Iterative equations in Banach spaces

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Abstract

Let $X$ be a Banach space, let
\[ K(X) := \{ f : X \to X \text{ Lipshitz: } \| f - \text{id} \|_{\sup} < \infty \}, \]
and let $P : K(X) \to K(X)$, $F \in K(X)$. By applying the Banach contraction principle we prove that

if $P$ is sufficiently close (in a certain sense) to the identity then the equation

$Pf = F$

has a unique solution $f$. As a corollary we obtain results on iterative equations of the types

$\sum_i A_i f^i(x) = F(x)$ or $\sum_i A_i f(\phi_i(x)) = F(x)$

with operator coefficients in Banach spaces.

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1. Introduction

One of the oldest and most important problems of iterative equation is the to find the iterative roots of a given function \( F \),

\[ f^n = F. \]  \hfill (1)

Its study goes back to N. Abel [1] and Ch. Babbage [5]. For some more recent results and further references on the subject of iterative equations we refer to the monographs [6,7] and to the survey article [3].

The polynomial-like iterative equation

\[ \sum_i \alpha_i f^i = F, \]

where \( \sum_i \alpha_i = 1 \) is a natural generalization of (1).

A successful approach to deal with this equation with the use of fixed point method was introduced by W. Zhang in [8] in the year 1987. We would like to quote the main theorem.

**Theorem Zh.** Let \( F : [a, b] \to [a, b] \), \( F(a) = a \), \( F(b) = b \) be such that there exist \( L, M > 0 \) with

\[ L(x_2 - x_1) \leq F(x_2) - F(x_1) \leq M(x_2 - x_1) \quad \text{for} \quad x_1 \leq x_2. \]

Let \( \alpha_1, \ldots, \alpha_n \geq 0 \), \( \alpha_1 > 0 \), be such that \( \sum_{i=1}^n \alpha_i = 1 \). Then the equation

\[ \sum_{i=1}^n \alpha_i f^i(x) = F(x) \quad \text{for} \quad x \in [a, b] \]  \hfill (2)

has a solution \( f : [a, b] \to [a, b] \), which is an a increasing function Lipschitz with constant \( M/\lambda_1 \).

Since we use its idea we would like to present the sketch of the proof.

**Idea of the proof.** To solve Eq. (2) it is enough to find a solution to the fixed point problem

\[ f = (Gf)^{-1} \circ F, \]

where \( G : f \to \sum_{i=1}^n \alpha_i f^i. \)

We define \( V \) as the set of increasing homeomorphisms of \([a, b]\) with \( \text{lip}(f) \leq M/\alpha_1 \) (lip denotes the Lipschitz constant). Let \( f \in V \). Since \( f \) is increasing we have

\[ Gf(y) - Gf(x) \geq \alpha_1(y - x) \quad \text{for} \quad x < y. \]  \hfill (3)

This implies that \( Gf \) is increasing (and therefore invertible) and \( \text{lip}((Gf)^{-1}) \leq 1/\alpha_1 \). Consequently \((Gf)^{-1} \circ F\) is increasing as a composition of increasing functions and \( \text{lip}((Gf)^{-1} \circ F) \leq \text{lip}((Gf)^{-1}) \cdot \text{lip}(F) \leq M/\alpha_1 \). This means that \((Gf)^{-1} \circ f \in V\).

By the Arzelà–Ascoli lemma \( V \) is compact in the space of continuous functions with the supremum norm. Schauder’s fixed point theorem applied to the operation \( V \ni f \to (Gf)^{-1} \circ F \in V \) makes the proof complete. \( \square \)
As we see the above proof does not give us uniqueness or continuous dependence of solutions on $F$ or coefficients. If we want to get this we need to apply the Banach contraction principle instead of Schauder’s theorem—however at a cost. Roughly speaking, to obtain that the mapping $f \rightarrow (Gf)^{-1} \circ F$ is a contraction we need to assume that $\text{lip}(F)$ is not big and $\lambda_1$ is close to one (see, for example, [9,10]).

Equation (2) can be generalized in many directions. For example, [2] deals with the case of variable coefficients.

In [4] the more dimensional case was treated—the domain of $F$ was a compact convex subset of $\mathbb{R}^n$. The following paper is a partial generalization of [4]—we assume that the domain is an arbitrary closed (not necessarily convex!) subset of a Banach space, and we allow operator coefficients. The main real “novelty” which allowed this is that instead of working with Lipschitz constant of $F$ we work with Lipschitz constant of $F - \text{id}$ (which is assumed to be small). It causes some additional work but in our opinion is worth the trouble.

2. Fixed-point theorem

As our results are based on the Banach contraction principle we first need to describe the space on which our results will take place.

Let $X$ be a Banach space. For functions $f : S \rightarrow X$, where $S$ is a subset of $X$, we put

$$\|f\|_{\sup} := \sup_{s \in S} \|f(s)\|,$$
$$k(f) := \text{lip}(f - \text{id}).$$

One can easily notice that $\|\cdot\|_{\sup}$ defines a metric (which can possibly attain $+\infty$) by the formula $d_{\sup}(f, g) := \|f - g\|_{\sup}.$

Let $V$ be a closed subset of $X$. We define

$$\mathcal{K}(V) := \{f : V \rightarrow X \mid f|_{\partial V} = \text{id}|_{\partial V}, \|f - \text{id}\|_{\sup} < \infty\},$$
$$\mathcal{K}_m(V) := \{f \in \mathcal{K}(V) \mid k(f) \leq m\} \quad \text{for} \ m \geq 0.$$

The space $(\mathcal{K}(V), d_{\sup})$ is complete metric space and $\mathcal{K}_m(V)$ is a closed subset of $\mathcal{K}(V)$.

Now we are ready to present our main tool which we will use further on.

**Theorem 1.** Let $m \in (0, 1)$ be given. Let $\mathcal{F}, \mathcal{G} : \mathcal{K}_m(V) \rightarrow \mathcal{K}(V)$ be fixed and let

$$K_{\mathcal{F}} := \sup\{k(\mathcal{F} f) : f \in \mathcal{K}_m(V)\},$$
$$K_{\mathcal{G}} := \sup\{k(\mathcal{G} f) : f \in \mathcal{K}_m(V)\}.$$

If

$$\frac{1 + k_{\mathcal{F}}}{1 - k_{\mathcal{G}}} \leq 1 + m,$$  
(4)
$$\text{lip}(\mathcal{F}) + \text{lip}(\mathcal{G}) < 1 - k_{\mathcal{G}}.$$  
(5)
Then the equation

$$G f(x) = F f(x) \quad \text{for } x \in V$$

has in $K_m(V)$ a unique solution $f$.

Before proceeding to the proof we first need to show some properties of the space $K(V)$.

**Proposition 1.** Let $f, g, f_1, g_1 \in K(V)$ and $f_1(V) \subset V$ and $g_1(V) \subset V$.

(i) If $\text{lip}(f) < \infty$ then

$$\|f \circ f_1 - g \circ g_1\|_{\text{sup}} \leq \text{lip}(f) \|f_1 - g_1\|_{\text{sup}} + \|f - g\|_{\text{sup}}.$$  

(ii) We assume that $f, g$ are invertible and that $\text{lip}(f^{-1}) < \infty$. Then

$$\|f^{-1} - g^{-1}\|_{\text{sup}} \leq \text{lip}(f^{-1}) \|f - g\|_{\text{sup}}.$$  

**Proof.** (i) We have

$$\|f(f_1(x)) - g(g_1(x))\| \leq \|f(f_1(x)) - f(g_1(x))\| + \|f(g_1(x)) - g(g_1(x))\| \leq \text{lip}(f) \|f_1 - g_1\|_{\text{sup}} + \|f - g\|_{\text{sup}}.$$  

(ii) We show the second inequality. We have

$$\|f^{-1} - g^{-1}\|_{\text{sup}} = \|f^{-1} - f^{-1} \circ f \circ g^{-1}\|_{\text{sup}} \leq \text{lip}(f^{-1}) \|g \circ g^{-1} - f \circ g^{-1}\|_{\text{sup}} = \text{lip}(f^{-1}) \|f - g\|_{\text{sup}}.$$  

If $q = 1$ then by $\frac{q^n - 1}{q - 1}$ we understand $n$. The next corollary follows in an easy inductive argument from Proposition 1(i).

**Corollary 1.** Let $f, g \in K(V)$, $f(V) \subset V$ and $g(V) \subset V$. If $\text{lip}(f) < \infty$ then

$$\|f^n - g^n\|_{\text{sup}} \leq \frac{\text{lip}(f)^n - 1}{\text{lip}(f) - 1} \|f - g\|_{\text{sup}} \quad \text{for } n \in \mathbb{N}.$$  

Now we are ready to show some properties of the function $k$.

**Proposition 2.**

(i) Let $f, g \in K(V)$, $g(V) \subset V$. Then $f \circ g \in K(V)$ and

$$k(f \circ g) \leq (1 + k(f))(1 + k(g)) - 1.$$  

(ii) Let $f \in K(V)$ be such that $k(f) < 1$. Then $f(V) \subset V$.

(iii) Let $f \in K(V)$ be such that $k(f) < 1$. Then $f^{-1} \in K(V)$ and

$$k(f^{-1}) \leq \frac{1}{1 - k(f)} - 1.$$  

(6)
Proof. We use the following notation: for $f \in K(V)$ by $p_f$ we denote the mapping $f - \text{id}$. 

(i) We have

$$k(f \circ g) = \text{lip}((\text{id} + p_f) \circ (\text{id} + p_g) - \text{id}) = \text{lip}(p_g + p_f \circ (\text{id} + p_g))$$

$$\leq \text{lip}(p_g) + \text{lip}(p_f \circ (\text{id} + p_g)) \leq \text{lip}(p_g) + \text{lip}(p_f) \cdot (1 + \text{lip}(p_g))$$

$$= k(g) + k(f) \cdot (1 + k(g)).$$

Moreover, by Proposition 1 we have

$$\|f \circ g - \text{id}\|_{\sup} \leq \text{lip}(f)\|g - \text{id}\|_{\sup} + \|f - \text{id}\|_{\sup} < \infty.$$ 

(ii) We consider the case $V \neq X$. Let $x \in V$ be arbitrary and let $x_0 \in \partial V$ be such that

$$\|x - x_0\| \leq \frac{1}{k(f)}d(x, \partial V).$$

Then

$$\|f(x) - x\| = \|(f(x) - x) - (f(x_0) - x_0)\| \leq k(f)\|x - x_0\| \leq d(x, \partial V),$$

which trivially yields that $f(x) \in V$.

(iii) We first show that $f$ is surjective. Let us fix an arbitrary $z \in V$. To show that $f$ is surjective we have to find $x_0 \in V$ such that $x_0 + p_f(x_0) = z$, or in other words that $z - p_f(x_0) = x_0$.

To do so we use Banach fixed point theorem. We prolong $p_f$ on the whole of $X$ by setting $p_f(x) = 0$ for $x \in X \setminus V$. One can easily check that since $f|_{\partial V} = \text{id}|_{\partial V}$ such prolonged $p_f$ is Lipschitz with constant $k(f) \leq m < 1$. Thus the mapping $x \to z - p_f(x)$ is a contraction, and therefore by the Banach contraction principle there exists $x_0 \in X$ such that $x_0 = z - p_f(x_0)$. We show that $x_0 \in V$. If not, then $p_f(x_0) = 0$, and consequently $x_0 = z$, a contradiction since by the assumptions $z \in V$.

Now we deal with injectivity of $f$. For $x, y \in V$ we have

$$\|f(x) - f(y)\| = \|(\text{id} + p_f)(x) - (\text{id} + p_f)(y)\| \geq \|x - y\| - \text{lip}(p_f)\|x - y\|$$

$$= (1 - k(f))\|x - y\|.$$ 

Since $k(f) < 1$ this yields that $f$ is injective. It follows that $g := f^{-1}$ is well defined and that

$$\|g(x) - g(y)\| \leq \frac{1}{1 - k(f)}\|x - y\| \quad \text{for} \quad x, y \in V.$$ 

We estimate $k(f^{-1}) = k(g)$ from above. For $x, y \in V$ we obtain

$$\|p_g(x) - p_g(y)\| = \|p_f(g(x)) - p_f(g(y))\| \leq k(f)\|g(x) - g(y)\|$$

$$\leq \frac{k(f)}{1 - k(f)}\|x - y\|. \quad \Box$$

Remark 1. We would like to explain here the one of the reasons why we work with the constant $k$ instead of only with Lipschitz constant. Looking at the proof of Theorem Zh one sees that it is essential to obtain that the function $Gf = a_0\text{id} + \sum_{i=1}^{n} a_i f^{-1}$ is invertible. In the paper [4] it was done with the use of Brouwer’s theorem—namely an injection $f$ from ball into itself which is identity on the boundary of the ball is automatically a surjection. Since this result does not hold in infinite dimensional Banach spaces we needed
an analogous condition which would ensure the invertibility of \( f \). That is why we arrived at the constant \( k \) (if \( k(f) < 1 \) then \( f \) is invertible, see Proposition 2).

As a direct corollary of the Proposition 2(i) we get

**Corollary 2.** Let \( f \in \mathcal{K}(V) \). Then
\[
k(f^n) \leq (1 + k(f))^n - 1 \quad \text{for } n \in \mathbb{N}.
\]

Since \( \text{lip}(g) = \text{lip}(\text{id} + (g - \text{id})) \leq 1 + k(g) \) as a consequence of Proposition 2(iii) we obtain

**Corollary 3.** Let \( f \in \mathcal{K}(V) \) be such that \( k(f) < 1 \). Then \( f^{-1} \in \mathcal{K}(V) \) and
\[
\text{lip}(f^{-1}) \leq \frac{1}{1 - k(f)}.
\]

Now we are ready to present the proof of Theorem 1.

**Proof of Theorem 1.** By (5), \( k(Gf) < 1 \) for \( f \in \mathcal{K}_m(V) \). Then Proposition 2(iii) yields that the mapping
\[
\mathcal{P} : \mathcal{K}_m(V) \ni f \mapsto (Gf)^{-1} \circ Ff \in \mathcal{K}(V)
\]
is well defined. For \( f \in \mathcal{K}_m(V) \) by Proposition 1 and (4) we obtain
\[
k(\mathcal{P}f) = k((Gf)^{-1} \circ Ff) \leq \frac{1 + kF}{1 - kG} - 1 \leq m.
\]
This means that \( \mathcal{P}(\mathcal{K}_m(V)) \subset \mathcal{K}_m(V) \). We show that \( \mathcal{P} \) is a contraction. Let \( f, g \in \mathcal{K}_m(V) \) be arbitrarily chosen. Then by Proposition 1 and Corollary 3,
\[
\|\mathcal{P}f - \mathcal{P}g\|_{\sup} = \|(Gf)^{-1} \circ Ff - (Gg)^{-1} \circ Fg\|
\leq \frac{1}{1 - kG} \|(Ff - Fg)\|_{\sup} + \|(Gf)^{-1} - (Gg)^{-1}\|_{\sup}
\leq \frac{1}{1 - kG} (\text{lip}(F) + \text{lip}(G)) \cdot \|f - g\|_{\sup},
\]
which by (5) implies that \( \mathcal{P} \) is a contraction. Banach contraction principle makes the proof complete. \( \square \)

We present a direct corollary of Theorem 1 in the case when \( F \) is a constant map.

**Corollary 4.** Let \( m \in (0, 1) \) be given. Let \( F \in \mathcal{K}_m(V) \) and \( G : \mathcal{K}_m(V) \to \mathcal{K}(V) \) be fixed and let
\[
k_G := \sup\{k(Gf) : f \in \mathcal{K}_m(V)\}.
\]
We assume that
Then the equation
\[ Gf \left( f(x) \right) = F(x) \quad \text{for } x \in V \]
has in \( K_m(V) \) a unique solution \( f \).

3. Polynomial-like iterative equation

We present application of the previous section for the polynomial iterative equation.

From now on we use the following convention: given a double sequence \( \{f_k\}_{k \in \mathbb{Z}} \) of functions, by \( \sum_{k \in \mathbb{Z}} f_k \) we understand a limit of the series in the topology of pointwise convergence.

By \( L(X) \) we denote the Banach space of bounded linear operators on \( X \) with the operator norm. We assume that we are given a sequence \( \{A_i\}_{i \in \mathbb{Z}} \subset L(X) \) such that
\[ \sum_{i \in \mathbb{Z}} A_i = \text{id}. \]

For \( I \subset \mathbb{Z} \) we define the function \( W_I : [0, 1] \to [0, \infty) \) by the following formula:
\[ W_I(r) := \sum_{i < 1, i \in I} \|A_i\| \cdot [(1 - r)^i - 1] + \sum_{i > 1, i \in I} \|A_i\| \cdot [(1 + r)^i - 1]. \]

**Theorem 2.** Let \( m \in (0, 1) \). Let \( V \) be a nonempty closed subset of a Banach space \( X \) and let \( F \in K(V) \). If
\[ (1 + m)(1 - W_Z(m)) > 1 + k(F) \]
then the iterative equation
\[ \sum_{i = -\infty}^{\infty} A_i f^i = F \]
has in \( K_m(V) \) a unique solution \( f \).

**Proof.** We apply Corollary 4. We put
\[ Gf := \sum_{i = -\infty}^{\infty} A_i f^{i-1}. \]

We first show that for each \( f \in K_m(V) \) the function \( Gf \) is well defined.

Let \( I \) be a finite subset of \( \mathbb{Z} \) and let \( f, g \in K_m(V) \) be fixed. Then by Proposition 1, Corollaries 1 and 3, we have
\[
\sum_{i \in I} \|A_i(f^{i-1} - g^{i-1})\|_{\sup} \\
\leq \sum_{i < 1, i \in I} \|A_i\| \cdot \|f^{i-1} - g^{i-1}\|_{\sup} + \sum_{i > 1, i \in I} \|A_i\| \cdot \|f^{i-1} - g^{i-1}\|_{\sup} \\
\leq \sum_{i < 1, i \in I} \|A_i\| \cdot \frac{(1/(1-m))^{-(i-1)} - 1}{1/(1-m) - 1} \|f - g\|_{\sup} \\
+ \sum_{i > 1, i \in I} \|A_i\| \cdot \frac{(1 + m)^{i-1} - 1}{1 + m - 1} \|f - g\|_{\sup} \\
= \frac{W_I(m)}{m} \|f - g\|_{\sup}.
\]

Thus we get
\[
\sum_{i \in I} \|A_i(f^{i-1} - g^{i-1})\|_{\sup} \leq \frac{W_I(m)}{m} \cdot \|f - g\|_{\sup}. \tag{10}
\]

Let us fix \(f \in \mathcal{K}_m(V)\) and \(x \in V\). We show that the sum appearing in the definition of \(\mathcal{G}f(x)\) is convergent. By (10) applied to the pair \(f, \text{id}\) and the fact that \(W_I(m)\) is convergent we have
\[
\limsup_{k,n \to \infty} \sum_{k < |i| < n} \|A_i(f^{i-1}(x))\| \\
\leq \limsup_{k,n \to \infty} \sum_{k < |i| < n} \|A_i(f^{i-1}(x) - x)\| + \limsup_{k,n \to \infty} \sum_{k < |i| < n} \|A_i\| \cdot \|x\| \\
\leq \limsup_{k,n \to \infty} \frac{W_{i: k < |i| < n}(m)}{m} \|f - \text{id}\|_{\sup} + 0 = 0,
\]

which means that \(\mathcal{G}f\) is a well-defined function.

Now we show some properties of the mapping \(\mathcal{G}\). For \(f, g \in \mathcal{K}_m(V)\) by (10) we get
\[
\|\mathcal{G}f - \mathcal{G}g\|_{\sup} \leq \frac{W(m)}{m} \|f - g\|_{\sup} \quad \text{for } f, g \in \mathcal{K}_m(V). \tag{11}
\]

If \(g = \text{id}\) then \(\mathcal{G}g = \text{id}\) and therefore this proves that \(\|\mathcal{G}f - \text{id}\|_{\sup} < \infty\), and consequently that \(\mathcal{G}f \in \mathcal{K}(V)\).

Now we direct our attention to (7). By applying Proposition 2 and Corollary 2 we get for \(x, y \in V\),
\[
\|\mathcal{G}(f(x) - x) - \mathcal{G}(f(y) - y)\| \\
\leq \sum_{i < 1} \|A_i\| \|f^{i-1}(x) - f^{i-1}(y)\| \\
+ \sum_{i > 1} \|A_i\| \|f^{i-1}(x) - f^{i-1}(y)\| \\
\leq \sum_{i < 1} \|A_i\| k(f^{i-1})\|x - y\| + \sum_{i > 1} \|A_i\| k(f^{i-1})\|x - y\|
\[ \leq \left( \sum_{i<1} \|A_i\| \left[ \frac{1 + \frac{k(f)}{1 - k(f)}}{1} \right]^{-(i-1)} - 1 \right) \\
+ \sum_{i>1} \|A_i\| \left( 1 + k(f) \right)^{i-1} \|x - y\| \\
= W(k(f)) \|x - y\| \leq W(m) \|x - y\| , \]

and therefore
\[ k_G \leq W(m). \tag{12} \]

By (9) this yields that (7) holds.

We prove (8). By (9), (11) and (12) we have
\[ \text{lip}(G) \leq \frac{W(m)}{m} \leq \frac{m - k(F)}{m(1 + m)} \leq \frac{m + mk(F)}{m(1 + m)} = \frac{1 + k(F)}{1 + m} \leq 1 - W(m) \leq 1 - K_G. \]

Corollary 4 makes the proof complete. \( \square \)

**Example 1.** To illustrate our results let us consider the quadratic iterative equation
\[ A(f^2(x)) + B(f(x)) + C(x) = F(x) \quad \text{for} \quad x \in V, \tag{13} \]
where \( A, B, C \in \mathcal{L}(X) \), \( A + B + C = \text{id} \) and \( F \in K(V) \). We assume that
\[ S := \|A\| + \|C\| < 1. \]

One can check that if \( k(F) \) is small enough, namely if
\[ k(F) < \left( \sqrt{1 + S} - \sqrt{2S} \right)^2 \]
then Eq. (13) has in \( K_m(V) \) a unique solution, with \( m = 1 - \frac{\sqrt{2S}}{1 + S} \).

4. Linear iterative equation

Analogously to the previous section we assume that \( \{A_i\}_{i=0,\ldots,\infty} \subset \mathcal{L}(X) \) is a sequence such that
\[ \sum_{i=0}^{\infty} A_i = \text{id}. \]

We assume that we are given \( \{\psi_i\}_{i=1,\ldots,\infty} \subset K(V) \) such that \( \psi_i(V) \subset V \). We have the following result.

**Theorem 3.** Let \( m \in (0, 1) \). Let \( V \) be a nonempty closed subset of a Banach space \( X \) and let \( F \in K(V) \). If
\[ C := \frac{1 + m}{1 - m} \sum_{i=1}^{\infty} \|A_i\| \left( 1 + k(\psi_i) \right) < \frac{1}{2}, \tag{14} \]
\[ C \leq \frac{m - k(f)}{1 + m} + \sum_{i=1}^{\infty} \|A_i\| \tag{15} \]
and
\[ \sum_{i=1}^{\infty} \|A_i\| \|\psi_i - \text{id}\|_{\sup} < \infty, \tag{16} \]
then the iterative equation
\[ A_0 f(x) + \sum_{i=1}^{\infty} A_i f(\psi_i(x)) = F(x) \]
has in \( K_m(V) \) a unique solution \( f \).

Proof. First we show that for all \( f \in K_m(V) \) and all \( x \in V \)
series
\[ \sum_{i=1}^{\infty} A_i \circ f \circ \psi_i \circ f^{-1}(x) \]
is convergent in the space \( X \).

For positive integer \( n \) by Propositions 1 and 2 we have
\[
\left\| \sum_{i>n} A_i f \psi_i f^{-1}(x) \right\|
\leq \left\| \sum_{i>n} A_i f \psi_i f^{-1}(x) - \sum_{i>n} A_i \text{id} \psi_i \text{id}^{-1}(x) \right\| + \left\| \sum_{i>n} A_i \psi_i (x) \right\|
\leq \sum_{i=1}^{\infty} \|A_i\| \left( (1 + k(f)) \|f^{-1} - \text{id^{-1}}\|_{\sup} + \|f - \text{id}\|_{\sup} \right) + \sum_{i>n} A_i \psi_i (x)
\leq \sum_{i>n} \|A_i\| \cdot \left( \frac{1 + k(f)}{1 - k(f)} (1 + k(\psi_i)) + 1 \right) \|f - \text{id}\|_{\sup}
+ \left\| \sum_{i>n} A_i(x) \right\| + \sum_{i>n} \|A_i\| \|\psi_i - \text{id}\|_{\sup}
\leq \frac{1 + m}{1 - m} \sum_{i>n} \|A_i\| \left( \frac{1 + k(f)}{1 - k(f)} (1 + k(\psi_i)) + 1 \right) \|f - \text{id}\|_{\sup} + \frac{1 + m}{1 - m} \sum_{i>n} \|A_i\| \|f - \text{id}\|_{\sup}
+ \left\| \sum_{i>n} A_i(x) \right\| + \sum_{i>n} \|A_i\| \|\psi_i - \text{id}\|_{\sup} \to 0
\]
if \( n \to \infty \).

We define the mapping \( G : K_m(V) \to K(V) \) by
\[ G f := A_0 + \sum_{i=1}^{\infty} A_i \circ f \circ \psi_i \circ f^{-1}. \]

We need the Lipschitz constant of \( G \). By Propositions 1 and 2 we have for \( f, g \in K_m(V) \),
\[
\|Gf(x) - Gg(x)\| \leq \sum_{i=1}^{\infty} \|A_i\| \cdot \|f(\psi_i(f^{-1}(x))) - g(\psi_i(g^{-1}(x)))\|
\]
\[
\leq \sum_{i=1}^{\infty} \|A_i\| (\text{lip}(f) \text{lip}(\psi_i) \|f^{-1} - g^{-1}\|_{\sup} + \|f - g\|_{\sup})
\]
\[
\leq \sum_{i=1}^{\infty} \|A_i\| \left( \frac{1 + k(f)}{1 - k(f)} (1 + k(\psi_i)) + 1 \right) \|f - g\|_{\sup}.
\]
This yields that
\[
\text{lip}(G) \leq \sum_{i=1}^{\infty} \|A_i\| - \left( \frac{1 + m}{1 - m} (1 + k(\psi_i)) + 1 \right). \tag{17}
\]

Now let \(f \in K_m(V)\). Then \(\|Gf - G(id)\|_{\sup} \leq \text{lip}(G) \|f - id\|_{\sup}\). Since clearly \(G(id) - id\) is bounded, we obtain that \(Gf - id\) is also bounded, and therefore \(Gf \in K(V)\).

Now we deal with \(k_G\). We have
\[
\|Gf(x) - x\| - (Gf(y) - y)\|
\]
\[
= \sum_{i=1}^{\infty} A_i \left[ (f(\psi_i(f^{-1}(x))) - x) - (f(\psi_i(f^{-1}(y))) - y) \right]
\]
\[
\leq \sum_{i=1}^{\infty} \|A_i\| \cdot \|f(\psi_i(f^{-1}(x))) - x - f(\psi_i(f^{-1}(y))) + y)\|
\]
\[
\leq \|x - y\| \sum_{i=1}^{\infty} \|A_i\| \cdot \left[ \frac{1 + k(f)}{1 - k(f)} (1 + k(\psi_i)) - 1 \right]
\]
\[
\leq \frac{1 + m}{1 - m} \|x - y\| \sum_{i=1}^{\infty} \|A_i\| \|1 + k(\psi_i)\| - \|x - y\| \sum_{i=1}^{\infty} \|A_i\|,
\]
which proves that
\[
k_G \leq \frac{1 + m}{1 - m} \sum_{i=1}^{\infty} (1 + k(\psi_i))\|A_i\| - \sum_{i=1}^{\infty} \|A_i\|. \tag{18}
\]

One can easily verify that (18) and (15) imply (7), (17) and (18) imply (8).

Corollary 4 completes the proof. \(\Box\)

**Example 2.** Let \(\{A_i\}_{i=0,\ldots,\infty} \subset L(X)\) be such that \(\sum_{i=0}^{\infty} A_i = id\) and let \(\{x_i\}_{i=1,\ldots,\infty} \subset X\) be arbitrary. We assume that
\[
s := \sum_{i=1}^{\infty} \|A_i\| < 0.5.
\]

We put \(m = (s + \sqrt{2s^2 + 2s})/(s + 2)\). If
\[
k(F) < \frac{1}{2} \left( \frac{m - s}{2m} \right) / \left( \frac{1 - m}{1 + m} \right),
\]
then one can check that the equation
\[ A_0(x) + \sum_{i=1}^{\infty} A_i f(x + x_i) = F(x) \quad \text{for } x \in X \]
has in \( K_m(V) \) a unique solution \( f \).

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