are two generally noncommuting operators and $\mathrm{e}_q^z$ is the Jackson $q$-exponential, and derived the expressions for $c_2$, $c_3$ and $c_4$. It is shown that one can also write $\mathrm{e}_q^{(A+B)} = \mathrm{e}_q^A \mathrm{e}_q^B \mathrm{e}_{q^2}^{\mathscr{C}_2} \mathrm{e}_{q^3}^{\mathscr{C}_3} \mathrm{e}_{q^4}^{\mathscr{C}_4} \mathrm{e}_{q^5}^{\mathscr{C}_5} \ldots$ . Explicit expressions for $\mathscr{C}_2$, $\mathscr{C}_3$ and $\mathscr{C}_4$ are given.


*Keywords:* Zassenhaus formula; $q$-analogue; $q$-functions

## 1. Introduction

As is well known, the Baker–Campbell–Hausdorff (BCH) formula

$$\mathrm{e}^A \mathrm{e}^B = \mathrm{e}^{A+B+(1/2)[A,B]+(1/12)([A,[A,B]]+[[A,B],B])+\cdots} \tag{1}$$

helps express the product of two noncommuting exponential operators as a single exponential operator in which the exponent is, in general, an infinite series in terms of repeated commutators and several hundred terms of the series have been calculated using the computer. The dual of the BCH formula is the Zassenhaus formula

$$\mathrm{e}^{A+B} = \mathrm{e}^A \mathrm{e}^B \mathrm{e}^{-1/2[A,B]} \mathrm{e}^{1/6[A,[A,B]]-1/3[[A,B],B]} \ldots \tag{2}$$

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0377-0427/$ - see front matter 
doi:10.1016/S0377-0427(03)00633-2

which helps disentangle an exponential operator into a product of, in general, an infinite series of exponential operators involving repeated commutators (see [12] for details). The BCH and the Zassenhaus formulas have several applications (see, for example, [14–16,7].

With the advent of $q$-deformed algebraic structures in physics (see, for example, [1,2,8] and references therein) there has been a growing interest in $q$-generalization of several results of classical analysis. Katriel and Solomon [11] first obtained the $q$-analogue of the BCH formula and later Katriel et al. [10] proposed a $q$-analogue of the Zassenhaus formula. Let us briefly recall their results.

The Jackson $q$-exponential is defined by

$$\mathrm{e}_q^x = \sum_{n=0}^{\infty} \frac{x^n}{[n]!}, \tag{3}$$

where

$$[n] = \frac{1-q^n}{1-q} \tag{4}$$

is the Heine basic number and

$$[n]! = [n]\,[n-1]\,[n-2]\cdots[1] \quad [0]! = 1, \tag{5}$$

such that

$$\lim_{q\to 1}[n] = n, \quad \lim_{q\to 1}\mathrm{e}_q^x = \mathrm{e}^x. \tag{6}$$

In the following $[n]$ will refer to the $q$-deformed $n$ defined by (4) corresponding to the base $q$. If the base is different then it will be indicated explicitly; for example, $[n]_{q^k}$ will mean $(1-q^{kn})/(1-q^k)$. The $q$-exponential function $\mathrm{e}_q^{\alpha x}$ is the eigenfunction of the Jackson $q$-differential operator

$$D_q f(x) = \frac{f(x)-f(qx)}{(1-q)x} \tag{7}$$

such that

$$D_q \mathrm{e}_q^{\alpha x} = \alpha \mathrm{e}_q^{\alpha x}. \tag{8}$$

The $q$-commutator is defined by

$$[X,Y]_q = XY - qYX \tag{9}$$

and it obeys the $q$-antisymmetry property

$$[Y,X]_q = -q[X,Y]_{q^{-1}}. \tag{10}$$

The $q$-BCH formula found in [11] reads

$$\mathrm{e}_q^A \mathrm{e}_q^B = \mathrm{e}_q^{A+B+(q/[2])[A,B]_{q^{-1}}+(q^2/[2]3!)([A,[A,B]_q]_{q^{-1}}+[[A,B]_q,B]_{q^{-1}})+\cdots}. \tag{11}$$

Katriel and Solomon [11] have given the explicit expressions for the terms involving up to 4-tuple $q$-commutator. In the limit $q\to 1$ the above $q$-BCH formula (11) is seen to agree with the classical BCH formula (1).

When $A$ and $B$ satisfy the relation $AB=q^{-1}BA$ it is found that $[A,B]_{q^{-1}}$, $[A,[A,B]_q]_{q^{-1}}$, $[[A,B]_q,B]_{q^{-1}}$, and all the higher $q$-commutators vanish leading to the result of Schützenberger [13] and Cigler [3] (see also, [5]):

$$e_q^A e_q^B = e_q^{A+B} \quad \text{if } AB = q^{-1}BA. \tag{12}$$

The $q$-analogue of the Zassenhaus formula proposed in [10] reads

$$e_q^{(A+B)} = e_q^A e_q^B e_q^{c_2} e_q^{c_3} e_q^{c_4} e_q^{c_5} \cdots \,, \tag{13}$$

where

$$\begin{aligned} c_2 &= [B,A]_q/[2], \\ c_3 &= ([[B,A]_q,A]_{q^2}/[3]!) + ([[B,A]_q,B]_q/[3]), \\ c_4 &= ([[[B,A]_q,A]_{q^2},A]_{q^3}/[4]!) + ([[[B,A]_q,B]_{q^2},B]_{q^3}/[2][4]) \\ &\quad + ([[[B,A]_q,A]_{q^2},B]_q/[2][4]) + (q[[B,A]_q,[B,A]_q]_q/[2]^2[4]), \\ &\quad \cdots . \end{aligned} \tag{14}$$

In this article we shall see that it is possible to have a $q$-Zassenhaus formula written also as

$$e_q^{(A+B)} = e_q^A e_q^B e_{q^2}^{\mathscr{C}_2} e_{q^3}^{\mathscr{C}_3} e_{q^4}^{\mathscr{C}_4} e_{q^5}^{\mathscr{C}_5} \cdots \tag{15}$$

and give the explicit expressions for the first few terms.

## 2. The Zassenhaus formula

A standard procedure to obtain the Zassenhaus formula is as follows. Set

$$e^{x(A+B)} = e^{xA} e^{xB} e^{x^2 C_2} e^{x^3 C_3} \cdots \,. \tag{16}$$

Differentiating both sides of (16) with respect to $x$ and multiplying it from the right by

$$e^{-x(A+B)} = \cdots e^{-x^3 C_3} e^{-x^2 C_2} e^{-xB} e^{-xA}, \tag{17}$$

one obtains

$$\begin{aligned} A + B &= A + e^{xA} B e^{-xA} + e^{xA} e^{xB} (2xC_2) e^{-xB} e^{-xA} \\ &\quad + e^{xA} e^{xB} e^{x^2 C_2} (3x^2 C_3) e^{-x^2 C_2} e^{-xB} e^{-xA} + \cdots \,. \end{aligned} \tag{18}$$

The expressions $e^{xA} B e^{-xA}$, $e^{x} e^{xB} (2xC_2) e^{-xB} e^{-xA}$, etc., are expanded again using the formula

$$e^{xA} B e^{-xA} = \sum_{n=0}^{\infty} \frac{x^n}{n!} \{A^n, B\}, \tag{19}$$

where the multiple commutator bracket $\{A^n, B\}$ is defined by

$$\{A^{n+1}, B\} = [A, \{A^n, B\}] \quad \{A^0, B\} = B. \tag{20}$$

Then, Eq. (18) becomes

$$0=\sum_{n=1}^{\infty}\frac{x^n}{n!}\{A^n,B\}$$
$$+2x\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{x^{m+n}}{m!n!}\{A^m,B^n,C_2\}$$
$$+3x^2\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{x^{l+m+2n}}{l!m!n!}\{A^l,B^m,C_2^n,C_3\}+\cdots \tag{21}$$

with

$$\begin{aligned}&\{A^0,B^n,C_2\}=\{B^n,C_2\}\\&\{A^{m+1},B^n,C_2\}=[A,\{A^m,B^n,C_2\}]\cdots\ .\end{aligned} \tag{22}$$

Equating the coefficients of $x^n$ to zero in (21) one obtains

$$\begin{aligned}&C_2=-\tfrac{1}{2}[A,B]\\&C_3=\tfrac{1}{6}[A,[A,B]]-\tfrac{1}{3}[[A,B],B]\cdots\ .\end{aligned} \tag{23}$$

There are also alternative methods of derivation of the above result (see, for example, [14,15] for more details). For our purpose of deriving a $q$-analogue of the Zassenhaus formula we shall follow Karplus and Schwinger [9] (see Appendix I where $\exp(A+B)$ is expanded in powers of $B$ up to the second term using a method we adopt here).

## 3. A $q$-analogue of the Zassenhaus formula

Let

$$F(x)=e_q^{x(A+B)},\quad F(0)=I \tag{24}$$

and write

$$e_q^{x(A+B)}=e_q^{xA}G(x),\quad G(0)=I. \tag{25}$$

On $q$-differentiating with respect to $x$

$$D_qF(x)=(A+B)e_q^{x(A+B)}. \tag{26}$$

Note that $q$-differentiation obeys the $q$-Leibniz rule

$$D_q(fg)=(D_qf)g+f(qx)(D_qg)=(D_qf)g(qx)+f(D_qg). \tag{27}$$

Now, $q$-differentiating the right-hand side of (25) using the $q$-Leibniz rule we have

$$\begin{aligned}(A+B)\mathrm{e}_q^{x(A+B)} &= A\mathrm{e}_q^{xA}G(x)+\mathrm{e}_q^{qxA}D_qG(x)\\ &= A\mathrm{e}_q^{x(A+B)}+\mathrm{e}_q^{qxA}D_qG(x).\end{aligned} \tag{28}$$

Thus,

$$B\mathrm{e}_q^{x(A+B)} = B\mathrm{e}_q^{xA}G(x) = \mathrm{e}_q^{qxA}D_qG(x) \tag{29}$$

or

$$D_qG(x) = (\mathrm{e}_q^{qxA})^{-1}B\mathrm{e}_q^{xA}G(x). \tag{30}$$

Recall that the inverse of $q$-differentiation is $q$-integration (see, for example, [4,6]), defined by

$$\int_0^x d_q\xi f(\xi) = (q-1)x\sum_{n=1}^{\infty}q^{-n}f(q^{-n}x), \tag{31}$$

such that

$$D_q\left(\int_0^x d_q\xi f(\xi)\right) = f(x). \tag{32}$$

Then, the formal solution of the $q$-differential equation (30) for $G(x)$ with the initial condition $G(0)=I$, is given by

$$\begin{aligned}G(x) &= I+\int_0^x d_q\xi(\mathrm{e}_q^{q\xi A})^{-1}B\mathrm{e}_q^{\xi A}G(\xi)\\ &= I+\int_0^x d_q\xi(\mathrm{e}_q^{q\xi A})^{-1}B\mathrm{e}_q^{\xi A}\\ &\quad +\int_0^x d_q\xi_1(\mathrm{e}_q^{q\xi_1 A})^{-1}B\mathrm{e}_q^{\xi_1 A}\int_0^{\xi_1} d_q\xi_2(\mathrm{e}_q^{q\xi_2 A})^{-1}B\mathrm{e}_q^{\xi_2 A}\\ &\quad +\cdots\ .\end{aligned} \tag{33}$$

Using the well known result

$$\mathrm{e}_q^{x}\mathrm{e}_{q^{-1}}^{-x} = 1, \tag{34}$$

we can rewrite (33) as

$$\begin{aligned}G(x) &= I+\int_0^x d_q\xi\,\mathrm{e}_{q^{-1}}^{-q\xi A}B\mathrm{e}_q^{\xi A}\\ &\quad +\int_0^x d_q\xi_1\mathrm{e}_{q^{-1}}^{-q\xi_1 A}B\mathrm{e}_q^{\xi_1 A}\int_0^{\xi_1} d_q\xi_2\mathrm{e}_{q^{-1}}^{-q\xi_2 A}B\mathrm{e}_q^{\xi_2 A}\\ &\quad +\cdots\ .\end{aligned} \tag{35}$$

We now require a $q$-analogue of the classical formula (19). By straightforward expansion and regrouping of terms in powers of $x$ we have

$$
\begin{aligned}
&(\mathrm{e}_q^{qxA})^{-1} B \mathrm{e}_q^{xA} \\
&\quad = \mathrm{e}_{q^{-1}}^{-qxA} B \mathrm{e}_q^{xA} = B + x[B,A]_q + \frac{x^2}{[2]!}[[B,A]_q,A]_{q^2} + \frac{x^3}{[3]!}[[[B,A]_q,A]_{q^2},A]_{q^3} + \cdots . \qquad (36)
\end{aligned}
$$

Let us write this equation as

$$\mathrm{e}_{q^{-1}}^{-qxA} B \mathrm{e}_q^{xA} = B + \sum_{n=1}^{\infty} \frac{x^n}{[n]!} X_n \qquad (37)$$

with the definition

$$X_n = [\cdots[[[B,A]_q,A]_{q^2},A]_{q^3}\cdots],A]_{q^n}, \quad n = 1,2,3,\ldots . \qquad (38)$$

Now, using (37) and (38) in (35), we get

$$
\begin{aligned}
G(x) = I &+ \int_0^x d_q\xi \left\{ B + \sum_{n=1}^{\infty} \frac{\xi^n}{[n]!} X_n \right\} \\
&+ \int_0^x d_q\xi_1 \left\{ B + \sum_{n=1}^{\infty} \frac{\xi_1^n}{[n]!} X_n \right\} \int_0^{\xi_1} d_q\xi_2 \left\{ B + \sum_{n=1}^{\infty} \frac{\xi_2^n}{[n]!} X_n \right\} + \cdots . \qquad (39)
\end{aligned}
$$

Collecting the first few terms of the resulting series in powers of $x$ we have

$$
\begin{aligned}
G(x) = &\left\{ I + xB + \frac{x^2B^2}{[2]!} + \frac{x^3B^3}{[3]!} + \cdots \right\} \\
&+ \left\{ \frac{x^2X_1}{[2]!} + \frac{x^4X_1^2}{[1][2]![4]} + \frac{x^6X_1^3}{[1]^2[2]![4][6]} + \cdots \right\} \\
&+ \left\{ x^3 \left( \frac{X_2}{[3]!} + \frac{BX_1}{[3]!} + \frac{X_1B}{[1][3]} \right) + x^6(\cdots) + \cdots \right\} + \cdots . \qquad (40)
\end{aligned}
$$

Realizing that the terms in the first bracket sum to $\mathrm{e}_q^{xB}$ let us rewrite (40) as

$$
\begin{aligned}
G(x) = \mathrm{e}_q^{xB} &\left( I + \mathrm{e}_{q^{-1}}^{-xB} \left\{ \frac{x^2X_1}{[2]!} + \frac{x^4X_1^2}{[1][2]![4]} + \frac{x^6X_1^3}{[1]^2[2]![4][6]} + \cdots \right\} \right. \\
&\left. + \mathrm{e}_{q^{-1}}^{-xB} \left\{ x^3 \left( \frac{X_2}{[3]!} + \frac{BX_1}{[3]!} + \frac{X_1B}{[1][3]} \right) + x^6(\cdots) + \cdots \right\} + \cdots \right). \qquad (41)
\end{aligned}
$$

Substituting the series expansion $\mathrm{e}_{q^{-1}}^{-xB}$, using the relation

$$[n]_{q^{-1}} = q^{1-n}[n]_q \qquad (42)$$

and after some straightforward algebra one can rewrite (41) further as

$$G(x) = \mathrm{e}_q^{xB}\mathrm{e}_{q^2}^{x^2X_1/[2]}\left(I + \mathrm{e}_{q^2}^{-x^2X_1/[2]}\left\{-xB + \frac{qx^2B^2}{[2]} - \frac{q^2x^3B^3}{[3]} + \cdots\right\}\cdots\right.$$
$$\left. + \mathrm{e}_{q^2}^{-x^2X_1/[2]}\left\{x^3\left(\frac{X_2}{[3]!} + \frac{BX_1}{[3]!} + \frac{X_1B}{[3]}\right) + \cdots\right\} + \cdots\right). \tag{43}$$

Now, substituting the series expression for $\mathrm{e}_{q^2}^{-x^2X_1/[2]}$ and simplifying, one recognizes that one can pull out a factor $\mathrm{e}_{q^3}^{x^3\{([X_1,B]_q/[3])+(X_2/[3]!)\}}$ from the above expression and write

$$G(x) = \mathrm{e}_q^{xB}\mathrm{e}_{q^2}^{x^2[B,A]_q/[2]}\mathrm{e}_{q^3}^{x^3\{([[B,A]_q,B]_q/[3])+([[B,A]_q,A]_{q^2}/[3]!)\}}\cdots$$
$$= \mathrm{e}_q^{xB}\mathrm{e}_{q^2}^{-qx^2[A,B]_{q-1}/[2]}\mathrm{e}_{q^3}^{x^3\{(q^3[A,[A,B]_{q-1}]_{q-2}/[3]!)-(q[[A,B]_{q-1},B]_q/[3])\}}\cdots\ . \tag{44}$$

This shows that the general expression for $G(x)$ can be assumed to be

$$G(x) = \mathrm{e}_q^{xB}\mathrm{e}_{q^2}^{x^2\mathscr{C}_2}\mathrm{e}_{q^3}^{x^3\mathscr{C}_3}\mathrm{e}_{q^4}^{x^4\mathscr{C}_4}\cdots\ . \tag{45}$$

The crucial point here is to note that the exponential factors in $G(x)$ have bases $q, q^2, q^3, q^4, \ldots$, respectively, unlike in the formula (13) introduced by Katriel et al. [10]. Thus, having recognized a new general form of $G(x)$ we can use the comparison method to determine $\mathscr{C}_2, \mathscr{C}_3, \mathscr{C}_4, \ldots$, in (45). To this end, we write (25) as

$$\sum_{n=0}^{\infty}\frac{x^n}{[n]!}(A+B)^n = \sum_{j,k,l,m,\ldots=0}^{\infty}\frac{x^{j+k+2l+3m+\cdots}}{[j]_q![k]_q![l]_{q^2}![m]_{q^3}!}A^jB^k\mathscr{C}_2^l\mathscr{C}_3^m\cdots \tag{46}$$

and compare the coefficients of equal powers of $x$. This leads to coupled equations for $\mathscr{C}_2, \mathscr{C}_3, \mathscr{C}_4, \ldots$, which can be solved in terms of $A$ and $B$ after straightforward algebra. Thus, solving the first few equations we confirm the forms of $\mathscr{C}_2$ and $\mathscr{C}_3$ already obtained above (compare (44) and (45)) and find that

$$\mathscr{C}_4 = \frac{1}{[4]!}(-q^6[A,[A,[A,B]_{q^{-1}}]_{q^{-2}}]_{q^{-3}} + q^3[3][[A,[A,B]_{q^{-1}}]_{q^{-2}},B]_q$$
$$- q[3][[[A,B]_{q^{-1}},B]_q,B]_{q^2}). \tag{47}$$

Now, using (25), (44), (45) and (47), we get the $q$-Zassenhaus formula as

$$\mathrm{e}_q^{(A+B)} = \mathrm{e}_q^A\mathrm{e}_q^B\mathrm{e}_{q^2}^{-q[A,B]_{q-1}/[2]}\mathrm{e}_{q^3}^{\{(q^3[A,[A,B]_{q-1}]_{q-2}/[3]!)-(q[[A,B]_{q-1},B]_q/[3])\}}$$
$$\times\mathrm{e}_{q^4}^{(1/[4]!)(-q^6[A,[A,[A,B]_{q-1}]_{q-2}]_{q-3}+q^3[3][[A,[A,B]_{q-1}]_{q-2},B]_q-q[3][[[A,B]_{q-1},B]_q,B]_{q^2})} \tag{48}$$

up to the first five terms. In the limit $q \to 1$ it is found that $\mathscr{C}_2$, $\mathscr{C}_3$ and $\mathscr{C}_4$ become the expressions for the classical Zassenhaus formula (see, for example, (7)–(9) of [15]). When $AB = q^{-1}BA$ it is found that $\mathscr{C}_2 = \mathscr{C}_3 = \mathscr{C}_4 = \cdots = 0$ leading to the Schützenberger–Cigler result (12).

In the literature on $q$-series there are two other definitions of the $q$-exponential:

$$\mathrm{e}_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{(1-q)(1-q^2)\cdots(1-q^n)}, \tag{49}$$

$$E_q(x) = \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}x^n}{(1-q)(1-q^2)\cdots(1-q^n)}, \tag{50}$$

such that

$$\lim_{q\to 1} \mathrm{e}_q((1-q)x) = \mathrm{e}^x, \tag{51}$$

$$\lim_{q\to 1} E_q((1-q)x) = \mathrm{e}^x, \tag{52}$$

$$\mathrm{e}_q(x)E_q(-x) = 1. \tag{53}$$

Let us note that the corresponding Zassenhaus formulae can be found in straightforward ways by using the relations:

$$\mathrm{e}_q(x) = \mathrm{e}_q^{x/(1-q)}, \tag{54}$$

$$E_q(x) = \mathrm{e}_{q^{-1}}^{x/(1-q)}. \tag{55}$$

Thus, we find, using (48) and (54),

$$\begin{aligned}\mathrm{e}_q(A+B) &= \mathrm{e}_q^{(A+B)/(1-q)} \\ &= \mathrm{e}_q(A)\mathrm{e}_q(B)\mathrm{e}_{q^2}(-q[A,B]_{q^{-1}}/(1-q))\cdots \end{aligned} \tag{56}$$

and

$$\mathrm{e}_q(A+B) = \mathrm{e}_q(A)\mathrm{e}_q(B) \quad \text{if } AB = q^{-1}BA. \tag{57}$$

Similarly,

$$\begin{aligned}E_q(A+B) &= \mathrm{e}_{q^{-1}}^{(A+B)/(1-q)} \\ &= E_q(A)E_q(B)E_{q^2}(-[A,B]_q/(1-q))\cdots \end{aligned} \tag{58}$$

and

$$E_q(A+B) = E_q(A)E_q(B) \quad \text{if } AB = qBA. \tag{59}$$

## 4. Conclusion

To summarize, it is found that the classical Zassenhaus formula

$$\begin{aligned}\mathrm{e}^{A+B} &= \mathrm{e}^A\mathrm{e}^B\mathrm{e}^{-(1/2)[A,B]}\mathrm{e}^{(1/6)[A,[A,B]]-(1/3)[[A,B],B]} \\ &\quad\times \mathrm{e}^{(1/4!)(-[A,[A,[A,B]]]+3[[A,[A,B]],B]-3[[[A,B],B],B])}\ldots \end{aligned} \tag{60}$$

has a $q$-analogue given by

$$\begin{aligned} e_q^{(A+B)} &= e_q^A e_q^B e_{q^2}^{-q[A,B]_{q^{-1}}/[2]} e_{q^3}^{\{(q^3[A,[A,B]_{q^{-1}}]_{q^{-2}}/[3]!)-(q[[A,B]_{q^{-1}},B]_q/[3])\}} \\ &\quad \times e_{q^4}^{(1/[4]!)(-q^6[A,[A,[A,B]_{q^{-1}}]_{q^{-2}}]_{q^{-3}}+q^3[3][[A,[A,B]_{q^{-1}}]_{q^{-2}},B]_q-q[3][[[A,B]_{q^{-1}},B]_q,B]_{q^2})} \\ &\quad \times \cdots \ , \end{aligned} \tag{61}$$

where the $q$-exponential is defined by

$$e_q^x = \sum_{n=0}^{\infty} \frac{x^n}{[n]!}. \tag{62}$$

Thus, we have shown that while Katriel et al. [10] have proposed a $q$-analogue of the Zassenhaus formula in the form

$$e_q^{(A+B)} = e_q^A e_q^B e_q^{c_2} e_q^{c_3} e_q^{c_4} e_q^{c_5} \cdots \ , \tag{63}$$

it is possible to have a $q$-Zassenhaus formula written also as

$$e_q^{(A+B)} = e_q^A e_q^B e_{q^2}^{\mathscr{C}_2} e_{q^3}^{\mathscr{C}_3} e_{q^4}^{\mathscr{C}_4} e_{q^5}^{\mathscr{C}_5} \cdots \ . \tag{64}$$

We have also explicitly found the first few terms of the disentanglement formula (64).

It should be noted that once a $q$-Zassenhaus formula is given for the $q$-exponential defined by (62) for the other two common definitions of the $q$-exponential (49) and (50) found in the literature the corresponding $q$-Zassenhaus formulae can be found using the relations (54) and (55).

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