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Tutte's 3-Flow Conjecture and Short Cycle Covers

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In this paper we prove: (i) If a graph G has a nowhere-zero 6-flow ϕ such that $|E_{odd}(\phi)| \geqslant \frac{2}{3}|E(G)|$, then G has a cycle cover in which the sum of the lengths of the cycles in the cycle cover is at most $\frac{44}{27}|E(G)|$, where $E_{odd}(\phi) = \{e \in E(G) : \phi(e)$ is odd$\}$; (ii) if Tutte's 3-Flow Conjecture is true, then every bridgeless graph G has a nowhere-zero 6-flow ϕ such that $|E_{odd}(\phi)| \geqslant \frac{2}{3}|E(G)|$. © 1993 Academic Press, Inc.

1. INTRODUCTION

For a graph G, $E(G)$ denotes the set of edges of G. An edge is said to be *contracted* if it is removed and its ends are identified. A *bridge* of G is an edge whose removal leaves a graph with more components than G. An *even graph* is one in which every vertex is of even degree. Sometimes, we identify a subgraph with its edge set. The *symmetric difference* of two even subgraphs Z_1 and Z_2, denoted by $Z_1 \oplus Z_2$, is the even subgraph $(Z_1 \cup Z_2) \backslash (Z_1 \cap Z_2)$. The empty set $\varnothing$ is regarded as an even subgraph of every graph.

A *cycle cover* of a graph G is a collection of cycles of G which covers all edges of G. Since a graph is even if and only if it has a decomposition into edge-disjoint cycles, we also regard a cycle cover as a collection of even subgraphs. The *size* of a cycle cover is the sum of the lengths of the cycles in the cover. The problem of finding a cycle cover of small size (a short cycle cover) has been studied by several authors [2, 8, 15]. The best known result on this subject is possibly the one by Bermond, Jackson, and Jaeger [2], and Alon and Tarsi [1], that every bridgeless graph G has a cycle cover of size at most $\frac{5}{3}|E(G)|$. A different proof of this result can be found in [4].

A *flow* in a graph G with an orientation D is an integer-valued function ϕ on $E(G)$ such that, for each vertex v, the sum of $\phi(e)$ over all edges e with

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head v is equal to the sum over all e with tail v. The *support* of ϕ is defined by $S(\phi) = \{e \in E(G) : \phi(e) \neq 0\}$. If there is a positive integer k such that $-k < \phi(e) < k$ for every $e \in E(G)$, then ϕ is called a k-flow, and furthermore, if $S(\phi) = E(G)$, then ϕ is called a *nowhere-zero k-flow*. An excellent survey on this topic is given by Jaeger [10]. For a flow ϕ in G, we define

$$E_{\text{odd}}(\phi) = \{e \in E(G) : \phi(e) \text{ is odd}\}$$

and

$$E_{\text{even}}(\phi) = \{e \in E(G) : \phi(e) \text{ is even}\}.$$

The following is an equivalent form of the 3-flow conjecture of Tutte (see unsolved problem 48 of [3]).

Conjecture 1.1. Every 4-edge-connected graph has a nowhere-zero 3-flow.

A theorem of Grötzsch [5] implies that the conjecture is true for planar graphs. Steinberg and Younger [14] proved that the conjecture is true for graphs that can be imbedded in the projective plane. Tutte [16] also conjectured that every bridgeless graph has a nowhere-zero 5-flow. Seymour [13] proved that every bridgeless graph has a nowhere-zero 6-flow (or see [18]), which improved the 8-flow theorem of Jaeger [9]. In his proof, Seymour [13] showed that a 3-connected cubic graph G has an even subgraph F and a 3-flow ϕ such that $E(G) \backslash E(F) \subseteq S(\phi)$. In this paper we show that if Tutte's 3-Flow Conjecture is true, then we can, in fact, choose F to be a 2-factor of G, which has an interesting application to the short cycle cover problem. This motivates the following conjecture.

Conjecture 1.2. Every bridgeless graph G has an even subgraph F and a 3-flow ϕ such that $E(G) \backslash E(F) \subseteq S(\phi)$ and $|E(F)| \geqslant \frac{2}{3} |E(G)|$.

Suppose that F is an even subgraph of a graph G. If G has a 3-flow ϕ with $E(G) \backslash E(F) \subseteq S(\phi)$, then we let f be a 2-flow in G with $S(f) = F$; it is easily seen that $g = 2\phi + f$ is a nowhere-zero 6-flow with $E_{\text{odd}}(g) = F$. Conversely, if G has a nowhere-zero 6-flow g with $E_{\text{odd}}(g) = F$, then by a classical result of Tutte (Lemma 2.1 in the next section), G has a 3-flow ϕ in which $\phi(e) = 0$ only if $|g(e)| = 3$, and thus, only if $e \in E(F)$. These show that Conjecture 1.2 is equivalent to

Conjecture 1.3. Every bridgeless graph G has a nowhere-zero 6-flow g with $|E_{\text{odd}}(g)| \geqslant \frac{2}{3} |E(G)|$.

We shall prove

THEOREM 1.4. *If Conjecture* 1.1 *is true, then Conjecture* 1.2 *is true.*

THEOREM 1.5. *If a graph has a nowhere-zero 6-flow g with* $|E_{\text{odd}}(g)| \geqslant \frac{2}{3}|E(G)|$, *then G has a cycle cover of size at most* $\frac{44}{27}|E(G)|$.

Combining Theorems 1.4 and 1.5 and taking into account the fact that Conjectures 1.2 and 1.3 are equivalent, we have that

THEOREM 1.6. *If Tutte's 3-Flow Conjectue is true, then every bridgeless graph G has a cycle cover of size at most* $\frac{44}{27}|E(G)|$.

Related to Conjecture 1.2, the following is a problem suggested by the referee.

PROBLEM 1.7. Let G be a bridgeless graph. Find the best possible lower bound on $|E(F)|$, where F is an even subgraph of G such that G has a 3-flow ϕ with $E(G)\setminus E(F) \subseteq S(\phi)$.

2. PRELIMINARIES

A basic result on integer flows is the following one due to Tutte ((6.3) of [17]).

LEMMA 2.1. *If G has a flow* ϕ, *then, for any integer* $k>0$, *G has a k-flow* ϕ' *such that* $\phi'(e) \equiv \phi(e) \pmod{k}$ *for every* $e \in E(G)$.

The following is an easy consequence of Lemma 2.1.

LEMMA 2.2. *Let* $F \subseteq E(G)$ *and let G' be obtained from G by contracting the edges in F. If G' has a k-flow* ϕ', *then G has a k-flow* ϕ *such that* $S(\phi') \subseteq S(\phi)$.

A flow in G is always associated with some orientation of G. By changing signs, one can arrange for two flows in G to have the same orientation. If ϕ_1 and ϕ_2 are two flows in G under the same orientation D, then for any integers l and m the linear combination $\phi = l\phi_1 + m\phi_2$ is a flow under D. Let ϕ be a flow in G. A flow f in G is called a *sub-flow* of ϕ if

(i) f has the same orientation as ϕ;

(ii) $|f(e)| \leqslant |\phi(e)|$ and $f(e)\,\phi(e) \geqslant 0$ for every $e \in E(G)$,

where the condition $f(e)\,\phi(e) \geqslant 0$ simply means that $f(e)$ and $\phi(e)$ have the same sign. Moreover, if f is a k-flow, then we call f a *sub-k-flow* of ϕ. By the definition, if f is a sub-flow of ϕ, then $\phi - f$ is also a sub-flow of ϕ. For technical reasons, we regard the everywhere-zero flow as a sub-2-flow of any flow. Note that we may make a flow nowhere-negative by changing the orientation. The construction of a feasible circulation described in [7, pp. 51–53] gives the following interesting result (or see [11]).

LEMMA 2.3. *Let ϕ be a flow in a graph G and r a real number, $r \geqslant 1$. Then G has a sub-flow f of ϕ in which $f(e) = \lfloor \phi(e)/r \rfloor$ or $\lceil \phi(e)/r \rceil$ for every edge $e \in E(G)$.*

By this result, if a graph G has a k-flow ϕ, where $k \geqslant 2$, then it has a sub-2-flow f of ϕ (using $r = k-1$). Since $\phi - f$ is a sub-$(k-1)$-flow of ϕ, we may apply the lemma to $\phi - f$ using $r = k-2$ (if $k \geqslant 3$). Repeatedly, we decompose the k-flow ϕ into $(k-1)$ sub-2-flows. This is a result found by Little, Tutte, and Younger [12]. We state it as Lemma 2.4 below.

LEMMA 2.4. *Every k-flow ϕ is the sum of $(k-1)$ sub-2-flows of ϕ.*

Lemma 2.4 plays a key role in the proof of the next lemma. For a flow ϕ in a graph G, we set

$$E_i(\phi) = \{e \in E(G) : \phi(e) = i\} \qquad \text{and} \qquad E_{\pm i}(\phi) = E_i(\phi) \cup E_{-i}(\phi).$$

LEMMA 2.5. *Let f be a k-flow in G. Then there is a k-flow ϕ in G such that $S(\phi) = S(f)$ and*

$$|E_{\pm 1}(\phi)| \geqslant \frac{k-1}{k} (|E_{+1}(f)| + |E_{\pm(k-1)}(f)|).$$

Proof. By Lemma 2.4, we have that $f = \sum_{i=1}^{k-1} f_i$, where f_i is a sub-2-flow of f. Let $\phi_i = kf_i - f$, $1 \leqslant i \leqslant k-1$. Since $f_i(e)$ and $f(e)$ have the same sign for each $e \in E(G)$, each ϕ_i is a k-flow in G with the same support as f. Let $e \in E(G)$: if $|f(e)| = 1$, then there is exactly one j such that $|f_j(e)| = 1$, and so $|\phi_i(e)| = 1$ for all $i \neq j$, $1 \leqslant i \leqslant k-1$; if $|f(e)| = k-1$, then for every i, $1 \leqslant i \leqslant k-1$, $|f_i(e)| = 1$ and so $|\phi_i(e)| = 1$. Therefore,

$$\sum_{i=1}^{k-1} |E_{\pm 1}(\phi_i)| \geqslant (k-2)|E_{\pm 1}(f)| + (k-1)|E_{\pm(k-1)}(f)|.$$

That is,

$$|E_{\pm 1}(f)| + \sum_{i=1}^{k-1} |E_{\pm 1}(\phi_i)| \geqslant (k-1)(|E_{\pm 1}(f)| + |E_{\pm(k-1)}(f)|).$$

Choosing $\phi \in \{f, \phi_1, ..., \phi_{k-1}\}$ with $|E_{\pm 1}(\phi)|$ maximum,

$$k\,|E_{\pm 1}(\phi)| \geqslant (k-1)(|E_{\pm 1}(f)| + |E_{\pm(k-1)}(f)|).$$

Dividing both sides by k yields the required result. ∎

LEMMA 2.6. *Let ϕ be a 4-flow in G. Then G contains two even subgraphs Z_1 and Z_2 such that*

$$Z_1 \cup Z_2 = S(\phi) \qquad \text{and} \qquad Z_1 \cap Z_2 = E_{\pm 2}(\phi).$$

Proof. We apply Lemma 2.3 to ϕ using $r = 2$ to obtain a sub-3-flow f, and let $f' = \phi - f$. Then $E_{\pm 1}(f)$ and $E_{\pm 1}(f')$ are two even subgraphs of the required properties. ∎

3. PROOF OF THEOREMS

A graph is called a *weighted graph* if each edge e is assigned a non-negative real number $w(e)$, called the *weight* of e. Let G be a weighted graph and H a subgraph of G. The *weight* of H is defined by

$$w(H) = \sum_{e \in E(H)} w(e).$$

An unweighted graph can be regarded as a weighted graph in which each edge e is assigned weight $w(e) = 1$. On the other hand, for a weighted graph with a weight function w, since the rationals are dense, we may assume that w is rational-valued. By multiplying out denominators, we have a weighted graph in which each edge has an integer weight $w'(e)$. Replacing each edge e by a path of length $w'(e)$ gives an unweighted graph. By these observations we see that Theorem 1.4 is equivalent to

THEOREM 3.1. *If Conjecture* 1.1 *is true, then every bridgeless weighted graph G has an even subgraph F and a 3-flow ϕ such that $E(G) \backslash E(F) \subseteq S(\phi)$ and $w(F) \geq \frac{2}{3} w(G)$.*

However, sometimes it seems more convenient to work on a weighted graph than on an unweighted one, in particular, when we use induction to reduce the size of the graph. This has been seen in [4] and will be seen again here. Instead of proving Theorem 1.4 directly, we prove Theorem 3.1 first. Before going to the proof, we need the following two observations.

(i) Let G be a loopless, 2-edge-connected graph. If G has a vertex y of degree more than three, then by a result of Fleischner [6] there are two edges xy and yz such that deleting xy, yz and joining x and z by a new edge yield a 2-edge-connected graph. Let G' be the new graph and assign to the new edge the weight $w(xy) + w(yz)$ so that $w(G') = w(G)$. It is clear that an even subgraph of G' can be extended to an even subgraph of G with the same weight and a 3-flow in G' can be extended to a 3-flow in G with the same support.

(ii) If G has a 2-edge-cut $\{e_1, e_2\}$, we let G' be the graph obtained by contracting e_1. Reassign to e_2 a new weight $w(e_1)+w(e_2)$ so that $w(G')=w(G)$. Consider an even subgraph F' of G' and a 3-flow ϕ' in G'. If $e_2 \in F'$, then $F' \cup \{e_1\}$ is an even subgraph of G with the same weight as F'; if $e_2 \notin F'$, then F' gives an even subgraph of G with the same set of edges. Moreover, in either case, the 3-flow ϕ' can be extended to a 3-flow ϕ in G in which $\phi(e)=\phi'(e)$ for all $e \in E(G)\setminus\{e_1\}$ and $\phi(e_1)=\phi'(e_2)$ or $-\phi'(e_2)$, according to the orientation of e_2.

From the above two observations, it follows that

LEMMA 3.2. *If G is a counterexample to the statement of Theorem* 3.1, *with a minimum number of edges, then G is simple, cubic, and* 3-*connected.*

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. Suppose, to the contrary, that the theorem is not true. Let G be a minimum counterexample. Then, by Lemma 3.2, G is simple, cubic, and 3-connected. As proved in [4], G contains a 2-factor F with

$$w(F) \geqslant \tfrac{2}{3} w(G)$$

such that the graph G', obtained by contracting each component of F, is 4-edge-connected. If Conjecture 1.1 is true, then G' has a nowhere-zero 3-flow. By Lemma 2.2, G has a 3-flow ϕ with $E(G') \subseteq S(\phi)$. But $E(G')=E(G)\setminus E(F)$, and so F and ϕ have the required properties. This contradiction proves Theorem 3.1. ∎

Now we return to unweighted graphs. The rest of the paper is devoted to the proof of Theorem 1.5. We first prove the following lemma.

LEMMA 3.3. *Let g be a 6-flow in G. Then there is a 6-flow ϕ in G such that $S(\phi)=S(g)$ and*

$$|E_{\pm 1}(\phi)| \geqslant \tfrac{5}{9} |E_{\text{odd}}(g)|.$$

Proof. Since $E_{\text{odd}}(g)$ is an even subgraph of G, we may define a 2-flow f_0 in G with $S(f_0)=E_{\text{odd}}(g)$. Applying Lemma 2.1 to the flows $g+2f_0$ and $g-2f_0$ yields two 6-flows in G, say f_1 and f_2, respectively, in which for every $e \in E(G)$

$$f_1(e) \equiv g(e)+2f_0(e) \pmod 6 \qquad \text{and} \qquad f_2(e) \equiv g(e)-2f_0(e) \pmod 6.$$

It is not difficult to see that

$$E_{\pm 3}(f_1) \cup E_{\pm 3}(f_2) = E_{\pm 1}(g) \cup E_{\pm 5}(g).$$

Adding $E_{\pm 3}(g)$ to both sides yields

$$E_{\pm 3}(g) \cup E_{\pm 3}(f_1) \cup E_{\pm 3}(f_2) = E_{\text{odd}}(g).$$

Let $f \in \{g, f_1, f_2\}$ with $|E_{\pm 3}(f)|$ smallest. Then

$$|E_{\pm 3}(f)| \leqslant \tfrac{1}{3}\, |E_{\text{odd}}(g)|.$$

Since f is a 6-flow in G, by Lemma 2.5 with $k=6$, there is a 6-flow ϕ in G such that $S(\phi) = S(f)$ $(= S(g))$ and

$$\begin{aligned}|E_{\pm 1}(\phi)| &\geqslant \tfrac{5}{6}(|E_{\pm 1}(f)| + |E_{\pm 5}(f)|) = \tfrac{5}{6}(|E_{\text{odd}}(f)| - |E_{\pm 3}(f)|)\\ &\geqslant \tfrac{5}{6}(|E_{\text{odd}}(f)| - \tfrac{1}{3}\,|E_{\text{odd}}(g)|).\end{aligned}$$

Since $E_{\text{odd}}(f) = E_{\text{odd}}(g)$,

$$|E_{\pm 1}(\phi)| \geqslant \tfrac{5}{6}(\tfrac{2}{3}\,|E_{\text{odd}}(g)|) = \tfrac{5}{9}\,|E_{\text{odd}}(g)|.$$

This completes the proof of Lemma 3.3. ∎

Proof of Theorem 1.5. Let g be a nowhere-zero 6-flow in G with $|E_{\text{odd}}(g)| \geqslant \tfrac{2}{3}\,|E(G)|$. It follows from Lemma 3.3 that G has a nowhere-zero 6-flow ϕ such that

$$|E_{\pm 1}(\phi)| \geqslant \tfrac{5}{9}\,|E_{\text{odd}}(g)| \geqslant \tfrac{10}{27}\,|E(G)|.$$

Applying Lemma 2.3 to ϕ with $r=2$, we have a sub-4-flow of ϕ, say ϕ_1, such that $\phi_1(e) = \lfloor \phi(e)/2 \rfloor$ or $\lceil \phi(e)/2 \rceil$, for every $e \in E(G)$. Set $\phi_2 = \phi - \phi_1$. Then ϕ_2 is also a sub-4-flow of ϕ. Let $A = \{e \in E(G) : \phi_1(e) = 0\}$ and $B = \{e \in E(G) : \phi_2(e) = 0\}$. Then $A \cap B = \varnothing$ and $A \cup B = E_{\pm 1}(\phi)$. Moreover, $|\phi_1(e)| = 1$ if $e \in B$ and $|\phi_2(e)| = 1$ if $e \in A$. Applying Lemma 2.6 to ϕ_1, we have two even subgraphs X_1 and X_2 such that $X_1 \cup X_2 = S(\phi_1)$ and $X_1 \cap X_2 = E_{\pm 2}(\phi_1)$. Set $X_3 = X_1 \oplus X_2$. Then $B \subseteq X_3$ and $\{X_1, X_2, X_3\}$ covers each edge of $S(\phi_1) = E(G) - A$ exactly twice. Similarly, ϕ_2 yields three even subgraphs, say Y_1, Y_2, and Y_3, such that $A \subseteq Y_3$ and $\{Y_1, Y_2, Y_3\}$ covers each edge of $S(\phi_2) = E(G) - B$ exactly twice. Let $C_1 = \{X_1, X_2, Y_3\}$ and $C_2 = \{Y_1, Y_2, X_3\}$. Then both C_1 and C_2 are cycle covers of G. Since

$$\begin{aligned}\sum_{i=1}^{3} |X_i| + \sum_{i=1}^{3} |Y_i| &= 2\,|S(\phi_1)| + 2\,|S(\phi_2)| = 4\,|E(G)| - 2\,|A \cup B|\\ &= 4\,|E(G)| - 2\,|E_{\pm 1}(\phi)|,\end{aligned}$$

either C_1 or C_2 is of size at most

$$2\,|E(G)| - |E_{\pm 1}(\phi)| \leqslant 2\,|E(G)| - \tfrac{10}{27}\,|E(G)| = \tfrac{44}{27}\,|E(G)|.$$

This completes the proof of Theorem 1.5. ∎

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