An International Journal
computers & mathematics with applications

PERGAMON Computers and Mathematics with Applications 39 (2000) 77–90
www.elsevier.nl/locate/camwa

Periodic Solutions of a Single Species Discrete Population Model with Periodic Harvest/Stock

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(Received and accepted May 1999)

Abstract—We discuss a discrete population model describing single species growth with periodic harvest/stock. The theory of coincidence degree is applied to show that the model equation admits two periodic solutions. Under minor technical assumptions, we show that one of these two periodic solutions is positive and attracts almost all positive solutions.

1. INTRODUCTION

Much has been done for the discrete model

$$x(n+1) = (1+\alpha)x(n)\left[1 - \frac{1}{K}x(n)\right] \tag{1.1}$$

describing the growth of a single species population, where $\alpha > 0$ is the parameter related to the growth rate and $K > 0$ is the carrying capacity [1–3].

In this paper, we consider model equation (1.1) subject to periodic harvest/stock. In particular, we consider the nonautonomous difference equation

$$x(n+1) = \mu x(n)\left[1 - \frac{1}{K}x(n)\right] + b(n), \qquad \text{for } n \in \mathbb{N}, \tag{1.2}$$

where $\mu = 1 + \alpha \in (1,2)$, $b(n)$ denotes the difference between the stock and the harvest rates at time $n+1$, and we assume that $b : \mathbb{N} \to \mathbb{R}$ is an ω-periodic number sequence with $\omega \geq 1$ and satisfies

$$|b(n)| < \frac{(\mu-1)^2}{4\mu}K, \qquad \text{for } n \in \mathbb{N}. \tag{1.3}$$

*Project partially supported by National Natural Science Funds of P.R. China.
†Project partially supported by Natural Sciences and Engineering Research Council of Canada.

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PII: S0898-1221(99)00315-6

A first step towards our complete description of the dynamics of (1.2) is the existence of multiple periodic solutions. We note that such an existence problem is highly nontrivial and, to the best of our knowledge, while some progress has been made for differential equations [4–7] and linear difference equations [8–10], little has been done for nonlinear problems [11]. Our approach to the existence problem here is based on the coincidence degree and the related continuation theorem as well as some *a priori* estimates. We will first formulate a general existence theorem which, when applied to equation (1.2), implies the existence of two periodic solutions $\overline{x}$ and x^*. We will then show that $\overline{x}$ is unstable and x^* is exponentially asymptotically stable and attracts all positive solutions x of (1.2) if $x(1) \neq \overline{x}(1)$.

2. EXISTENCE OF PERIODIC SOLUTIONS OF GENERAL DIFFERENCE EQUATIONS

In this section, we apply the theory of the composite coincidence degree to a general difference equation to obtain a general existence result for periodic solutions of prescribed periods.

Let $\mathbb{R}$, $\mathbb{N}$, and $\mathbb{Z}$ denote the sets of all real numbers, nonnegative integers, and integers, respectively. We fix two integers $\omega \geq 1$ and $q \geq 1$. Define

$$l_q = \{x = \{x(n)\} : x(n) \in \mathbb{R}^q,\ n \in \mathbb{Z}\}.$$

For a sequence of mappings $\{G_n : n \in \mathbb{N}\}$ with $G_n : l_q \to \mathbb{R}^q$, we use $G = \{G_n\}$ to denote the mapping $G : l_q \to l_q$ defined by

$$G(x) = \{G_n(x)\}, \qquad \text{for } x \in l_q. \tag{2.1}$$

For $a = (a_1, \dots, a_q) \in \mathbb{R}^q$, define $|a| = \max_{1 \leq j \leq q} |a_j|$. Let $l^\omega \subseteq l_q$ denote the subspace of all ω-periodic sequences equipped with the usual supremum norm $\|\cdot\|$, i.e.,

$$\|x\| = \max_{0 \leq n \leq \omega - 1} |x(n)|, \qquad \text{for } x = \{x(n) : n \in \mathbb{Z}\} \in l^\omega.$$

It is easy to see that l^ω is a finite-dimensional Banach space.

Let the linear operator $S : l^\omega \to \mathbb{R}^q$ be defined by

$$S(x) = \frac{1}{\omega} \sum_{n=0}^{\omega-1} x(n), \qquad x = \{x(n)\} \in l^\omega. \tag{2.2}$$

Then we obtain two subspaces l_0^ω and l_c^ω of l^ω defined by

$$l_0^\omega = \{x = \{x(n)\} \in l^\omega : S(x) = 0\} \tag{2.3}$$

and

$$l_c^\omega = \{x = \{x(n)\} \in l^\omega : x(n) \equiv \beta, \text{ for some } \beta \in \mathbb{R}^q \text{ and for all } n \in \mathbb{N}\}, \tag{2.4}$$

respectively. Denote by $L : l^\omega \to l^\omega$ the difference operator given by $Lx = \{(Lx)(n)\}$ with

$$(Lx)(n) = x(n+1) - x(n), \qquad \text{for } x \in l^\omega \text{ and } n \in \mathbb{Z}. \tag{2.5}$$

Let a linear operator $K : l^\omega \to l_c^\omega$ be defined by $Kx = \{(Kx)(n)\}$ with

$$(Kx)(n) \equiv S(x), \qquad \text{for } x \in l^\omega \text{ and } n \in \mathbb{Z}. \tag{2.6}$$

Then we have the following lemma.

LEMMA 2.1.

(i) *Both* l_0^ω *and* l_c^ω *are closed linear subspaces of* l^ω *and*

$$l^\omega = l_0^\omega \oplus l_c^\omega, \qquad \dim l_c^\omega = q. \tag{2.7}$$

(ii) L *is a bounded linear operator with*

$$\operatorname{Ker} L = l_c^\omega \qquad \text{and} \qquad \operatorname{Im} L = l_0^\omega. \tag{2.8}$$

(iii) K *is a bounded linear operator with*

$$\operatorname{Ker}(L+K) = \{0\} \qquad \text{and} \qquad \operatorname{Im}(L+K) = l^\omega,$$

that is, $L + K : l^\omega \to l^\omega$ *is a bijection.*

PROOF. It is easy to check that both l_0^ω and l_c^ω are closed. To show (2.7), we associate each $x = \{x(n)\} \in l^\omega$ with two sequences $x_c = \{x_c(n)\}$ and $x_0 = \{x_0(n)\}$ defined by

$$x_c(n) = S(x) \quad \text{and} \quad x_0(n) = x(n) - S(x), \qquad \text{for } n \in \mathbb{Z}. \tag{2.9}$$

Obviously, $x_c \in l_c^\omega$ and $x_0 \in l_0^\omega$. Hence, $l^\omega = l_0^\omega + l_c^\omega$. On the other hand, if $x = \{x(n)\} \in l_c^\omega \cap l_0^\omega$, then $x \in l_c^\omega$ implies that there exists $\beta \in \mathbb{R}^q$ so that $x(n) \equiv \beta$ for $n \in \mathbb{Z}$ and $x \in l_0^\omega$ gives $\beta = S(x) = 0$, and hence, $x = 0$. This completes the proof of (i).

It is trivial to see from (2.3) and (2.5) that $\operatorname{Ker} L = l_c^\omega$, so we only need to prove $\operatorname{Im} L = l_0^\omega$. Observe that $S(Lx) = 0$ for any $x \in l^\omega$. Therefore, $\operatorname{Im} L \subset l_0^\omega$. We now prove $l_0^\omega \subset \operatorname{Im} L$. For any $y = \{y(n)\} \in l_0^\omega$, define $x = \{x(n)\}$ by

$$x(n) = \begin{cases} \sum\limits_{i=0}^{n-1} y(i), & \text{if } n \ge 1, \\ 0, & \text{if } n = 0, \\ -\sum\limits_{i=n}^{-1} y(i), & \text{if } n \le -1. \end{cases}$$

Then it is easy to check that $x(n+\omega) - x(n) = \omega S(y)$ for $n \ge -\omega$ and $x(n+\omega) - x(n) = -\omega S(y)$ for $n \le -(\omega+1)$. This, combined with $y \in l_0^\omega$, implies that $x(n+\omega) = x(n)$ for all $n \in \mathbb{Z}$, i.e., $x \in l^\omega$. Clearly, $L(x) = y$. Therefore, $l_0^\omega \subset \operatorname{Im} L$, and hence, (ii) is proved.

Finally, to prove (iii), we first observe from (2.2), (2.5), and (2.6) the following basic relations:

$$S(Lx) = 0, \quad S(Kx) = S(x), \quad L(Kx) = 0, \qquad \text{for } x \in l^\omega. \tag{2.10}$$

If $x \in \operatorname{Ker}(L+K)$, then $Lx + Kx = 0$. Applying S to both sides and using (2.10), we get $S(x) = 0$. This, combined with (2.6), gives $Kx = 0$, and hence, $Lx = 0$. Thus, (ii) and (2.3) imply that $x \in l_0^\omega \cap l_c^\omega$. Therefore, by (i), $x = 0$, i.e., $\operatorname{Ker}(K+L) = \{0\}$. On the other hand, for any $x \in l^\omega$, we can decompose x uniquely as $x = x_c + x_0$ with $x_c \in l_c^\omega$ and $x_0 \in l_0^\omega$. Then (2.8) implies that there exists $\hat{y} \in l^\omega$ so that $L\hat{y} = x_0$. Let $y = \hat{y} - K\hat{y} + Kx$, then $y \in l^\omega$. We have by (2.10) that $Ly = L\hat{y} = x_0$ and $S(y) = S(x)$. From $S(y) = S(x)$, we get $Ky = x_c$. Thus, we have $(L+K)y = x_0 + x_c = x$, and hence, $\operatorname{Im}(L+K) = l^\omega$. This completes the proof.

We now recall a general continuation theorem related to coincidence problems. Let X and Y be two Banach spaces. A closed linear operator $\hat{L} : \operatorname{Dom}(\hat{L}) \cap X \to Y$ is a Fredholm operator of index zero if

(i) $\operatorname{Ker} \hat{L}$ is a finite-dimensional subspace of X;
(ii) $\operatorname{Im} \hat{L}$ is closed and has finite codimension;
(iii) $\dim \operatorname{Ker} \hat{L} = \operatorname{codim} \operatorname{Im} \hat{L}$.

For such an operator $\hat{L}$, a compact resolvent is a compact linear operator $\hat{K} : X \to Y$ such that $\hat{L} + \hat{K} : \mathrm{Dom}(\hat{L}) \to Y$ is a bijection. We denote by $CR(\hat{L})$ the set of all compact resolvents of $\hat{L}$. For any $\hat{K} \in CR(\hat{L})$, we define $R_{\hat{K}} = (\hat{L} + \hat{K})^{-1} : Y \to X$. Let $\Omega \subseteq X$ be a bounded, open subset such that $\mathrm{Dom}(\hat{L}) \cap \Omega \neq \emptyset$ and $\hat{G} : \overline{\Omega} \to Y$ be a continuous mapping. For any $\hat{K} \in CR(\hat{L})$, we define a transformation $H_{\hat{K}}(\hat{G}) : \overline{\Omega} \to X$ by $H_{\hat{K}}(\hat{G}) = R_{\hat{K}}[\hat{G} + \hat{K}]$. We say that $\hat{G}$ is $\hat{L}$-condensing if $H_{\hat{K}}(\hat{G}) : \overline{\Omega} \to X$ with some $\hat{K} \in CR(\hat{L})$ is a condensing mapping. We denote by $C_{\hat{L}}(\overline{\Omega}, \partial\Omega)$ the class of all $\hat{L}$-condensings $\hat{G} : \overline{\Omega} \to Y$ such that $x \neq H_{\hat{K}}(\hat{G})(x)$ for every $x \in \partial\Omega$. The following general result will be needed (see [12, Theorem 4.6.9]).

THEOREM 2.2. *Suppose that Ω is an open, bounded, convex, and symmetric neighborhood of $0 \in X$. Assume that $\hat{G} \in C_{\hat{L}}(\overline{\Omega}, \partial\Omega)$ and there is no point $x \in \partial\Omega$ such that $x + \lambda x \in \mathrm{Dom}(\hat{L})$ and $\hat{L}(x + \lambda x) = \hat{G}(x) - \lambda\hat{G}(-x)$ for $\lambda \in [0,1]$. Then $\deg_{\hat{L}}(\hat{G}, \Omega) \neq 0$ and the composite coincidence problem*

$$\hat{L}x = \hat{G}(x), \qquad x \in \overline{\Omega}$$

has at least one solution in Ω; here, $\deg_{\hat{L}}(\hat{G}, \Omega) := \deg(Id - H_{\hat{K}}(\hat{G}), \Omega)$ with $\hat{K} \in CR(\hat{L})$ is the composite coincidence degree of $\hat{G}$ and the operator $\hat{L}$ in the set Ω, and $\deg(Id - H_{\hat{K}}(\hat{G}), \Omega)$ is the degree for a condensing field.

To apply Theorem 2.2 to difference equations, we let $X = Y = l^\omega$, $\Omega \subset l^\omega$, be an open, bounded subset and $G : \Omega \to l^\omega$ a continuous mapping which maps every bounded subset of Ω into bounded set of l^ω. For $r > 0$, we set

$$\Omega_r = \{x = \{x(n)\} \in l^\omega : \|x\| < r\}.$$

It is clear that Ω_r is an open, bounded, convex, and symmetric neighborhood of $0 \in l^\omega$.

Using Lemma 2.1, we have the following corollary.

COROLLARY 2.3.

(i) *The bounded linear operator $L : l^\omega \to l^\omega$ is a Fredholm operator of index zero.*
(ii) *The bounded linear operator $K : l^\omega \to l^\omega_c$ is a compact resolvent of L, i.e., $K \in CR(L)$.*
(iii) *$R_K = (L + K)^{-1}$ and $H_K = R_K[G + K]$ are both compact operators, i.e., G is L-condensing.*
(iv) *$G \in C_L(\overline{\Omega}, \partial\Omega)$ provided that $Lx \neq G(x)$ for every $x \in \partial\Omega$.*

As an immediate consequence of Theorem 2.2 and Corollary 2.3, we have the following existence theorem for ω-solutions of the difference equation:

$$\Delta x(n) = G_n(x), \tag{2.11}$$

where $x = \{x(n)\} \in l_q$, $\Delta x(n) = x(n+1) - x(n)$, and $G_n : l_q \to \mathbb{R}^q$ for $n \in \mathbb{Z}$.

THEOREM 2.4. *Assume that there exists $r > 0$ such that*

(i) *$G = \{G_n\} : \overline{\Omega}_r \to l^\omega$ is a continuous mapping whose image is a bounded set of l^ω;*
(ii) *there is no point $x \in \partial\Omega_r$ such that*

$$\Delta x(n) = \frac{1}{1+\lambda} G_n(x) - \frac{\lambda}{1+\lambda} G_n(-x), \tag{2.12}$$

for $\lambda \in [0,1]$.

Then (2.11) has an ω-periodic solution $x_\omega = \{x_\omega(n)\}$ satisfying $\|x_\omega\| < r$.

We now consider a special case of (2.11):

$$\Delta x(n) = \gamma x(n) + f_n(x) + c(n), \qquad \text{for } x \in l_q, \tag{2.13}$$

with $|\gamma| \in (0,1)$, $c = \{c(n)\} \in l^\omega$, and $f_n : l_q \to \mathbb{R}^q$ for $n \in \mathbb{Z}$. The following existence theorem of ω-periodic solutions will be crucial in our investigation of model equation (1.2).

THEOREM 2.5. *If there exists $r \in (0,1)$ such that*

(i) $f = \{f_n\} : \Omega_r \to l^\omega$ *is a continuous mapping and satisfies*

$$|f_n(x)| \le |\gamma| r^2, \qquad \text{for } x \in \overline{\Omega}_r \text{ and } n \in \mathbb{Z},$$

(ii) $\|c\| < |\gamma| r(1-r)$,

then (2.13) has an ω-periodic solution $x_\omega = \{x_\omega(n)\}$ satisfying $\|x_\omega\| < r$.

PROOF. Let $G_n(x) = \gamma x(n) + f_n(x) + c(n)$ for $x = \{x(n)\} \in l_q$, and $G(x) = \{G_n(x)\}$ be defined by (2.1). It is clear that $G : \overline{\Omega}_r \to l^\omega$ is a continuous mapping whose image is a bounded set of l^ω. By Theorem 2.4, we only need to prove that there is no point $x \in \partial\Omega_r$ such that (2.12) holds.

Assume, on the contrary, there exists $x = \{x(n)\} \in \partial\Omega_r$ such that (2.12) holds, that is,

$$x(n+1) = (1+\gamma)x(n) + \left[\frac{1}{1+\lambda_0} f_n(x) - \frac{\lambda_0}{1+\lambda_0} f_n(-x)\right] + \frac{1-\lambda_0}{1+\lambda_0} c(n), \tag{2.14}$$

for some $\lambda_0 \in [0,1]$. Let $|x(n_0)| = r$ for some $n_0 \ge 0$. We will complete the proof in two cases.

CASE A. $0 < \gamma < 1$. We have by (2.14) and Condition (i) that

$$|x(n)| \le \frac{1}{1+|\gamma|} |x(n+1)| + \frac{1}{1+|\gamma|}\xi, \tag{2.15}$$

where $\xi = |\gamma| r^2 + \|c\|$. From (2.15), it follows that

$$|x(n_0)| \le \left(\frac{1}{1+|\gamma|}\right)^\omega |x(n_0+\omega)| + \frac{1}{1+|\gamma|}\xi \left[\sum_{j=0}^{\omega-1} \left(\frac{1}{1+|\gamma|}\right)^j\right]. \tag{2.16}$$

Noting that $|x(n_0)| = |x(n_0+\omega)| = r$ and $\xi = |\gamma| r^2 + \|c\|$, we obtain from (2.16) that $\|c\| \ge |\gamma| r(1-r)$, which contradicts Condition (ii).

CASE B. $-1 < \gamma < 0$. We have by (2.14) and Condition (i) that

$$|x(n+1)| \le (1-|\gamma|)\,|x(n)| + \xi,$$

where ξ is defined as in Case A. Hence, we have $|x(n_0+\omega)| \le (1-|\gamma|)^\omega |x(n_0)| + \xi[\sum_{j=0}^{\omega-1} (1-|\gamma|)^j]$, which implies $\|c\| \ge |\gamma| r(1-r)$, a contradiction. This completes the proof.

In Theorem 2.5, if we let $q = 1$ and $f_n(x) = f(x(n))$ for $x = \{x(n)\} \in l_q$, then we obtain the following conclusion.

COROLLARY 2.6. *If $c = \{c(n)\} \in l^\omega$ satisfies $\|c\| < \alpha r(1-r)$ for some $\alpha \in (0,1)$ and $r \in (0,1/2)$, then the following difference equations:*

$$\Delta y(n) = \alpha y(n)\,(1 - y(n)) + c(n)$$

and

$$\Delta z(n) = -\alpha z(n)\,(1 - z(n)) - c(n)$$

have ω-periodic solutions $\overline{y} = \{\overline{y}(n)\}$ and $\overline{z} = \{\overline{z}(n)\}$, respectively, satisfying

$$\|\overline{y}\| < r \qquad \text{and} \qquad \|\overline{z}\| < r.$$

3. EXISTENCE AND ATTRACTIVITY OF PERIODIC SOLUTIONS OF THE POPULATION MODEL WITH HARVEST/STOCK

With the preparation in Section 2, we can now consider the following difference equation:

$$x(n+1) = \mu x(n)\left[1 - \frac{1}{K}x(n)\right] + b(n), \tag{3.1}$$

with $\mu \in (1,2)$ and $b = \{b(n)\} \in l^\omega$. We assume throughout this section that

$$\|b\| < \frac{(\mu-1)^2}{4\mu}K. \tag{h}$$

Define

$$r_0(b) = \frac{1}{2} - \frac{1}{2}\left[1 - \frac{4\mu}{(\mu-1)^2K}\|b\|\right]^{1/2}. \tag{3.2}$$

Clearly, $0 \le r_0(b) < 1/2$ and $\|b\| = ((\mu-1)^2/\mu)Kr_0(1-r_0)$, here and in what follows, for the sake of simplicity, we write r_0 for $r_0(b)$.

THEOREM 3.1. *If condition (h) holds, then (3.1) has two ω-periodic solutions $\overline{x} = \{\overline{x}(n)\}$ and $x^* = \{x^*(n)\}$ satisfying*

$$|\overline{x}(n)| \le \left(1 - \frac{1}{\mu}\right)Kr_0 \tag{3.3}$$

and

$$\left|x^*(n) - \left(1 - \frac{1}{\mu}\right)K\right| \le \left(1 - \frac{1}{\mu}\right)Kr_0, \tag{3.4}$$

respectively, where r_0 is defined by (3.2). x^* is always a positive solution. Moreover, if $b(n) \le 0$ and $S(b) < 0$, then $\overline{x}$ is a positive solution; if $S(b) > 0$, then $\overline{x}$ is not a positive solution.

PROOF. Set $\alpha = \mu - 1$ and $c = \{c(n)\}$ with $c(n) = (\mu/(\mu-1))(1/K)\,b(n)$ for $n \in \mathbb{N}$. We know from (3.2) that

$$\|c\| = \alpha r_0(1-r_0) < \alpha r(1-r), \tag{3.5}$$

for any $r \in (r_0, 1/2)$. Making the change of variables $x(n) \to y(n)$ with $x(n) = ((\mu-1)/\mu)\,Ky(n)$, we can rewrite (3.1) as

$$\Delta y(n) = \alpha y(n)\,[1 - y(n)] + c(n). \tag{3.6}$$

By virtue of (3.5) and Corollary 2.6, (3.6) has an ω-periodic solution $\overline{y}_{(r)} = \{\overline{y}_{(r)}(n)\}$ satisfying $\|\overline{y}_{(r)}\| < r$. We claim that $\overline{y}_{(r)}$ is independent of the choice of $r \in (r_0, 1/2)$. For any $r_0(b) < r_2 < r_1 < 1/2$, set

$$\eta = 1 - \alpha + 2r_1\alpha.$$

It is easy to see that $0 < \eta < 1$ and

$$\left|(1-\alpha) - \alpha\left[\overline{y}_{(r_1)}(n) + \overline{y}_{(r_2)}(n)\right]\right| \le \eta, \qquad \text{for } n \ge 0.$$

This, combined with (3.6), gives us

$$\begin{aligned}\left|\overline{y}_{(r_1)}(n+1) - \overline{y}_{(r_2)}(n+1)\right| &= \left|\left[\overline{y}_{(r_1)(n)} - \overline{y}_{(r_2)}(n)\right]\left\{(1-\alpha) - \alpha\left[\overline{y}_{(r_1)}(n) + \overline{y}_{(r_2)}(n)\right]\right\}\right| \\ &\le \eta\left|\overline{y}_{(r_1)}(n) - \overline{y}_{(r_2)}(n)\right|.\end{aligned}$$

Since both $\overline{y}_{(r_1)}$ and $\overline{y}_{(r_2)}$ are ω-periodic, we have $\overline{y}_{(r_1)} = \overline{y}_{(r_2)}$. This proves the claim. Denote this common ω-periodic solution by $\overline{y} = \{\overline{y}(n)\}$. Then $\|\overline{y}\| < r$ for any $r \in (r_0, 1/2)$. Thus, $\|\overline{y}\| \le r_0$. Consequently, $\overline{x} = ((\mu-1)/\mu)K\overline{y}$ is an ω-periodic solution of (3.1) satisfying (3.3).

If $b(n) \le 0$ and $S(b) < 0$, then $c(n) \le 0$ and $S(c) < 0$. Let $i_0 \in \{0, \dots, \omega - 1\}$ be given so that $\overline{y}(i_0) = \min_{0 \le i \le \omega-1} \overline{y}(i)$. To prove that $\overline{x}$ is positive, we only need to prove that $\overline{y}(i_0) > 0$. If this were false, then $\overline{y}(i_0) \le 0$. From (3.6), we have

$$\begin{aligned}\overline{y}(i_0+1) &= \overline{y}(i_0) + \alpha\overline{y}(i_0)[1-\overline{y}(i_0)] + c(i_0)\\ &\le \overline{y}(i_0) + c(i_0).\end{aligned}$$

Thus, $\overline{y}(i_0+1) \le 0$. Continuing in this fashion, we obtain $\overline{y}(i_0+\omega) \le \overline{y}(i_0) + c(i_0) + c(i_0+1) + \cdots + c(i_0+\omega-1) < \overline{y}(i_0)$, since $S(c) < 0$. This contradicts $\overline{y} \in l^\omega$.

If $S(b) \ge 0$, then $S(c) \ge 0$. Assuming, by way of contradiction, that $x(n) > 0$ (or, equivalently, $y(n) > 0$ for $n \ge 0$), then (3.6) and $|y(n)| \le r_0$ give us

$$\begin{aligned}0 &= \alpha S(\overline{y}) - \alpha\frac{1}{\omega}\sum_{j=0}^{\omega-1}[\overline{y}(j)]^2 + S(c)\\ &\ge \alpha S(\overline{y}) - \alpha\frac{1}{\omega}\sum_{j=0}^{\omega-1} r_0\overline{y}(j)\\ &= \alpha(1-r_0)S(\overline{y}),\end{aligned}$$

which contradicts $S(\overline{y}) > 0$, since $r_0 \in [0, 1/2)$ and $\alpha \in (0,1)$.

To obtain x^*, we introduce the change of variables $z(n) = 1 - (\mu/(\mu-1))(1/K)x(n)$ and transform (3.1) into

$$\Delta z(n) = -\alpha z(n)[1-z(n)] - c(n). \tag{3.7}$$

The same argument used to get $\overline{y}$ shows that (3.7) has an ω-periodic solution $z^* = \{z^*(n)\}$ satisfying $\|z^*\| \le r_0$. Set $x^*(n) = (1-z^*(n))(1-1/\mu)K$ for $n \ge 0$ and $x^* = \{x^*(n)\}$. Then x^* is an ω-periodic solution of (3.1) satisfying (3.4). Inequality (3.4) and $r_0(b) \in [0, 1/2)$ guarantee that x^* is always positive. This completes the proof.

We now consider the stability of the above periodic solutions. For any $n_0 \ge 0$ and $a \in \mathbb{R}$, let $x(n_0, a) = \{x(n; n_0, a)\}$ denote the solution of

$$\begin{aligned}x(n+1) &= \mu x(n)\left[1 - \frac{1}{K}x(n)\right] + b(n), \qquad n \ge n_0,\\ x(n_0) &= a.\end{aligned}$$

Define a mapping $\theta_0 : (0,1) \to (0,1)$ by

$$\theta_0(\xi) = \max\{|3-2\mu|, 1-(1-\xi)(1-2r_0)(\mu-1)\}, \qquad \text{for } \xi \in (0,1). \tag{3.8}$$

Then we have the following result on the stability of x^*.

THEOREM 3.2. *If (h) holds, then x^* is exponentially asymptotically stable. Indeed, for any $a \in \mathbb{R}$, $\xi \in (0,1)$ and $n_0 \ge 0$, if $|a - x^*(n_0)| \le \xi(1-2r_0)(1-1/\mu)K$ then*

$$|x(n_0+j; n_0, a) - x^*(n_0+j)| \le [\theta_0(\xi)]^j |a - x^*(n_0)|, \qquad \text{for } j \ge 0,$$

where $\theta_0(\xi)$ is defined by (3.8).

PROOF. It is sufficient to show that, for any chosen $n \ge n_0$,

$$|x(n; n_0, a) - x^*(n)| \le \xi(1-2r_0)\left(1-\frac{1}{\mu}\right)K \tag{3.9}$$

implies

$$|x(n+1; n_0, a) - x^*(n+1)| \le \theta_0(\xi)|x(n; n_0, a) - x^*(n)|.$$

We know from (3.1) that

$$|x(n+1;n_0,a)-x^*(n+1)| = \left|\mu - \frac{1}{K}\mu\left[x(n;n_0,a)+x^*(n)\right]\right| |x(n;n_0,a)-x^*(n)|.$$

It is easy to check that

$$\left|\mu - \frac{1}{K}\mu\left[x(n;n_0,a)+x^*(n)\right]\right| \le \theta_0(\xi)$$

by using (3.4), (3.8), and (3.9). This completes the proof.

In what follows, we extend the above (local) stability result to the global attractivity of x^* and we also want to obtain more precise estimates on the convergence rate of other solutions. For these purposes, we need several lemmas to estimate the rate between $|x(n+1;n_0,a)-x^*(n+1)|$ and $|x(n;n_0,a)-x^*(n)|$ for $n \ge n_0$ and $a \in \mathbb{R}$. Let

$$Q_0 = \left\{a \in \mathbb{R} : \left|a - \left(1-\frac{1}{\mu}\right)K\right| \le \left(1-\frac{1}{\mu}\right)Kr_0\right\}, \tag{3.10}$$

$$\sigma = \max\{\mu-1, 1-(1-2r_0)(\mu-1)\}. \tag{3.11}$$

Then $\sigma \in (0,1)$. For any $r \in (r_0, 1-r_0)$, let

$$\tau(r) = \max\{\sigma, 1-(\mu-1)(r-r_0)\}, \tag{3.12}$$

$$Q_r = \left\{a \in \mathbb{R} : \left(1-\frac{1}{\mu}\right)Kr \le a \le \left(1-\frac{1}{\mu}\right)K(1+r_0)\right\}, \tag{3.13}$$

$$B_r^0 = \left\{a \in \mathbb{R} : \left(1-\frac{1}{\mu}\right)Kr \le a \le K - \left(1-\frac{1}{\mu}\right)Kr\right\}. \tag{3.14}$$

It is easy to see that $Q_0 \subseteq Q_r \subseteq B_r^0$. Define the set A_0 by

$$A_0 = \{a \in \mathbb{R} : \overline{x}(0) < a < K - \overline{x}(0)\}, \tag{3.15}$$

where $\overline{x} = \{\overline{x}(n)\}$ is the ω-periodic solution of (3.1) obtained in Theorem 3.1.

LEMMA 3.3. *For any $n_0 \ge 0$ and $a \in Q_0$, we have $x(n;n_0,a) \in Q_0$ for $n \ge n_0$ and*

$$|x(n_0+j;n_0,a) - x^*(n_0+j)| \le \sigma^j |a - x^*(n_0)|, \qquad \text{for } j \ge 0, \tag{3.16}$$

where Q_0 and σ are defined by (3.10) and (3.11), respectively.

PROOF. First, we want to prove that for any $n_0 \ge 0$ if $a \in Q_0$, then $x(n;n_0,a) \in Q_0$ for any $n \ge n_0$. Equivalently, we want to prove that $|z(n_0+j)| \le r_0$ for any $j \ge 1$ and $n_0 \ge 0$ provided that $|z(n_0)| \le r_0$. Obviously, it is enough to show that for any $n \ge n_0$ if $|z(n)| \le r_0$, then $|z(n+1)| \le r_0$. From (3.9), we have

$$z(n+1) = z(n) - \alpha z(n)[1-z(n)] - c(n). \tag{3.17}$$

We estimate $|z(n+1)|$ in two cases.

CASE A. $z(n) = r \in [0, r_0]$. Then (3.5) and (3.17) give

$$r - \alpha r(1-r) - \alpha r_0(1-r_0) \le z(n+1) \le r - \alpha r(1-r) + \alpha r_0(1-r_0).$$

It is clear that $z(n+1) \le r_0$ provided that

$$[r_0 - \alpha r_0(1-r_0)] - [r - \alpha r(1-r)] \ge 0.$$

Set $\varphi(t) = t - \alpha t(1-t)$ for $t \in (0,1)$. Then $\varphi'(t) = 1 - \alpha + 2\alpha t > 0$ implies that $\varphi(r_0) - \varphi(r) \geq 0$. Thus, $z(n+1) \leq r_0$. On the other hand, $z(n+1) \geq r - \alpha r(1-r) - \alpha r_0(1-r_0) \geq (1-\alpha)r - r_0 \geq -r_0$. Therefore, $|z(n+1)| \leq r_0$.

CASE B. $z(n) = -r \in [-r_0, 0]$. We obtain from (3.5) and (3.17) that

$$\begin{aligned} z(n+1) &\leq \alpha r(1+r) + \alpha r_0 (1-r_0) - r \\ &= \alpha r + \alpha r^2 + \alpha r_0 - \alpha r_0^2 - r \\ &= \alpha r - r + \alpha r^2 - \alpha r_0^2 + \alpha r_0 - r_0 + r_0 \\ &\leq r_0 \end{aligned}$$

and that

$$\begin{aligned} z(n+1) &\geq \alpha r(1+r) - \alpha r_0 (1-r_0) - r \\ &\geq \alpha r(1-r) - \alpha r_0 (1-r_0) - r \\ &= \varphi(r_0) - \varphi(r) - r_0 \\ &\geq -r_0. \end{aligned}$$

Thus, $|z(n+1)| \leq r_0$.

In the second step, we want to show that (3.16) holds. In view of the fact that $a \in Q_0$ implies $x(n; n_0, a) \in Q_0$ for any $n \geq n_0$, we only need to prove that for any $n \geq n_0$ if $x(n; n_0, a) \in Q_0$, then

$$|x(n+1; n_0, a) - x^*(n+1)| \leq \sigma |x(n; n_0, a) - x^*(n)|.$$

Noting that (3.1) gives

$$x(n+1; n_0, a) - x^*(n+1) = \mu \left[1 - \frac{1}{K}(x(n; n_0, a) + x^*(n))\right][x(n; n_0, a) - x^*(n)],$$

we only need to show

$$\left|1 - \frac{1}{K}[x(n; n_0, a) + x^*(n)]\right| \leq \frac{\sigma}{\mu},$$

which follows from (3.4) and (3.11) easily. This completes the proof.

LEMMA 3.4. *For any $n_0 \geq 0$ and $a \in Q_r$, we have $x(n; n_0, a) \in Q_r$ for $n \geq n_0$ and*

$$|x(n_0+j; n_0, a) - x^*(n_0+j)| \leq [\tau(r)]^j |a - x^*(n_0)|, \qquad \text{for } j \geq 0, \tag{3.18}$$

where $\tau(r) \in (0,1)$ and Q_r are defined by (3.12) and (3.13), respectively.

PROOF. If $a \in Q_0$, then Lemma 3.3 and (3.12) imply (3.18). So, without loss of generality, we may assume that $(1 - 1/\mu)Kr \leq a < (1 - 1/\mu)K(1 - r_0)$.

Note that if $x(\hat{n}; n_0, a) \in Q_0$ for some $\hat{n} \geq n_0$, then Lemma 3.1 and (3.11),(3.12) imply that $x(n; n_0, a) \in Q_0$ for $n \geq \hat{n}$, and hence, (3.18) holds for $j \geq \hat{n} - n_0$. Thus, to complete the proof we only need to show that if

$$\left(1 - \frac{1}{\mu}\right) Kr \leq x(n; n_0, a) < \left(1 - \frac{1}{\mu}\right) K(1 - r_0), \tag{3.19}$$

then

$$0 < x^*(n+1) - x(n+1; n_0, a) \leq \tau(r)[x^*(n) - x(n; n_0, a)] \tag{3.20}$$

and

$$x(n; n_0, a) < x(n+1; n_0, a) < \left(1 - \frac{1}{\mu}\right) K(1 + r_0). \tag{3.21}$$

By (3.1),

$$x^*(n+1) - x(n+1; n_0, a) = \mu \left[1 - \frac{1}{K}(x^*(n) + x(n; n_0, a))\right] [x^*(n) - x(n; n_0, a)] .$$

We observe that $x^*(n) \geq (1 - 1/\mu)K(1 - r_0) > x(n; n_0, a)$ from (3.4) and (3.19). To show (3.20), we only need to show

$$0 < \mu - \frac{\mu}{K}[x^*(n) + x(n; n_0, a)] \leq \tau(r). \tag{3.22}$$

In fact, (3.4) and (3.19) give

$$\left(1 - \frac{1}{\mu}\right)(1 + r - r_0) K \leq x^*(n) + x(n; n_0, a) < 2\left(1 - \frac{1}{\mu}\right) K,$$

from which (3.22) follows.

Using the change of variables $x(n; n_0, a) = ((\mu - 1)/\mu)Ky(n)$, we can rewrite (3.21) as

$$y(n) < y(n+1) < 1 + r_0,$$

where $y(n)$ satisfies $r \leq y(n) < 1 - r_0$ and (3.6). We know from $r_0 < r \leq y(n) < 1 - r_0$ that $y(n)[1 - y(n)] > r_0(1 - r_0)$. This, together with (3.5) and (3.6), yields $\Delta y(n) > \alpha r_0(1 - r_0) - \alpha r_0(1 - r_0) = 0$. Thus, $y(n+1) > y(n)$. On the other hand, we obtain from (3.20) that $x(n+1; n_0, a) < x^*(n+1) \leq (1 - 1/\mu)K(1 + r_0)$. Thus, (3.21) holds. This completes the proof.

If $r_0 < (1/2)(1/(\mu - 1) - 1)$, we define the set Q^0 by

$$Q^0 = \left\{a \in \mathbb{R} : \left(1 - \frac{1}{\mu}\right) K(1 + r_0) < a < K - \left(1 - \frac{1}{\mu}\right) K(1 + r_0)\right\}. \tag{3.23}$$

LEMMA 3.5. *If $r_0 < (1/2)(1/(\mu - 1) - 1)$, $n_0 \geq 0$ and $a \in Q^0$, then*

$$|x(n_0 + j; n_0, a) - x^*(n_0 + j)| \leq \sigma^j |a - x^*(n_0)|, \qquad \text{for } j \geq 0, \tag{3.24}$$

where σ is defined by (3.11).

PROOF. First, if $x(\hat{n}; n_0, a) \in Q_0$ for some $\hat{n} > n_0$, then it follows from Lemma 3.3 that

$$|x(\hat{n} + j; n_0, a) - x^*(\hat{n} + j)| \leq \sigma^j |x(\hat{n}; n_0, a) - x^*(\hat{n})|, \qquad \text{for } j \geq 0. \tag{3.25}$$

In view of (3.11), $\sigma \geq 2 - \mu + 2r_0(\mu - 1) \geq 2 - \mu$. Thus, the lemma is proved if we can show that $r_0 < (1/2)(1/(\mu - 1) - 1)$ and $x(n; n_0, a) \in Q^0$ for $n \geq n_0$ imply

$$0 < x(n+1; n_0, a) - x^*(n+1) \leq (2 - \mu)[x(n; n_0, a) - x^*(n)] \tag{3.26}$$

and

$$x(n+1; n_0, a) < x(n; n_0, a) . \tag{3.27}$$

Indeed, if (3.26) and (3.27) hold, then by (3.4) and setting $n = n_0$, we get

$$x(n_0 + 1; n_0, a) < x(n_0; n_0, a) = a < K - \left(1 - \frac{1}{\mu}\right) K(1 + r_0) \tag{3.28}$$

and

$$x(n_0 + 1; n_0, a) \geq x^*(n_0 + 1) \geq \left(1 - \frac{1}{\mu}\right) K(1 - r_0) .$$

Consequently, if $x(n_0+1;n_0,a) \le (1-1/\mu)K(1+r_0)$, then (3.25) implies (3.24); and if $x(n_0+1;n_0,a) > (1-1/\mu)K(1+r_0)$, then (3.28) implies $x(n_0+1;n_0,a) \in Q^0$. In the latter case, we can continue the same procedure for $n = n_0+1, n_0+2, \dots$ to yield (3.24).

It remains to verify the relations (3.26),(3.27). Using the change of variables $y(n) = (\mu/(\mu-1))\,(1/K)\,x(n;n_0,a)$ for $n \ge n_0$, we rewrite (3.27) as

$$y(n+1) < y(n), \qquad \text{if } y(n) > 1+r_0, \tag{3.29}$$

where $y(n)$ satisfies $\Delta y(n) = \alpha y(n)[1-y(n)] + c(n)$. From (3.5) and $y(n) > 1+r_0$, it is easy to see that $\Delta y(n) < \alpha(1+r_0)(-r_0) + \alpha r_0(1-r_0) \le 0$, which implies (3.29). On the other hand, combining (3.4) with (3.23), one obtains

$$2\left(1-\frac{1}{\mu}\right)K < x\left(n;n_0,a\right) + x^*(n) < K.$$

Accordingly,

$$0 < \mu - \frac{\mu}{K}\left[x\left(n;n_0,a\right) + x^*(n)\right] < 2-\mu.$$

Now, (3.26) can be verified directly from $x(n;n_0,a) > (1-1/\mu)K(1+r_0) \ge x^*(n)$ and

$$x\left(n+1;n_0,a\right) - x^*(n+1) = \mu\left\{1 - \frac{1}{K}\left[x\left(n;n_0,a\right) + x^*(n)\right]\right\}\left[x\left(n;n_0,a\right) - x^*(n)\right].$$

This completes the proof.

LEMMA 3.6. *For any $n_0 \ge 0$ and $a \in \mathbb{R}$,*

$$x\left(n_0+j;n_0,a\right) = x\left(n_0+j;n_0,K-a\right), \qquad \text{for } j \ge 1.$$

PROOF. It is sufficient to show $x(n_0+1;n_0,a) = x(n_0+1;n_0,K-a)$, which follows directly from (3.1).

For $r \in (r_0, 1-r_0)$, define

$$\hat{Q}_r = \left\{a \in \mathbb{R} : K - \left(1-\frac{1}{\mu}\right)K\left(1+r_0\right) \le a \le K - \left(1-\frac{1}{\mu}\right)Kr\right\}. \tag{3.30}$$

LEMMA 3.7. *For any $n_0 \ge 0$ and $a \in \hat{Q}_r$,*

$$\left|x\left(n_0+j;n_0,a\right) - x^*\left(n_0+j\right)\right| \le [\tau(r)]^j\left|a - x^*\left(n_0\right)\right|, \qquad \text{for } j \ge 0. \tag{3.31}$$

PROOF. First, from (3.1), (3.4), and (3.30), we have

$$\left|x\left(n_0+1;n_0,a\right) - x^*\left(n_0+1\right)\right| \le (\mu-1)\left|a - x^*\left(n_0\right)\right|. \tag{3.32}$$

Then using Lemmas 3.4 and 3.6 and (3.32), we get

$$\begin{aligned}
\left|x\left(n_0+j;n_0,a\right) - x^*\left(n_0+j\right)\right| &= \left|x\left(n_0+j;n_0,K-a\right) - x^*\left(n_0+j\right)\right| \\
&\le [\tau(r)]^{j-1}\left|x\left(n_0+1;n_0,K-a\right) - x^*\left(n_0+1\right)\right| \\
&= [\tau(r)]^{j-1}\left|x\left(n_0+1;n_0,a\right) - x^*\left(n_0+1\right)\right| \\
&\le [\tau(r)]^{j-1}(\mu-1)\left|a - x^*\left(n_0\right)\right|,
\end{aligned}$$

for $j \ge 1$. From (3.11) and (3.12), we have $(\mu-1) \le \tau(r)$. Therefore, (3.31) holds. This completes the proof.

Summarizing the above results, we obtain the following conclusion.

THEOREM 3.8. *If (h) holds, then for any $n_0 \geq 0$ and $a \in B_r^0$ with $r \in (r_0, 1-r_0)$, we have $x(n; n_0, a) \in B_r^0$ for $n \geq n_0$ and*

$$|x(n; n_0, a) - x^*(n)| \leq [\tau(r)]^{n-n_0} |a - x^*(n_0)|, \qquad \text{for } n \geq n_0,$$

where B_r^0 and $\tau(r) \in (0,1)$ are defined by (3.14) and (3.12), respectively.

We need the following technical result for our discussion about the instability of $\overline{x}$.

LEMMA 3.9. *For any $n_0 \geq 0$ and $a < -(1-1/\mu)Kr_0$, we have $x(n; n_0, a) \to -\infty$ as $n \to \infty$.*

PROOF. Let $\hat{y}(n) = -(\mu/(\mu-1))(1/K)x(n; n_0, a)$ for $n \geq n_0$. Then $\hat{y}(n)$ satisfies

$$\Delta\hat{y}(n) = \alpha\hat{y}(n)[1+\hat{y}(n)] - c(n). \tag{3.33}$$

So we only need to prove that $\hat{y}(n_0) > r_0$ implies $\hat{y}(n) \to \infty$ as $n \to \infty$. Substituting (3.5) into (3.33), we obtain

$$\begin{aligned}\hat{y}(n_0+1) - \hat{y}(n_0) &\geq \alpha\hat{y}(n_0)[1+\hat{y}(n_0)] - \alpha r_0(1-r_0) \\ &\geq \alpha r_0^2.\end{aligned}$$

Thus, $\hat{y}(n_0+1) > \hat{y}(n_0) > r_0$. Repeating the above argument leads to

$$\hat{y}(n_0+j) - \hat{y}(n_0) \geq j\alpha r_0^2,$$

which implies that $\hat{y}(n) \to \infty$ as $n \to \infty$. This completes the proof.

We are now ready to state our main results regarding the stability of the periodic solutions $\overline{x}$ and x^*.

THEOREM 3.10. *Assume condition (h) holds, $n_0 \geq 0$ and $a \in \mathbb{R}$ such that $|a| \leq (1-1/\mu)Kr_0$. Then we have*

(i) *$|x(n; n_0, a) - x^*(n)| \to 0$ as $n \to \infty$ if $a > \overline{x}(n_0)$;*
(ii) *$x(n; n_0, a) \to -\infty$ as $n \to \infty$ if $a < \overline{x}(n_0)$ (this implies that $\overline{x} = \{\overline{x}(n)\}$ is unstable).*

PROOF. We can derive easily from (3.3) that for any $n \geq n_0$ if $|x(n; n_0, a)| \leq (1-1/\mu)Kr_0$, then

$$\mu\left\{1 - \frac{1}{K}[x(n; n_0, a) + \overline{x}(n)]\right\} \geq 1 + \eta_0, \tag{3.34}$$

where $\eta_0 = (\mu-1)(1-2r_0)$.

(i) $a > \overline{x}(n_0)$. In this case, we claim that we can find a $j_0 \geq 1$ such that

$$\left(1 - \frac{1}{\mu}\right)Kr_0 < x(n_0 + j_0; n_0, a) < \left(1 - \frac{1}{\mu}\right)K.$$

Indeed, from (3.1), we have

$$x(n+1; n_0, a) - \overline{x}(n+1) = \frac{1}{\mu}\left\{1 - \frac{1}{K}[x(n; n_0, a) + \overline{x}(n)]\right\}[x(n; n_0, a) - \overline{x}(n)], \tag{3.35}$$

for $n \geq n_0$. Letting $n = n_0$ in (3.35) and using (3.34) and the fact that $a > \overline{x}(n_0)$, we obtain

$$x(n_0+1; n_0, a) - \overline{x}(n_0+1) \geq (1+\eta_0)[a - \overline{x}(n_0)],$$

if $|x(n; n_0, a)| \leq (1-1/\mu)Kr_0$. Thus, $x(n_0+1; n_0, a) > \overline{x}(n_0+1)$. Continuing in this fashion, we can find a $j_0 \geq 1$ such that $|x(n_0+j; n_0, a)| \leq (1-1/\mu)Kr_0$, for $j = 0, 1, \ldots, j_0 - 1$ and $x(n_0+j_0; n_0, a) > (1-1/\mu)Kr_0$. To prove that $x(n_0+j; n_0, a) <$

$(1 - 1/\mu)K$, we introduce the change of variables $x(n; n_0, a) = ((\mu - 1)/\mu) K y(n)$ for $n \geq n_0$. Then $y(n_0 + j_0)$ satisfies

$$y(n_0 + j_0) = y(n_0 + j_0 - 1) + (\mu - 1) y(n_0 + j_0 - 1)[1 - y(n_0 + j_0 - 1)] + c(n_0 + j_0 - 1)$$

and $|y(n_0 + j_0 - 1)| \leq r_0$. This and (3.5) give

$$\begin{aligned} y(n_0 + j_0) &\leq r_0 + 2(\mu - 1) r_0 (1 - r_0) \\ &< r_0 + 1 - r_0 \\ &= 1. \end{aligned}$$

Consequently, $x(n_0 + j_0; n_0, a) < (1 - 1/\mu)K$. This proves the claim.

Using the result of the claim, we can find an $r \in (r_0, 1 - r_0)$ such that $x(n_0 + j_0; n_0, a) \in Q_r$. Then (i) follows from Theorem 3.8 and $Q_r \subseteq B_r^0$.

(ii) $a < \overline{x}(n_0)$. The proof is similar to that of (i) above. Namely, we first find a $j_0 \geq 1$ such that $x(n_0 + j_0; n_0, a) < -(1 - 1/\mu)K r_0$, and then derive (ii) from Lemma 3.9. This completes the proof.

The following two lemmas, which can be proved by using Lemmas 3.6, 3.9, and Theorem 3.10, are important in our description of the basin of attraction of the periodic solution x^*.

LEMMA 3.11. *Let $n_0 \geq 0$ and assume $K - (1 - 1/\mu)K r_0 \leq a \leq K + (1 - 1/\mu)K r_0$. Then*

(i) $|x(n; n_0, a) - x^*(n)| \to 0$ *as* $n \to \infty$ *if* $K - a > \overline{x}(n_0)$;
(ii) $x(n; n_0, a) \to -\infty$ *as* $n \to \infty$ *if* $K - a < \overline{x}(n_0)$.

LEMMA 3.12. *For any $n_0 \geq 0$ and $a > K + (1 - 1/\mu)K r_0$, we have $x(n; n_0, a) \to -\infty$ as $n \to \infty$.*

Define a set D_0 by

$$D_0 = \left\{ a \in \mathbb{R} : \left(1 - \frac{1}{\mu}\right) K r_0 < a < K - \left(1 - \frac{1}{\mu}\right) K r_0 \right\}. \tag{3.36}$$

THEOREM 3.13. *Let (h) hold. We have*

(i) $|x(n; 0, a) - x^*(n)| \to 0$ *as* $n \to \infty$ *if and only if* $a \in A_0$, *where* A_0 *is defined by (3.15).*
(ii) $x(n; 0, a)$ *is a positive solution if* $a \in D_0$. *Moreover, if* $a \in B_r^0$ *for some* $r \in (r_0, 1 - r_0)$, *then*

$$|x(n; 0, a) - x^*(n)| \leq [\tau(r)]^n |a - x^*(0)|, \qquad \text{for } n \geq 0,$$

where D_0 *and* $\tau(r)$ *are defined by (3.36) and (3.12), respectively.*
(iii) *If* $x(n; 0, a)$ *is a positive solution of (3.1) and* $x(1; 0, a) \neq \overline{x}(1)$, *then* $a \in A_0$.

PROOF.

(i) If $|x(n; 0, a) - x^*(n)| \to 0$ as $n \to \infty$, then $\{x(n; 0, a)\}$ is bounded. Combining Lemma 3.9 with Theorem 3.10, we get $\overline{x}(0) \leq a \leq K - \overline{x}(0)$. If $a = \overline{x}(0)$ or $a = K - \overline{x}(0)$, then Lemma 3.6 implies that $x(n; 0, a) = \overline{x}(n)$ for $n \in \mathbb{N}$. This contradicts $|x(n; 0, a) - x^*(n)| \to 0$ as $n \to \infty$. Thus, we have $\overline{x}(0) < a < K - \overline{x}(0)$, i.e., $a \in A_0$. On the other hand, if $a \in A_0$, applying Lemma 3.11(i) and Theorems 3.10 and 3.8 with $n_0 = 0$ yield $|x(n; 0, a) - x^*(n)| \to 0$ as $n \to \infty$.
(ii) We first note that $a \in D_0$ implies that there exists $r \in (r_0, 1 - r_0)$ such that $a \in B_r^0$. Then (ii) follows immediately from Theorem 3.8.

The proof of (iii) is similar to that of (i), and hence, is omitted.

We conclude with a brief biological interpretation of our results for the case where (3.1) denotes a single species population model with harvest, i.e., $b(n) \leq 0$ and $b(n) \not\equiv 0$. In this case, $\overline{x} = \{\overline{x}(n)\}$ can be regarded as a critical periodic solution. If $a < \overline{x}(0)$ or $a > K - \overline{x}(0)$, then we know from Lemmas 3.6 and 3.9 and Theorem 3.10 that the species is on the way to extinction. If, however, $\overline{x}(0) \leq a \leq K - \overline{x}(0)$, then Theorem 3.13 shows that the species is persistent.

REFERENCES

1. R.P. Agarwal, *Difference Equations and Inequalities*, Marcel Dekker, New York, (1992).
2. W.G. Kelley and A.C. Peterson, *Difference Equations: An Introduction with Applications*, Academic Press, New York, (1991).
3. E.P. Odum, *Fundamental Ecology*, 3rd edition, W.B. Saunders, (1971).
4. H.I. Freedman and H. Xia, Periodic solutions of single species models with delay, differential equations, dynamical systems and control science, *Lecture Notes in Pure and Appl. Math.* **152**, 55–74, (1994).
5. K. Gopalsamy, M.R.S. Kulenovic and G. Ladas, Environmental periodicity and time delays in a "food-limited" population model, *J. Math. Anal. Appl.* **147**, 545–555, (1990).
6. Y. Kuang, *Delay Differential Equations with Applications in Population Dynamics*, Academic Press, Boston, MA, (1993).
7. W. Wang, P. Fergola and C. Tenneriello, Global attractivity of periodic solutions of population models, *J. Math. Anal. Appl.* **211**, 498–511, (1997).
8. R.P. Agarwal and P.Y.H. Pang, On a generalized difference system, *Nonlinear Anal.* **30**, 365–376, (1997).
9. I. Katsunori, Asymptotic analysis for linear difference equations, *Trans. Amer. Math. Soc.* **349**, 4107–4142, (1997).
10. Z.Y. Zhang, An algebraic principle for the stability of difference operators, *J. Diff. Eqns.* **136**, 236–247, (1997).
11. M. Ryszard and P. Jerzy, On periodic solutions of a first order difference equation, *An. Stiint. Univ. "Al.I. Cuza" Iasi Sect. I a Mat. (N.S.)* **34** (2), 125–133, (1998).
12. W. Krawcewicz and J. Wu, *Theory of Degrees with Applications to Bifurcations and Differential Equations*, CMS Series of Monographs, Wiley, New York, (1997).