The lower and upper bounds on Perron root of nonnegative irreducible matrices

Guang-Xin Huang\textsuperscript{a,∗}, Feng Yin\textsuperscript{b}, Ke Guo\textsuperscript{a}

\textsuperscript{a}College of Information and Management, Chengdu University of Technology, Chengdu 610059, PR China
\textsuperscript{b}Department of Mathematics, Sichuan University of Science and Engineering, Zigong 643000, PR China

Received 9 August 2006; received in revised form 26 March 2007

Abstract

Let \( A \) be an \( n \times n \) nonnegative irreducible matrix, let \( A[z] \) be the principal submatrix of \( A \) based on the nonempty ordered subset \( z \) of \( \{1, 2, \ldots, n\} \), and define the generalized Perron complement of \( A[z] \) by \( P_t(A/A[z]) \), i.e.,

\[
P_t(A/A[z]) = A[\beta] + A[\beta, z](tI - A[z])^{-1}A[z, \beta], \quad t > \rho(A[z]).
\]

This paper gives the upper and lower bounds on the Perron root of \( A \). An upper bound on Perron root is derived from the maximum of the given parameter \( t_0 \) and the maximum of the row sums of \( P_{t_0}(A/A[z]) \), synchronously, a lower bound on Perron root is expressed by the minimum of the given parameter \( t_0 \) and the minimum of the row sums of \( P_{t_0}(A/A[z]) \). It is also shown how to choose the parameter \( t \) after \( z \) to get tighter upper and lower bounds of \( \rho(A) \). Several numerical examples are presented to show that our method compared with the methods in [L.Z. Lu, M.K. Ng, Locations of Perron roots, Linear Algebra Appl. 392 (2004) 103–117.] is more effective.

© 2007 Elsevier B.V. All rights reserved.

MSC: 15A48; 05C50

Keywords: Nonnegative irreducible matrix; Perron root; Lower and upper bounds; Generalized Perron complement

1. Introduction

In this paper the following notations are considered and used. Let \( R^{n \times n}, R^{n \times n}_{\geq} \) and \( R^{n \times n}_> \) denote the sets of all \( n \times n \) real matrices, all \( n \times n \) real nonnegative matrices and all \( n \times n \) real positive matrices, respectively. For \( A, B \in R^{n \times n}_> \), we denote by \( A > B \) that each entry of the matrix \( A - B \) is nonnegative, and \( A - B \) has at least one positive entry. For an arbitrary matrix \( A = (a_{ij}) \in R^{n \times n}_> \), let \( A^T \) denote the transpose of \( A \) and

\[
r_i(A) = \sum_{k=1}^{n} a_{ik}, \quad i = 1, 2, \ldots, n,
\]

\[
r(A) = (r_1(A), r_2(A), \ldots, r_n(A))^T,
\]

\[
r_{\min}(A) = \min_{1 \leq i \leq n} r_i(A), \quad r_{\max}(A) = \max_{1 \leq i \leq n} r_i(A).
\]

+ This work was supported in part by the Foundation of the Education Council of Chongqing and the Key Science Fund of Chengdu University of Technology.

∗ Corresponding author.

E-mail address: huangx@cdut.edu.cn (G.-X. Huang).

0377-0427/$ - see front matter © 2007 Elsevier B.V. All rights reserved.
doi:10.1016/j.cam.2007.06.034
Let $N = \{1, 2, \ldots, n\}$. Let $x$ denote a nonempty ordered subset of $N$ and $\beta = N \setminus x$, both consisting of strictly increasing integers. We also denote the submatrix of the matrix $A$ whose rows and columns are determined by $x$ and $\beta$, respectively, in $A[x, \beta]$. The matrix $A[x]$ is just equal to the matrix $A[x, x]$, the principal submatrix of $A$ based on $x$.

For a nonnegative irreducible matrix $A \in \mathbb{R}_{\geq}^{n \times n}$, a fundamental matrix problem is to locate the Perron root $\rho(A)$ of $A$. It is well known that for such a matrix $A$, the following inequality [1] holds:

$$r_{\min}(A) \leq \rho(A) \leq r_{\max}(A)$$ \hspace{1cm} (2)

and the equality holds in one of the bounds if and only if it holds in both. For $A \in \mathbb{R}_{\geq}^{n \times n}$, the bounds of $\rho(A)$ were improved by Brauer [6]. Meyer [7] defined the Perron complement and used it to compute the unique normalized Perron eigenvector of a nonnegative irreducible $A$. Neumann [8] used it to analyze the properties of inverse $M$-matrices. Fan [3] used it to derive the bounds of the Perron root of symmetric irreducible nonnegative matrices and $Z$-matrices. For a nonnegative irreducible matrix $A$, in order to obtain the bounds on $\rho(A)$, $P_t(A/A[x])$ for $t \geq \rho(A)$ was first defined by Neumann [8], followed by Lu [4] who defined and used the generalized Perron complement $P_t(A/A[z])$ of $A[z]$, which is given by

$$P_t(A/A[z]) = A[\beta] + A[\beta, x](tI - A[z])^{-1}A[x, \beta], \quad t > \rho(A[z]).$$ \hspace{1cm} (3)

It has been shown in [4] that the use of the generalized Perron complement of $A[z]$ can give tight bounds on $\rho(A)$. Lu [5] has given a new localization method that utilizes the relationship between the Perron root of a nonnegative matrix and the estimates of the row sums of its generalized Perron complement. The main results in [5] can only obtain a tight upper bound or a tight lower bound of $\rho(A)$, respectively. In this paper, however, we aim to solve the problems as follows. It has always been supposed that matrix $A \in \mathbb{R}_{\geq}^{n \times n}$ is irreducible without special specification.

**Problem 1.** How to obtain a tighter lower upper bounds of $\rho(A)$ together by the estimates of the row sums of its generalized Perron complement?

**Problem 2.** How to properly choose parameters $x$ and $t$ after $x$ to get an “optimal” lower bound and an “optimal” upper bound of $\rho(A)$, respectively?

This paper is organized as follows. In Section 2, we will give the lower and upper bounds on Perron root by the minimum of the given parameter $t_0$ and the minimum row sum of the row sums of $P_t(A/A[z])$ and the maximum of the given parameter $t_0$ and the maximum row sum of the row sums of $P_t(A/A[z])$, respectively. Then in Section 3, we will properly choose the parameter $t$ after $x$ to get an “optimal” lower bound and an “optimal” upper bound $\rho(A)$, respectively. In Sections 2 and 3, some numerical examples are also given to show the application of the corresponding results.

2. The upper and lower bounds on Perron root

In this section, we will show tighter lower and upper bounds on Perron root by the minimum of the given parameter $t_0$ and the minimum row sum of the row sums of $P_{t_0}(A/A[z])$ and the maximum of the given parameter $t_0$ and the maximum row sum of the row sums of $P_{t_0}(A/A[z])$, respectively. Three numerical examples are provided to illustrate the results.

For the generalized Perron complement matrix $P_t(A/A[z])$, the following results are needed.

**Lemma 2.1** (See [5, Theorem 5]). Assume that $l$ and $u$ are found such that

$$l \leq \rho(P_t(A/A[z])) \leq u, \quad t > \rho(A[z]),$$ \hspace{1cm} (4)

then

$$\min\{t, l\} \leq \rho(A) \leq \max\{t, u\}.$$ \hspace{1cm} (5)

Let

$$z(t, x) = r_{\min}(P_t(A/A[z])), \quad \bar{z}(t, x) = r_{\max}(P_t(A/A[z])),$$ \hspace{1cm} (6)
By Lemma 2.1, we have

\[ \min\{t, z(t, x)\} \leq \rho(A) \leq \max\{t, \hat{z}(t, x)\}. \]  

(7)

**Lemma 2.2 (See [5, Theorem 7]).** If \( A \geq 0 \) and \( t_0 > r_{\max}(A[z]) \), then

\[ z(t_0, x) \geq \min_j \{ r_j(A[\beta]) + v_1(t_0, x)r_j(A[\beta, x]) \}, \]

and

\[ \hat{z}(t_0, x) \leq \max_j \{ r_j(A[\beta]) + v_2(t_0, x)r_j(A[\beta, x]) \}, \]  

where

\[ v_1(t_0, x) = \min_i \frac{r_i(A[z, \beta])}{t_0 - r_i(A[z])} \quad \text{and} \quad v_2(t_0, x) = \max_i \frac{r_i(A[z, \beta])}{t_0 - r_i(A[z])}. \]

A tighter lower bound of \( \rho(A) \) can be obtained by Lemmas 2.1 and 2.2.

**Corollary 2.1.** Let \( A \) be an \( n \times n \) irreducible nonnegative matrix with \( n \geq 3 \) and \( r_{\max}(A) > r_{\min}(A) \). If \( x \) (or \( \beta = N \setminus x \)) and \( t_0 \) are chosen, respectively, such that

\[ \max \left\{ \max_{j \in \beta} r_j(A), r_{\max}(A[z]) \right\} < \min_{i \in \beta} r_i(A), \quad r_{\min}(A[\beta, x]) > 0 \]  

(10)

and

\[ \max \left\{ \max_{j \in \beta} r_j(A), r_{\max}(A[z]) \right\} < t_0 < \min_{i \in \beta} r_i(A), \]  

(11)

then

\[ \rho(A) \geq \min\{t_0, z(t_0, x)\} > r_{\min}(A). \]

(12)

**Proof.** Let \( x \) and \( t_0 \) be chosen such that \( v_1(t_0, x) > 1 \), where \( v_1(t_0, x) \) is defined in (9). Note that

\[ r_{\max}(A[z]) < t_0 < \min_{i \in \beta} r_i(A), \]

for any \( 1 \leq i \leq |x| \), so

\[ t_0 - r_i(A[z]) - r_i(A[z, \beta]) = t_0 - r_i(A[z, N]) \leq t_0 - \min_{i \in \beta} r_i(A) < 0 \]

and

\[ 0 < t_0 - r_{\max}(A[z]) \leq t_0 - r_i(A[z]) < r_i(A[z, \beta]). \]

Therefore

\[ v_1(t_0, x) = \min_i \frac{r_i(A[z, \beta])}{t_0 - r_i(A[z])} > 1, \]

it follows from Lemma 2.2 and (10) that

\[ z(t_0, x) \geq \min_j \{ r_j(A[\beta]) \} + v_1(t_0, x)r_j(A[\beta, x]) > \min_j r_j(A) = r_{\min}(A). \]

(13)

Thus (7), (11) and (13) give (12). This completes the proof. \( \square \)
Remark. Corollary 2.1 was first provided by Lu [5, Theorem 8]; however, it is obvious that the condition $r_{\min}(A[x, \beta]) > 0$ is not necessary for it is implied in inequality (14) in [5], i.e., $r_{\min}(A[x, \beta]) > 0$ is implied in (10).

The following result gives a tighter upper bound of $\rho(A)$.

**Lemma 2.3.** Let $A$ be an $n \times n$ irreducible nonnegative matrix with $n \geq 3$ and $r_{\max}(A) > r_{\min}(A)$. If $x$ (or $\beta = N \setminus x$) and $t_0$ are chosen, respectively, such that

\[
\max \left\{ \max_{j \in \beta} r_j(A), r_{\max}(A[x]) \right\} < \min_{i \in x} r_i(A), r_{\min}(A[\beta, x]) > 0
\]  

(14)

and

\[
\max_{j \in \beta} r_j(A) < t_0 < r_{\max}(A),
\]  

(15)

then

\[\rho(A) \leq \max\{t_0, \hat{z}(t_0, \beta)\} < r_{\max}(A).\]  

(16)

**Proof.** Let $x$ and $t_0$ be chosen such that $0 < v_2(t_0, \beta) < 1$, where

\[v_2(t_0, \beta) = \max_{j} \frac{r_j(A[\beta, x])}{t_0 - r_j(A[\beta])}.
\]

Note that

\[t_0 > \max_{j \in \beta} r_j(A),
\]

\[t_0 - r_j(A[\beta]) - r_j(A[\beta, x]) = t_0 - r_j(A[\beta, N]) \geq t_0 - \max_{j \in \beta} r_j(A) > 0,
\]

and from (14), it follows that

\[t_0 - r_j(A[\beta]) > r_j(A[\beta, x]) > 0.
\]

So

\[0 < v_2(t_0, \beta) = \max_{j} \frac{r_j(A[\beta, x])}{t_0 - r_j(A[\beta])} < 1,
\]

since $r_{\min}(A[x, \beta]) > 0$ is implied in (14). By using Lemma 2.2 and (14) again, we have

\[\hat{z}(t_0, \beta) \leq \max_{i} [r_i(A[x])] + v_2(t_0, \beta)r_i(A[\beta, \beta]) < \max_{i \in x} r_i(A) = r_{\max}(A).
\]  

(17)

Therefore, (16) is implied by (7), (15) and (17). This completes the proof. □

By using Corollary 2.1 and Lemma 2.3, we have the following result.

**Theorem 2.1.** Let $A$ be an $n \times n$ irreducible nonnegative matrix with $n \geq 3$ and $r_{\max}(A) > r_{\min}(A)$. If $x$ (or $\beta = N \setminus x$) and $t_0$ are chosen, respectively, such that

\[
\max \left\{ \max_{j \in \beta} r_j(A), r_{\max}(A[x]) \right\} < \min_{i \in x} r_i(A), r_{\min}(A[\beta, x]) > 0
\]  

(18)

and

\[
\max \left\{ \max_{j \in \beta} r_j(A), r_{\max}(A[x]) \right\} < t_0 < \min_{i \in x} \left\{ r_{\max}(A), \min_{i \in x} r_i(A) \right\},
\]  

(19)
then
\[ r_{\min}(A) < \min\{t_0, z(t_0, x)\} \leq \rho(A) \leq \max\{t_0, \hat{z}(t_0, \beta)\} < r_{\max}(A). \]  

(20)

**Proof.** When (18) and (19) hold, it follows that (10), (11) and (14), (15) hold, respectively. By using Corollary 2.1 and Lemma 2.3, we have that (20) holds. This completes the proof. □

It can be seen that (2) is improved by (20) in Theorem 2.1. From the proof of Corollary 2.1, Lemma 2.3 and Theorem 2.1, we have the following result.

**Corollary 2.2.** With the conditions of Theorem 2.1, we have
\[ r_{\max}(A) < \min\left\{t_0, \min_{j \in \mathbb{Z}} [r_j(A[\beta]) + v_1(t_0, \alpha)r_j(A[\alpha,\alpha])]\right\} \leq \rho(A) \leq \max\left\{t_0, \max_{i \in \mathbb{Z}} [r_i(A[\alpha]) + v_2(t_0, \beta)r_i(A[\alpha,\beta])]\right\} < r_{\max}(A), \]  

(21)

where
\[ v_1(t_0, \alpha) = \min_i \frac{r_i(A[\alpha,\beta])}{t_0 - r_i(A[\alpha])} \quad \text{and} \quad v_2(t_0, \beta) = \max_j \frac{r_j(A[\beta,\alpha])}{t_0 - r_j(A[\beta])}. \]

Next we will consider the following examples to illustrate the results of Theorem 2.1.

**Example 1.** Consider the positive matrix (see [6] or [5]):
\[ A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 2 & 3 & 5 \end{pmatrix}, \]

we can compute that \( r(A) = (4, 6, 10)^T, r_{\max}(A) = 10, r_{\min}(A) = 4 \). Let \( \alpha = \{3\}, \beta = N \setminus \alpha = \{1, 2\}, \) then
\[ \min_{i \in \mathbb{Z}} r_i(A) = 10 > 6 = \max_{j \in \beta} r_j(A), \quad \min_{i \in \mathbb{Z}} r_i(A) = 10 > 5 = r_{\max}(A[\alpha]), \]

\[ \max_{j \in \beta} \left\{ \min_{i \in \mathbb{Z}} r_i(A), r_{\max}(A[\alpha]) \right\} = 5 < t_0 < 10 = \min_{i \in \mathbb{Z}} \left\{ \min_{i \in \mathbb{Z}} r_i(A), r_{\max}(A) \right\}. \]

According to Theorem 2.1, let \( t_0 = 7 \), then \( z(t_0, \alpha) = 7, \hat{z}(t_0, \beta) = 7.8235 \),
\[ \min\{t_0, z(t_0, \alpha)\} = 7 \leq \rho(A) \leq \max\{t_0, \hat{z}(t_0, \beta)\} = 7.8325 \]

let \( t_0 = 7.5 \), then \( z(t_0, \alpha) = 6, \hat{z}(t_0, \beta) = 7.5466 \),
\[ \min\{7.5, 6\} = 6 \leq \rho(A) \leq \max\{7.5, 7.5466\} = 7.5466. \]

It follows that \( 7 \leq \rho(A) \leq 7.5466 \). Note that \( \rho(A) \approx 7.5311 \) and the upper and lower bounds are better than those stated in [5, Examples 2 and 3] and are better than those given in [6, p. 158].

**Example 2.** Consider the following \( 8 \times 8 \) matrix (see [7] or [5]):
\[ A = \begin{pmatrix} 8 & 6 & 3 & 5 & 7 & 0 & 7 & 1 \ 0 & 7 & 3 & 8 & 5 & 6 & 4 & 1 \ 1 & 2 & 6 & 1 & 3 & 8 & 8 & 7 \ 2 & 8 & 4 & 0 & 7 & 7 & 8 & 2 \ 2 & 4 & 6 & 2 & 5 & 7 & 6 & 5 \ 4 & 1 & 0 & 4 & 8 & 4 & 8 & 2 \ 3 & 1 & 6 & 6 & 4 & 5 & 5 & 0 \ 0 & 1 & 1 & 6 & 7 & 0 & 3 & 4 \ \end{pmatrix}. \]
We can compute that \( r_{\text{max}}(A) = 38 \), \( r_{\text{min}}(A) = 22 \) and \( r(A) = (37, 34, 36, 38, 37, 31, 30, 22)^T \). Let \( \beta = N \setminus \{6, 7, 8\} \), then

\[
\min_{i \in \beta} r_i(A) = 34 > 31 = \max_{j \in \beta} r_j(A), \quad \min_{i \in \beta} r_i(A) = 34 > 29 = r_{\text{max}}(A[z])
\]

and

\[
\max \left\{ \max_{j \in \beta} r_j(A), r_{\max}(A[z]) \right\} = 31 < t_0 < 34 = \min \left\{ \min_{i \in \beta} r_i(A), r_{\max}(A) \right\}.
\]

According to Theorem 2.1, if we let \( t_0 = 31.01 \), then \( z(t_0, \beta) = 29.2713, \hat{z}(t_0, \beta) = 36.7673 \), \( \min \{t_0, z(t_0, \beta)\} = 29.2713 \leq \rho(A) \leq \max \{t_0, \hat{z}(t_0, \beta)\} = 36.7673 \). Let \( \alpha = \{1, 3, 4, 5\} \), \( \beta = N \setminus \{2, 6, 7, 8\} \), then

\[
\min_{i \in \beta} r_i(A) = 36 > 34 = \max_{j \in \beta} r_j(A), \quad \min_{i \in \beta} r_i(A) = 36 > 16 = r_{\max}(A[z]),
\]

and

\[
\max \left\{ \max_{j \in \beta} r_j(A), r_{\max}(A[z]) \right\} = 34 < t_0 < 36 = \min \left\{ \min_{i \in \beta} r_i(A), r_{\max}(A) \right\}.
\]

According to Theorem 2.1, if we let \( t_0 = 34.575 \), then \( z(t_0, \beta) = 23.7308, \hat{z}(t_0, \beta) = 34.5760 \), \( \min \{t_0, z(t_0, \beta)\} = 23.7308 \leq \rho(A) \leq \max \{t_0, \hat{z}(t_0, \beta)\} = 34.5760 \).

So we have \( 29.2713 \leq \rho(A) \leq 34.5760 \). Noting that \( \rho(A) \approx 33.2418 \), it can be seen that the upper and lower bounds are better than the bounds given in [5, Example 4].

**Example 3.** Consider an \( n \times n \) positive matrix ([7] or [5]):

\[
A_n = \begin{pmatrix}
1 & 1 & 1 & \cdots & \cdots & 1 \\
1 & 2 & 2 & \cdots & \cdots & 2 \\
1 & 2 & 3 & \ddots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
1 & 2 & \cdots & n-1 & n-1 \\
1 & 2 & \cdots & n-2 & n-1 & n
\end{pmatrix}
\]

Let \( n = 20 \), then \( \rho(A_{20}) \approx 170.404 \), \( r_{\max}(A) = 210 \), \( r_{\min}(A) = 20 \). Let \( \alpha = \{11, \ldots, 20\} \), \( \beta = N \setminus \{1, \ldots, 10\} \), then

\[
\min_{i \in \beta} r_i(A) = 165 > 155 = \max_{j \in \beta} r_j(A) \quad \text{and} \quad \min_{i \in \beta} r_i(A) = 165 > 155 = r_{\max}(A[z]),
\]

\[
\max \left\{ \max_{j \in \beta} r_j(A), r_{\max}(A[z]) \right\} = 155 < t_0 < 165 = \min \left\{ \min_{i \in \beta} r_i(A), r_{\max}(A) \right\}.
\]

According to Theorem 2.1, if we let \( t_0 = 155.01 \), then \( z(t_0, \alpha) = 44.6665, \hat{z}(t_0, \beta) = 189.7912 \), \( \min \{t_0, z(t_0, \alpha)\} = 44.6665 \leq \rho(A) \leq \max \{t_0, \hat{z}(t_0, \beta)\} = 189.7912 \). Let \( \alpha = \{13, \ldots, 20\} \), \( \beta = N \setminus \{1, \ldots, 12\} \), then

\[
\min_{i \in \beta} r_i(A) = 182 > 174 = \max_{j \in \beta} r_j(A) \quad \text{and} \quad \min_{i \in \beta} r_i(A) = 182 > 132 = r_{\max}(A[z]),
\]

\[
\max \left\{ \max_{j \in \beta} r_j(A), r_{\max}(A[z]) \right\} = 174 < t_0 < 182 = \min \left\{ \min_{i \in \beta} r_i(A), r_{\max}(A) \right\}.
\]

According to Theorem 2.1, if we let \( t_0 = 177.4 \), then \( z(t_0, \alpha) = 23.2679, \hat{z}(t_0, \beta) = 177.4019 \), \( \min \{t_0, z(t_0, \alpha)\} = 23.2679 \leq \rho(A) \leq \max \{t_0, \hat{z}(t_0, \beta)\} = 177.4019 \).
So we have $44.6665 \leq \rho(A) \leq 177.4019$. Noting that when $x = \{10, \ldots, 20\}$, $\beta = N \setminus x = \{1, \ldots, 9\}$, we have

$$\min_{i \in x} r_i(A) = 155 > 144 = \max_{j \in \beta} r_j(A),$$

but $\min_{i \in x} r_i(A) = 155 < 165 = r_{\max}(A[z])$, so this do not satisfying inequality (18) in Theorem 2.1.

From the previous examples we can see that an appropriate choice of $x$ and $t_0$ such that conditions (18) and (19) hold in Theorem 2.1 makes us get a tighter upper bound and a lower bound of Perron root.

3. The optimal choice of $t$ after $x$

To get better bounds, we can see from Examples 1–3 that there still exists a problem as to how the parameters $t$ after $x$ should be chosen. In this section, we will discuss this problem.

**Lemma 3.1.** If $A$ is a nonnegative irreducible matrix, then

1. $z(t, z) = r_{\min}(P_t(A/A[z]))$ is a strictly decreasing function of $t$ on $(\rho(A[z]), +\infty))$.

2. $\hat{z}(t, \hat{z}) = r_{\max}(P_t(A/A[\hat{z}]))$ is a strictly decreasing function of $t$ on $(\rho(A[\hat{z}]), +\infty))$.

**Proof.** We will only prove the first part. The second part can be proved similarly. Suppose $t_2 > t_1 > \rho(A[z])$, then

$$t_2 I - A[z] > t_1 I - A[z],$$

it follows by the characters of $M$ matrices that

$$(t_1 I - A[z])^{-1} > (t_2 I - A[z])^{-1} > 0,$$

so we have that $P_{t_2}(A/A[z]) > P_{t_1}(A/A[z]) > 0$, therefore

$$r_{\min}(P_{t_2}(A/A[z])) < r_{\min}(P_{t_1}(A/A[z])).$$

This completes the proof. \(\square\)

By Lemma 3.1, we have

**Lemma 3.2.** Suppose $A$ is a nonnegative irreducible matrix, then

1. $\min\{t, z(t, z)\}$ is a strictly increasing function of $t$ when $\rho(A[z]) < t \leq z(t, z)$ and is strictly decreasing function of $t$ when $t \geq z(t, z)$.

2. $\max\{t, \hat{z}(t, \hat{z})\}$ is a strictly decreasing function of $t$ when $\rho(A[\hat{z}]) < t \leq \hat{z}(t, \hat{z})$ and is a strictly increasing function of $t$ when $t \geq \hat{z}(t, \hat{z})$.

**Proof.** As for the first part, when $\rho(A[z]) < t \leq z(t, z)$, we have $\min\{t, z(t, z)\} = t$ a strictly increasing function of $t$. If $t \geq z(t, z)$, then $\min\{t, z(t, z)\} = z(t, z)$ is a strictly decreasing function of $t$ by using Lemma 3.1. We can prove the second part similarly and so it is omitted. This completes the proof. \(\square\)

From Lemmas 3.1, 3.2 and Theorem 2.1, we can easily have

**Theorem 3.1.** Suppose $A$ satisfies the conditions in Theorem 2.1, then the lower bound $\min\{t, z(t, z)\}$ of $\rho(A)$ is tightest when $t$ satisfies $t = z(t, z)$ and the upper bound $\max\{t, \hat{z}(t, \hat{z})\}$ of $\rho(A)$ is tightest when $t$ satisfies $t = \hat{z}(t, \hat{z})$.

Several examples are given as follows to show the application of Theorems 2.1 and 3.1.

**Example 4.** Consider the nonnegative matrix (see [5, Example 6] or [2, Example 3]):

$$A = \begin{pmatrix} 2 & 5 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 1 & 4 & 1 & 2 \\ 1 & 1 & 0 & 1 \end{pmatrix},$$
Example 6. Consider the positive matrix in Example 3. Choose bound of $\hat{t}$ which satisfies the conditions in Theorem 2.1 and makes $\hat{t}$.

Estimates of the bounds on $\rho(A)$ with $\alpha = \{1, 3\}$

<table>
<thead>
<tr>
<th>$t_0$</th>
<th>$z(t_0, \alpha)$</th>
<th>$\hat{z}(t_0, \beta)$</th>
<th>$\min{t_0, z(t_0, \alpha)} \leq \rho(A) \leq \max{t_0, \hat{z}(t_0, \beta)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.53</td>
<td>4.5445</td>
<td>4.9763</td>
<td>4.53 \leq \rho(A) \leq 4.9763</td>
</tr>
<tr>
<td>4.54</td>
<td>4.5327</td>
<td>4.9685</td>
<td>4.5327 \leq \rho(A) \leq 4.9685</td>
</tr>
<tr>
<td>4.55</td>
<td>4.5210</td>
<td>4.9608</td>
<td>4.5210 \leq \rho(A) \leq 4.9608</td>
</tr>
<tr>
<td>4.7</td>
<td>4.3582</td>
<td>4.8519</td>
<td>4.3582 \leq \rho(A) \leq 4.8519</td>
</tr>
<tr>
<td>4.79</td>
<td>4.2707</td>
<td>4.7921</td>
<td>4.2707 \leq \rho(A) \leq 4.7921</td>
</tr>
<tr>
<td>4.8</td>
<td>4.2614</td>
<td>4.7857</td>
<td>4.2614 \leq \rho(A) \leq 4.8</td>
</tr>
</tbody>
</table>

Estimates of the bounds on $\rho(A)$ with $\alpha = \{3\}$

<table>
<thead>
<tr>
<th>$t_0$</th>
<th>$z(t_0, \alpha)$</th>
<th>$\hat{z}(t_0, \beta)$</th>
<th>$\min{t_0, z(t_0, \alpha)} \leq \rho(A) \leq \max{t_0, \hat{z}(t_0, \beta)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.5</td>
<td>8.6667</td>
<td>8.1681</td>
<td>6.5 \leq \rho(A) \leq 8.1681</td>
</tr>
<tr>
<td>7.0</td>
<td>7</td>
<td>7.8235</td>
<td>7.0 \leq \rho(A) \leq 7.8235</td>
</tr>
<tr>
<td>7.5</td>
<td>6</td>
<td>7.5466</td>
<td>6 \leq \rho(A) \leq 7.5466</td>
</tr>
<tr>
<td>7.6</td>
<td>5.8462</td>
<td>7.4796</td>
<td>5.8 \leq \rho(A) \leq 7.6</td>
</tr>
</tbody>
</table>

$\rho(A) \approx 4.6182$, $r_{\max}(A) = 8$, $r_{\min}(A) = 3$. Let $\alpha = \{1, 3\}$, $\beta = N \setminus \alpha = \{2, 4\}$, then

$$\min_{i \in \alpha} r_i(A) = 8 > 3 = \max_{j \in \beta} r_j(A) \quad \text{and} \quad \min_{i \in \alpha} r_i(A) = 8 > 3 = r_{\max}(A[\alpha]),$$

choose different $t_0$ satisfying

$$\max_{j \in \beta} \left\{ \max_{i \in \alpha} r_j(A), r_{\max}(A[\alpha]) \right\} = 3 < t_0 < 8 = \min_{i \in \alpha} \left\{ \min_{i \in \alpha} r_i(A), r_{\max}(A) \right\}.$$  \hspace{1cm} (22)

by computing we get different values of $z(t, \alpha)$ and $\hat{z}(t, \beta)$ shown in Table 1. From Table 1 we can see the following three facts. Firstly, $4.5327 \leq \rho(A) \leq 4.7921$. We see that the bounds improve the bounds obtained in [5, Example 6]. Secondly, we get lower bound $4.5327$ of $\rho(A)$ when $t_0 = 4.54 \approx 4.5327 = z(t_0, \alpha)$ and the upper bound 4.7921 of $\rho(A)$ when $t_0 = 4.79 \approx 4.7921 = \hat{z}(t_0, \beta)$, which suggests that Theorem 3.1 provides us with good results. Finally, $z(t_0, \alpha)$ decreases strictly from 4.5445 to 4.2614 with $t_0$ from 4.53 to 4.8 and $\hat{z}(t_0, \beta)$ decrease strictly from 4.9763 to 4.7857 with $t_0$ from 4.53 to 4.8, which supported the results in Lemma 3.1.

Example 5. Consider the matrix in Example 1. Choose $\alpha = \{3\}$, when $t_0$ is evaluated differently which satisfies $6 < t_0 < 10$. By Theorem 2.1 we have different values of $z(t, \alpha)$ and $\hat{z}(t, \beta)$ shown in Table 2. Noting that $7.0 \leq \rho(A) \leq 7.5466$, we get the lower bound 7.0 of $\rho(A)$ when $t_0 = 7.0 = z(t_0, \alpha)$ and the upper bound 7.5466 of $\rho(A)$ when $t_0 = 7.5 \approx 7.5466 = \hat{z}(t_0, \beta)$, which suggest that the results in Lemmas 3.1 and 3.2 are perfect.

Remark 3.1. It is a pity that the parameter $t$ in Theorem 3.1 satisfying $t = z(t, \alpha)$ or $t = \hat{z}(t, \beta)$ cannot always reach, respectively, for the conditions in Theorem 2.1. However, we should choose $t$ which satisfies the conditions in Theorem 2.1 and make $|t - z(t, \alpha)|$ smallest so that we can get a much tighter lower bound of $\rho(A)$. Similarly, we should choose $t$ which satisfies the conditions in Theorem 2.1 and makes $|t - \hat{z}(t, \beta)|$ smallest so that we can get a much tighter upper bound of $\rho(A)$.

Example 6. Consider the positive matrix in Example 3. Choose $\alpha = \{11, \ldots, 20\}$, $155 < t_0 < 165$ and $\alpha = \{13, \ldots, 20\}$, $174 < t_0 < 182$. By Theorem 2.1 we have Table 3. From Table 3, we get the lower bound 44.665 of $\rho(A)$ when $t_0 = 155.01$ makes $|t_0 - z(t_0, \alpha)| = 110.345$ the smallest value in Table 3 and satisfies the conditions in Theorem 2.3, and the upper bound 177.4019 of $\rho(A)$ when $t_0 = 177.4 \approx 177.4019 = \hat{z}(t_0, \beta)$ holds, which suggest that Theorem 3.1 provides us with a good method to choose $t$ after $\alpha$. 
Table 3
Estimates of the bounds on $\rho(A)$ with $\alpha = \{11, \ldots, 20\}$ and $\alpha = \{13, \ldots, 20\}$

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$t_0$</th>
<th>$z(t_0, \alpha)$</th>
<th>$\hat{z}(t_0, \beta)$</th>
<th>$\min{t_0, z(t_0, \alpha)} \leq \rho(A) \leq \max{t_0, \hat{z}(t_0, \beta)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha = {11, \ldots, 20}$</td>
<td>155.01</td>
<td>44.665</td>
<td>189.7912</td>
<td>44.6665 $\leq \rho(A) \leq 189.7912$</td>
</tr>
<tr>
<td>$\alpha = {11, \ldots, 20}$</td>
<td>156</td>
<td>44.3744</td>
<td>189.4824</td>
<td>44.3744 $\leq \rho(A) \leq 189.4824$</td>
</tr>
<tr>
<td>$\alpha = {11, \ldots, 20}$</td>
<td>160</td>
<td>37.448</td>
<td>188.2887</td>
<td>37.448 $\leq \rho(A) \leq 188.2887$</td>
</tr>
<tr>
<td>$\alpha = {13, \ldots, 20}$</td>
<td>176</td>
<td>23.5611</td>
<td>177.9647</td>
<td>23.5611 $\leq \rho(A) \leq 177.9647$</td>
</tr>
<tr>
<td>$\alpha = {13, \ldots, 20}$</td>
<td>177.4</td>
<td>23.2679</td>
<td>177.4019</td>
<td>23.2679 $\leq \rho(A) \leq 177.4019$</td>
</tr>
<tr>
<td>$\alpha = {13, \ldots, 20}$</td>
<td>177.5</td>
<td>23.2475</td>
<td>177.3622</td>
<td>23.2475 $\leq \rho(A) \leq 177.3622$</td>
</tr>
</tbody>
</table>

Acknowledgement

The authors are grateful to the anonymous referee for the constructive and helpful comments and Prof. Lothar Reichel for all the communication.

References