Quiver representations with an irreducible null cone

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ABSTRACT
Let \( \mathbf{d} \) be a prehomogeneous dimension vector for a quiver \( \mathbf{Q} \). There is an action of the product \( \text{Gl}(\mathbf{d}) \) of linear groups on the vector space \( \text{rep}(\mathbf{Q}, \mathbf{d}) \) of representations of \( \mathbf{Q} \) with dimension vector \( \mathbf{d} \), and there is a representation \( \mathbf{T} \) with a dense \( \text{Gl}(\mathbf{d}) \)-orbit in \( \text{rep}(\mathbf{Q}, \mathbf{d}) \). We give a construction for a dense subset \( \mathcal{F}_{\mathbf{Q}, \mathbf{d}} \) of the variety \( \mathcal{Z}_{\mathbf{Q}, \mathbf{d}} \) of common zeros of all semi-invariants in \( k[\text{rep}(\mathbf{Q}, \mathbf{d})] \) of positive degree, and we show that this set is stable for big dimension vectors, i.e. \( \mathcal{F}_{\mathbf{Q}, N\mathbf{d}} = \{ X \oplus T^{N-1}: X \in \mathcal{F}_{\mathbf{Q}, \mathbf{d}} \} \). Moreover, we show that the existence of a dense orbit in \( \mathcal{Z}_{\mathbf{Q}, \mathbf{d}} \) depends on a quiver \( \mathbf{Q}^\perp \) such that the category of representations of \( \mathbf{Q}^\perp \) is equivalent to the right perpendicular category \( \mathbf{T}^\perp \).

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1. Introduction

We fix an algebraically closed field \( k \) and a quiver \( \mathbf{Q} \) with a finite set of vertices \( \mathbf{Q}_0 = \{ 1, \ldots, n \} \) and a finite set \( \mathbf{Q}_1 \) of arrows \( \alpha : t_\alpha \to h_\alpha \), where \( t_\alpha \) and \( h_\alpha \) denote the tail and the head of \( \alpha \), respectively. A representation \( X \) of \( \mathbf{Q} \) consists of a family \( \{ X(i): i \in \mathbf{Q}_0 \} \) of finite dimensional \( k \)-vector spaces and a family \( \{ X(\alpha) : X(t_\alpha) \to X(h_\alpha) : \alpha \in \mathbf{Q}_1 \} \) of \( k \)-linear maps. A morphism \( f : X \to Y \) between two representations of \( \mathbf{Q} \) is a family \( \{ f(i) : X(i) \to Y(i) : i \in \mathbf{Q}_0 \} \) of \( k \)-linear maps satisfying \( f(h_\alpha) \circ X(\alpha) = Y(\alpha) \circ f(t_\alpha) \) for all \( \alpha \in \mathbf{Q}_1 \). The vector space of morphisms from \( X \) to \( Y \) will be denoted by \( \text{Hom}(X, Y) \).

For a fixed dimension vector \( \mathbf{d} \in \mathbb{N}^{\mathbf{Q}_0} \), we consider the vector space

\[
\text{rep}(\mathbf{Q}, \mathbf{d}) = \prod_{\alpha \in \mathbf{Q}_1} \text{Mat}(d_{h_\alpha} \times d_{t_\alpha}, k)
\]
of representations of $Q$ with $X(i) = k^{d_i}$, $i \in Q_0$, and the product of general linear groups $\text{Gl}(\mathbf{d}) = \prod_{i=1}^n \text{Gl}(d_i, k)$. The algebraic group $\text{Gl}(\mathbf{d})$ acts on $\text{rep}(Q, \mathbf{d})$ by conjugation

$$((g_1, \ldots, g_n) \ast X)(\alpha) = g_{h_\alpha} \circ X(\alpha) \circ g_{\alpha}^{-1},$$

and we get an induced action of $\text{Gl}(\mathbf{d})$ on the coordinate algebra $k[\text{rep}(Q, \mathbf{d})]$, given by $(g \cdot f)(X) := f(g^{-1} \ast X)$, for $g \in \text{Gl}(\mathbf{d})$, $f \in k[\text{rep}(Q, \mathbf{d})]$ and $X \in \text{rep}(Q, \mathbf{d})$. The algebra $k[\text{rep}(Q, \mathbf{d})]^{\text{gl}(\mathbf{d})}$ of invariant polynomials under this action has been determined in [13] for fields of characteristic 0 and in [9] for fields of arbitrary characteristic, and it is trivial if $Q$ does not contain oriented cycles.

We consider the algebra $\text{Sl}(Q, \mathbf{d})$ of semi-invariant polynomial functions on $\text{rep}(Q, \mathbf{d})$, which is the algebra $k[\text{rep}(Q, \mathbf{d})]^{\text{sl}(\mathbf{d})}$ of polynomial functions invariant under the action of the group $\text{Sl}(\mathbf{d}) = \prod_{i=1}^n \text{Sl}(d_i, k)$, product of special linear groups. This algebra has been much studied in the last three decades (see [2] for a recent review), however the null cone $Z_{Q, \mathbf{d}}$ (see [5,8] for a definition), which is the set of common zeros of all polynomials in $\text{Sl}(Q, \mathbf{d})$ without constant term, has first been studied (in 2004) by Chang and Weyman [6], in the special case where the quiver $Q$ is of Dynkin type $A_n$.

Recall that a dimension vector $\mathbf{d}$ is called prehomogeneous if there exists a representation $T$ with dimension vector $\mathbf{d}$, such that its $\text{Gl}(\mathbf{d})$-orbit is open (for the Zariski topology) in $\text{rep}(Q, \mathbf{d})$. Such a representation $T$ is unique up to isomorphism since there is at most one open orbit in $\text{rep}(Q, \mathbf{d})$, and is characterized by $\text{Ext}(T, T) = 0$ [18]. Let $T = \bigoplus_{i=1}^r T_i^{\lambda_i}$ be a decomposition of $T$ where the direct summands $T_i$ are indecomposable and pairwise non-isomorphic. Riedtmann and Zwara proved in [16] the existence of an integer $N$ such that the variety $Z_{Q, \mathbf{d}}$ is irreducible, provided $\lambda_i \geq N$ for all $i = 1, \ldots, r$.

The aim of this paper is to give some conditions about the existence of a $\text{Gl}(\mathbf{d})$-orbit which is dense in $Z_{Q, \mathbf{d}}$, and to give a construction, as a direct sum of indecomposables, for a representation $X$ lying in this orbit. We will only consider dimension vectors $\mathbf{d}$ which are prehomogeneous, since in the non-prehomogeneous case, we cannot guarantee that the variety $Z_{Q, N, \mathbf{d}}$ is irreducible, for $N$ big.

In [19], Schofield gives a construction for semi-invariants, and as a consequence, the variety $Z_{Q, \mathbf{d}}$ can be characterized as follows: let $T^\perp$ be the full subcategory of $\text{rep}(Q)$, called the right perpendicular category, whose objects $A$ satisfy

$$\text{Hom}(T, A) = 0 = \text{Ext}(T, A).$$

The support of a representation $X$ is the full subquiver of $Q$ whose vertices are $\{i \in Q_0 : X(i) \neq 0\}$. We assume that the support of $T$ is all of $Q$. Therefore, the category $T^\perp$ is equivalent to the category $\text{rep}(Q^\perp)$ of representations of a quiver $Q^\perp$ having $n - r$ vertices and not containing oriented cycles [19]. Let $S_{r+1}, \ldots, S_n$ be the simple objects of $T^\perp$. Then we have

$$Z_{Q, \mathbf{d}} = \{X \in \text{rep}(Q, \mathbf{d}) : \text{Hom}(X, S_j^\prime) \neq 0, \ for \ j = r + 1, \ldots, n\}. $$

We will see that the existence of a dense orbit in $Z_{Q, \mathbf{d}}$ depends on this quiver $Q^\perp$. Recall that a (non-oriented) tree is a connected quiver whose underlying graph does not contain any cycle. We say that a tree is admissible if it has no subquiver of type $\mathbb{D}_m$, $m \geq 4$, with the following orientation
If $Z_{Q,d} = \overline{\text{Gl}(d) \ast X}$ is the closure of the $\text{Gl}(d)$-orbit of $X$, we say that the representation $X$ is generic in $Z_{Q,d}$. Moreover, if $X \oplus \bigoplus_{i=1}^{r} T_i$ is generic in $Z_{Q,d'}$ for any non-negative integers $m_1, \ldots, m_r$, where $d' = d + \sum_{i=1}^{r} m_i \dim T_i$, we say that the representation $X$ is stable. The orbit of a stable representation is called a stable orbit.

**Theorem 1.1.** Let $Q$ be a finite quiver and let $T_1, \ldots, T_r$ be pairwise non-isomorphic indecomposable representations of $Q$ such that the support of $T_1 \oplus \cdots \oplus T_r$ is all of $Q$ and $\text{Ext}(T_i, T_j) = 0$ for any $i, j \in \{1, \ldots, r\}$. Assume that the quiver $Q^\perp$ is a disjoint union of admissible trees. Then there are integers $v_1, \ldots, v_r$ such that if $\lambda_i \geq v_i$ for all $i = 1, \ldots, r$ and $d = \sum_{i=1}^{r} \lambda_i \dim T_i$, the null cone $Z_{Q,d}$ is the closure of one stable $\text{Gl}(d)$-orbit.

It will become clear that a generic representation $X$ in $Z_{Q,d}$ is stable if and only if $\text{Ext}(T, X) = 0 = \text{Ext}(X, T)$.

**Example 1.2.** If $Q$ is the quiver

```
1 ---- 2
|     |
3 ---- 4
```

and $d = \lambda \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$ with $\lambda \geq 3$, we have $Z_{Q,d} = \overline{\text{Gl}(d) \ast X}$ for the stable representation $X = Z \oplus T_1^{\lambda-3}$, where $Z$ is the direct sum of indecomposables

```
k 0 0 k 0 0 0 k k 0 k k
1 1 1 0 0 1 0 1 1 1
1 1 0 0 0 1 1
```

and where $T_1$ is the following indecomposable representation:
If $\lambda = 2$, the variety $Z_{Q,d}$ has two irreducible components of codimension 3 in $\text{rep}(Q,d)$. If $\lambda = 1$, the variety $Z_{Q,d}$ is irreducible of codimension 2. It is the closure of one $\text{Gl}(d)$-orbit, but this orbit is not stable.

The aim of this paper is also to give a construction for this stable representation $X$. Of course, if the quiver $Q$ is of infinite representation type, i.e. $Q$ is not a disjoint union of Dynkin quivers of type $A_n, D_n, E_6, E_7$ and $E_8$ [10], it may happen that the null cone $Z_{Q,d}$ cannot be written as the closure of one $\text{Gl}(d)$-orbit, even if the integers $\lambda_i$ are very big. In this case, we need infinitely many orbits to describe $Z_{Q,d}$. Our construction still works in this case and it gives us a family of representations of $Q$ which is dense in $Z_{Q,d}$.

**Example 1.3.** We consider the following quiver:

$$
\begin{array}{ccc}
1 & \rightarrow & 3 \\
\downarrow & & \downarrow \\
5 & \leftarrow & 2
\end{array}
$$

We put $T_1 = P_1$ and $T_2 = P_2$, where $P_i \in \text{rep}(Q)$ denotes the projective indecomposable representation attached to the vertex $i$ of $Q$. Thus the dimension vector of $T = T_1^{\lambda_1} \oplus T_2^{\lambda_2}$, $\lambda_1, \lambda_2 \geq 1$, is $d = \lambda_1(1 \ 1 \ 0 \ 0) + \lambda_2(0 \ 0 \ 1 \ 1)$. The category $T^\perp = \{X \in \text{rep}(Q): \text{Hom}(P_1, X) = 0 = \text{Hom}(P_2, X)\} = \{X \in \text{rep}(Q): X(1) = 0 = X(2)\}$ is equivalent to the category $\text{rep}(Q^\perp)$, where the quiver

$$
\begin{array}{ccc}
3 & \rightarrow & 5 \\
\downarrow & & \downarrow \\
6 & \leftarrow & 5
\end{array}
$$

is a non-oriented cycle. We can prove that (this is a special case of a construction given in Section 3) $Z_{Q,d}$ is not the closure of one $\text{Gl}(d)$-orbit, for any positive integers $\lambda_1, \lambda_2$. In fact, if $\lambda_1, \lambda_2 \geq 2$, we have

$$Z_{Q,d} = \bigcup_{\mu \in k^*} \text{Gl}(d) \ast (D_\mu \oplus S_1 \oplus S_2 \oplus S_4 \oplus S_6 \oplus T_1^{\lambda_1-2} \oplus T_2^{\lambda_2-2}),$$

where $S_i$ is the simple representation of $Q$ corresponding to the vertex $i$, and where
We will also discuss the following question: how big are the integers \( v_1, \ldots, v_r \) which appear in Theorem 1.1? We must have \( v_i \geq N \) for all \( i = 1, \ldots, r \), if we want to guarantee that the variety \( \mathcal{Z}_{Q,d} \) is irreducible. It has been proved in [17] that this integer \( N \) can be chosen to be quite small in case \( Q \) is a disjoint union of Dynkin or Euclidean quivers. Indeed, we can choose \( N = 3i \) if \( Q \) is a disjoint union of Dynkin quivers, and \( N = 2i \) if \( Q \) is a disjoint union of Dynkin quivers of type \( A \). The integer \( N \) may be very large if \( Q \) is a wild quiver.

We have a similar result:

**Theorem 1.4.** Let \( Q \) be a disjoint union of Dynkin quivers and \( d \) be a dimension vector for \( Q \). We write \( d = \sum_{i=1}^r \lambda_i \text{dim} T_i \), where the \( \text{GL}(d) \)-orbit of the representation \( T = \bigoplus_{i=1}^r T_i^{\lambda_i} \) is open in \( \text{rep}(Q,d) \) and where the representations \( T_i \) are indecomposable and pairwise non-isomorphic. Then, if \( \lambda_i \geq 3 \) for all \( i = 1, \ldots, r \), the null cone \( \mathcal{Z}_{Q,d} \) is the closure of one stable \( \text{GL}(d) \)-orbit.

Moreover, in the particular case where \( Q \) is a disjoint union of Dynkin quivers of type \( A \), the same holds if the condition \( \lambda_i \geq 2 \), \( i = 1, \ldots, r \), is satisfied.

The following example shows that the integers \( v_1, \ldots, v_r \) may be very large if there is no condition on the quiver \( Q \).

**Example 1.5.** We consider the following quiver \( Q \) with \( n \geq 2 \) vertices:

\[
\begin{align*}
2 & \xrightarrow{\alpha_2} 1 & & 3 \xrightarrow{\alpha_3} \cdots \xrightarrow{\alpha_{n-1}} n-1 \xleftarrow{\alpha_n} n \\
& 0 & & k & 0 \\
\end{align*}
\]

and the dimension vector \( d = \lambda \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \end{pmatrix} = \lambda \text{dim} P_1 \), with \( \lambda \geq 1 \). For a representation \( Y \) in \( \text{rep}(Q,d) \), we have \( Y \in \mathcal{Z}_{Q,d} \) if \( \det(Y(\alpha_i)) = 0 \) for all \( i = 2, \ldots, n \). Therefore, the variety \( \mathcal{Z}_{Q,d} \) is irreducible of codimension \( n-1 \) in \( \text{rep}(Q,d) \), for any choice of \( \lambda \geq 1 \), and we may choose \( N = 1 \). However, the null cone \( \mathcal{Z}_{Q,d} \) is the closure of one stable orbit if \( \lambda \geq n-1 \). And then, as follows from a construction in Section 3, a stable representation \( X \) in \( \mathcal{Z}_{Q,d} \) is given by the direct sum of indecomposables

\[
X = P_2 \oplus Z_2 \oplus P_3 \oplus Z_3 \oplus \cdots \oplus P_n \oplus Z_n \oplus P_1^{\lambda-(n-1)},
\]

where

\[
Z_2 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ k & k & \cdots & k \\ k & k & \cdots & k \end{pmatrix}, \quad Z_n = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ k & k & \cdots & k \\ k & k & \cdots & k \end{pmatrix}
\]

This example shows also that the integers \( v_1, \ldots, v_r \) can be arbitrary large, even if \( N = 1 \).
2. Preliminaries and notations

We will keep the following notations and assumptions throughout the paper.

We fix an algebraically closed field $k$ and a finite quiver $Q$. A vertex of $Q$ which is the head of no arrow in $Q_1$ is called a source, and a vertex which is the tail of no arrow is called a sink. A path $\pi$ in $Q$ of length $l(\pi) = m$ is a sequence $\alpha_1\alpha_2\cdots\alpha_m$ of $m$ arrows in $Q_1$ such that $t_{\alpha_i} = h_{\alpha_{i+1}}$ for all $i = 1, \ldots, m − 1$ (i.e. $\overrightarrow{\alpha_1} \overrightarrow{\alpha_2} \cdots \overrightarrow{\alpha_m}$). We define $t(\pi) = t_{\alpha_m}$ and $h(\pi) = h_{\alpha_1}$ to be the tail and the head of $\pi$, respectively. We denote by $e_i$ the trivial path at the vertex $i \in Q_0$. A non-trivial path $\pi$ such that $t(\pi) = h(\pi)$ is called an oriented cycle.

Assume that $Q$ does not contain oriented cycles. For any vertex $i \in Q_0$, the projective indecomposable $P_i$ is defined by $P_i(j) = \bigoplus k\pi$, where $\pi$ ranges over all paths in $Q$ from $i$ to $j$, and $P_i(\alpha(\pi)) = \alpha\pi$ for any path $\pi : i \rightarrow j$ and any arrow $\alpha : j \rightarrow l$. A path $\pi : i \rightarrow j$ defines a morphism $\theta_{\pi} : P_j \rightarrow P_i$, given by sending a path $\rho : j \rightarrow l$ to $\rho\pi : i \rightarrow l$.

We denote by $\text{rep}(Q)$ the category of representations of $Q$. It is an abelian category of global dimension at most 1. The only possibly non-trivial extension group $\text{Ext}^1$ will be denoted by $\text{Ext}$ (see [11, 18]). The dimension vector of a representation $X$ of $Q$ is the vector $\dim\,X = (\dim X(1), \ldots, \dim X(m))$ in $\mathbb{N}^m$. We fix representations $T_1, \ldots, T_r$ of $Q$, which are indecomposable, pairwise non-isomorphic and satisfy $\text{Ext}(T_i, T_j) = 0$ for all $i, j \in \{1, \ldots, r\}$. We choose positive integers $\lambda_1, \ldots, \lambda_r$, and we put $T = \bigoplus_{i=1}^r T_i^{\lambda_i}$ and $d = \dim\,T$. We suppose that the support of $T$ is all of $Q$ (we can always consider the support of $T$ instead of $Q$). We have $\text{Ext}(T, T) = 0$, thus the orbit $\text{Gl}(d) \cdot T$ is open in $\text{rep}(Q, d)$. Note that the support of a representation having an open orbit never contains oriented cycles. (For an oriented cycle $j_1 \overrightarrow{a_1} j_2 \rightarrow \cdots \rightarrow j_t \overrightarrow{a_t} j_1$ assume $d_{j_i} = \min(d_{j_i}) : i = 1, \ldots, t \neq 0$. The non-constant polynomial $f(X) = \det(X(\alpha_1) \cdots X(\alpha_t))$, $X \in \text{rep}(Q, d)$, is invariant under the action of $\text{Gl}(d)$, and it is constant on closures of $\text{Gl}(d)$-orbits. If there is an open $\text{Gl}(d)$-orbit in $\text{rep}(Q, d)$, $f$ is constant on $\text{rep}(Q, d)$, hence a contradiction.)

Riedtmann and Zwara showed that there is a positive integer $N$ (which depends on the quiver $Q$ and on the representations $T_1, \ldots, T_r$) such that $Z_{Q, d}$ is an irreducible variety of codimension $n − r$ in $\text{rep}(Q, d)$, for any dimension vector $d = \sum_{i=1}^r \lambda_i \dim\,T_i$ satisfying $\lambda_i \geq N$ for all $i = 1, \ldots, r$. It follows that we may always assume, taking the integers $\lambda_i$ big enough, that the set $Z_{Q, d + \sum_{i=1}^r m_i \dim\,T_i}$ is an irreducible variety, for any non-negative integers $m_1, \ldots, m_r$.

We recall that a representation $X$ is generic in $Z_{Q, d}$ if $\text{Gl}(d) \cdot X = Z_{Q, d}$, and a representation $X \in Z_{Q, d}$ is stable if $X \oplus \bigoplus_{i=1}^r T_i^{m_i}$ is generic in $Z_{Q, d'}$ for any non-negative integers $m_1, \ldots, m_r$, where $d' = d + \sum_{i=1}^r m_i \dim\,T_i$.

Remark 2.1. For any quiver $Q$ and any dimension vector $v$ for $Q$, the codimension in $\text{rep}(Q, v)$ of the $\text{Gl}(v)$-orbit of a representation $A \in \text{rep}(Q, v)$ is given by the formula [18]:

$$\text{codim}\,\text{Gl}(v) \cdot A = \dim\text{Ext}(A, A).$$

As a consequence, if $\lambda_i \geq N$ for all $i = 1, \ldots, r$, a representation $X$ is generic in $Z_{Q, d}$ iff $X \in Z_{Q, d}$ and $\dim\text{Ext}(X, X) = n − r$.

Lemma 2.2. Assume $\lambda_i \geq N$ for all $i = 1, \ldots, r$. Then, a generic representation $X$ in $Z_{Q, d}$ is stable iff $\text{Ext}(X, T) = 0 = \text{Ext}(T, X)$.

Proof. We chose positive integers $m_1, \ldots, m_r$ and put $d' = d + \sum_{i=1}^r m_i \dim\,T_i$. The variety $Z_{Q, d'}$ is irreducible of codimension $n − r$ in $\text{rep}(Q, d')$. It follows that the representation $X' := X \oplus \bigoplus_{i=1}^r T_i^{m_i}$ is generic in $Z_{Q, d'}$ iff $n − r = \dim\text{Ext}(X', X')$ iff $n − r = \dim\text{Ext}(X, X) + \dim\text{Ext}(\bigoplus_{i=1}^r T_i^{m_i}, X) + \dim\text{Ext}(X, \bigoplus_{i=1}^r T_i^{m_i})$ iff $\text{Ext}(X, T) = 0 = \text{Ext}(T, X)$.  \[\Box\]
3. Representations in \( \mathcal{Z}_{\mathbb{Q}, \mathbf{d}} \)

In this section we introduce the tools we need to construct a dense subset of \( \mathcal{Z}_{\mathbb{Q}, \mathbf{d}} \). Let \( E_T \) be the full subcategory of \( \text{rep}(Q) \) whose objects \( A \) satisfy

\[
\text{Ext}(A, T) = 0 = \text{Ext}(T, A).
\]

Note that \( \text{Ext}(A, T) = 0 \) and \( \text{Ext}(T, A) = 0 \) are open conditions on \( \text{rep}(Q, \mathbf{d}) \).

For \( A \in E_T \), we denote by \( \text{tr} A \) the trace of \( T \) in \( A \), i.e., the sum of the images of all maps from \( T \) to \( A \), and we set \( \bar{A} = A / \text{tr} A \). Note that \( \bar{A} \) lies in \( T^\perp \), since from the exact sequence

\[
0 \to \text{tr} A \to A \to \bar{A} \to 0
\]

we get the long exact sequence

\[
0 \to \text{Hom}(T, \text{tr} A) \to \text{Hom}(T, A) \to \text{Hom}(T, \bar{A}) \to 0,
\]

where the first map is an isomorphism and \( \text{Ext}(T, A) = 0 = \text{Ext}(T, \text{tr} A) \). (The last equality holds because there is a surjective morphism \( \bigoplus_{i=1}^{r} T_i^{a_i} \to \text{tr} A \), for some integers \( a_1, \ldots, a_r \), and \( \text{Ext}(T, T) = 0 \).)

**Lemma 3.1.** Let \( A, B \) be in \( E_T \). Then \( \text{Ext}(A, \text{tr} B) = 0 \), \( \text{Hom}(\text{tr} A, \bar{B}) = 0 \), and the following two sequences are exact:

\[
\begin{align*}
0 & \to \text{Hom}(\bar{A}, \bar{B}) \to \text{Hom}(A, B) \to \text{Hom}(A, \bar{B}) \to 0, \\
0 & \to \text{Ext}(\bar{A}, \bar{B}) \to \text{Ext}(A, B) \to \text{Ext}(\text{tr} A, \bar{B}) \to 0, \\
0 & \to \text{Hom}(A, \text{tr} B) \to \text{Hom}(A, B) \to \text{Hom}(A, \bar{B}) \to 0, \\
0 & \to \text{Ext}(A, B) \to \text{Ext}(A, \bar{B}) \to 0.
\end{align*}
\]

**Proof.** There is a surjective morphism \( \bigoplus_{i=1}^{r} T_i^{b_i} \to \text{tr} B \), for some integers \( b_1, \ldots, b_r \), and we have \( \text{Ext}(A, \bigoplus_{i=1}^{r} T_i^{b_i}) = 0 \). Thus we obtain \( \text{Ext}(A, \text{tr} B) = 0 \). Similarly, there is a surjective morphism \( \bigoplus_{i=1}^{r} T_i^{a_i} \to \text{tr} A \), moreover we have \( \text{Hom}(\bigoplus_{i=1}^{r} T_i^{a_i}, \bar{B}) = 0 \), thus \( \text{Hom}(\text{tr} A, \bar{B}) = 0 \). □

We recall a construction introduced in [4] by Bongartz. Let \( kQ \) be the quiver algebra of \( Q \), viewed as a projective representation of \( Q \). We put

\[
e_i := \dim \text{Ext}(T_i, kQ),
\]

\( i = 1, \ldots, r \), and we choose an exact sequence

\[
0 \to kQ \to M \to \bigoplus_{i=1}^{r} T_i^{e_i} \to 0
\]

such that the induced map

\[
\text{Hom}\left( T_i, \bigoplus_{i=1}^{r} T_i^{e_i} \right) \to \text{Ext}(T_i, kQ)
\]
is surjective for all \( l = 1, \ldots, r \). It is easy to check that \( M \) is independent of the choice of such a sequence, up to isomorphism, and that \( \text{Ext}(T \oplus M, T \oplus M) = 0 \). Therefore \( T \oplus M \) is a tilting module, and has \( n \) non-isomorphic indecomposable direct summands [4]. The direct sum \( T_\mathcal{B} = T_{r+1} \oplus \cdots \oplus T_n \) of the direct summands of \( M \) which are not isomorphic to \( T_1, \ldots, T_r \) will be called the Bongartz completion of \( T \). Note that the indecomposables \( T_{r+1}, \ldots, T_n \) lie in \( E_T \).

The following proposition is proved in [19].

**Proposition 3.2.** \( T_{r+1}, \ldots, T_n \) are representatives for the indecomposable projectives in \( T^\perp \).

We fix an equivalence of categories

\[
H : \text{rep}(Q^\perp) \sim \sim T^\perp.
\]

The set of vertices of \( Q^\perp \) will be denoted by \( \{r + 1, \ldots, n\} \), in such a way that the projective indecomposable \( \overline{T}_j \) of \( T^\perp \) corresponds to the projective indecomposable \( P^\perp_j \) of \( \text{rep}(Q^\perp) \), i.e., \( H(P^\perp_j) \cong \overline{T}_j \) for all \( j = r + 1, \ldots, n \).

Let \( A, B_1, \ldots, B_s \) be pairwise non-isomorphic indecomposable representations of \( Q \). We denote by \( \text{add}(B_1 \oplus \cdots \oplus B_s) \) the full subcategory of \( \text{rep}(Q) \) consisting of representations \( Y \) such that \( Y \cong B_{b_1}^{b_1} \oplus \cdots \oplus B_{b_s}^{b_s} \) for some integers \( b_1, \ldots, b_s \).

**Definition 3.3.** A map \( f : A \to \bigoplus_{l=1}^s B_l^{b_l} \) is a source map from \( A \) to \( \text{add}(B_1 \oplus \cdots \oplus B_s) \) provided

1. any map from \( A \) to some \( B_l \) factors through \( f \),
2. if \( \alpha \circ f \) has property (1) for an endomorphism \( \alpha \) of \( \bigoplus_{l=1}^s B_l^{b_l} \), then \( \alpha \) is an automorphism.

It is easy to see that source maps exist and that they are unique up to isomorphism, i.e., if \( f_1 : A \to \bigoplus_{l=1}^s B_l^{b_l} \) and \( f_2 : A \to \bigoplus_{l=1}^s B_l^{b_l} \) are two source maps from \( A \) to \( \text{add}(B_1 \oplus \cdots \oplus B_s) \), then \( b_l' = b_l \) for all \( l = 1, \ldots, s \), and there is an automorphism \( \alpha \) of \( \bigoplus_{l=1}^s B_l^{b_l} \) with \( f_1 = \alpha \circ f_2 \).

We consider, for every \( j = r + 1, \ldots, n \), a source map

\[
g_j : T_j \to T_j^{++}
\]

from \( T_j \) to \( \text{add}(T) \). The following proposition is proved in [15]:

**Proposition 3.4.** If the support of \( T \) is \( Q \), the source map \( g_j \) is injective, for \( j = r + 1, \ldots, n \).

We denote by \( Z_j \) the cokernel of \( g_j \), thus we have an exact sequence

\[
0 \to T_j \to T_j^{++} \to Z_j \to 0,
\]

where \( T_j^{++} \) is some representation in \( \text{add}(T) \), \( j = r + 1, \ldots, n \). Note that \( Z_j \) lies in \( E_T \) (by property (1) of the source map \( g_j \)), and that \( Z_j = 0 \) since there is a surjective map from \( \text{add}(T) \) to \( Z_j \). The following facts are essentially proved in [15]:

**Lemma 3.5.** Let \( j, k \) be in \( \{r + 1, \ldots, n\} \). Then we have:

1. \( \text{Ext}(Z_j, Z_k) = 0 \),
2. \( \text{Ext}(T_j, Z_k) = 0 \),
3. \( \dim \text{Ext}(Z_j, S_k') = \dim \text{Hom}(T_j, S_k') = \delta_{j,k} \),
4. \( \dim \text{End}(Z_j) = 1 \).
Remark 3.6. It follows from the previous lemma (parts (3) and (4)) that the representations $Z_j$ are indecomposable and pairwise non-isomorphic. Therefore, the direct sum $Z := T_1 \oplus \cdots \oplus T_r \oplus Z_{r+1} \oplus \cdots \oplus Z_n$ is a tilting module.

Definition 3.7. We denote by $G$ the functor from $E_T$ to $T^\perp$, where $G(A) = \overline{A}$ for all objects $A$ of $E_T$ and where the morphism $G(f) : G(A) \to G(B)$ is given by the surjective map

$$\text{Hom}(A, B) \to \text{Hom}(\overline{A}, \overline{B}) \cong \text{Hom}(A, B)/\text{Hom}(A, \text{tr}B),$$

for all objects $A, B$ of $E_T$ and all morphisms $f : A \to B$.

Definition 3.8. We denote by $I \subseteq \text{rep}(Q)$ the ideal generated by the representations $T_1, \ldots, T_r, Z_{r+1}, \ldots, Z_n$, i.e., the ideal of all morphisms which factor through some representation in $\text{add}(Z)$.

Proposition 3.9. The functor $G : E_T \to T^\perp$ is isomorphic to the quotient functor $E_T \to E_T/I$.

Proof. a) It is clear that the functor $G$ is full and $k$-linear, since for any $A, B$ objects of $E_T$, the map

$$G_{A,B} : \text{Hom}(A, B) \to \text{Hom}(G(A), G(B)),$$

$$\varphi \mapsto G(\varphi)$$

is surjective and $k$-linear.

b) Let $A \in E_T$ be indecomposable. We show that

$$G(A) = 0 \iff A \cong B \; \text{ for some } B \in \{T_1, \ldots, T_r, Z_{r+1}, \ldots, Z_n\}.$$

If $A \cong B$ for some $B$ in $\text{add}(Z)$, it is clear that $G(A) = 0$. Conversely, if $G(A) = 0$, we put $Y := T_1 \oplus \cdots \oplus T_r \oplus Z_{r+1} \oplus \cdots \oplus Z_n \oplus A$, and we have $\text{Ext}(Y, Y) = 0$. Indeed, there is a surjective morphism $U \to Y$ for some $U \in \text{add}(T)$ and we have $\text{Ext}(Y, U) = 0$. Therefore, $Y$ is a tilting module and $A$ must be isomorphic to one of the first $n$ direct summands of $Y$.

c) Let $A, B \in E_T$ and $\varphi \in \text{Hom}(A, B)$. We show that

$$G(\varphi) = 0 \iff \varphi \in I.$$

If $\varphi$ lies in $I$, it is clear by b) that $G(\varphi) = 0$.

If $G(\varphi) = 0$, then $\varphi$ factors through $\text{tr}B$. It is easy to check that $\text{tr}B$ lies in $E_T$. (Apply $\text{Hom}(T, \cdot)$ and $\text{Hom}(\cdot, T)$ successively to the exact sequence

$$0 \to \text{tr}B \to B \to \overline{B} \to 0.$$

We have $G(\text{tr}B) = 0$, therefore $\text{tr}B$ lies in $\text{add}(Z)$.

d) Let $Y$ be in $T^\perp$. We show that there exists a representation $\overline{Y}$ in $E_T$ such that $G(\overline{Y}) \cong Y$.

For each $i \in \{1, \ldots, r\}$ we put $\mu_i = \dim \text{Ext}(Y, T_i)$ and we choose an exact sequence

$$0 \to \bigoplus_{i=1}^{r} T_i^{\mu_i} \xrightarrow{f} \overline{Y} \to Y \to 0$$

such that the induced map $\text{Hom}(\bigoplus_{i=1}^{r} T_i^{\mu_i}, T_i) \to \text{Ext}(Y, T_i)$ is surjective, for all $i = 1, \ldots, r$. This implies $\text{Ext}(\overline{Y}, T) = 0$. We have $\text{Ext}(T, \overline{Y}) = 0$, since $\text{Ext}(T, \bigoplus_{i=1}^{r} T_i^{\mu_i}) = 0 = \text{Ext}(T, Y)$. This proves that
\( \tilde{Y} \) lies in \( E_T \). Since \( Y \) lies in \( T^\perp \), the induced map \( \text{Hom}(T, \bigoplus_{i=1}^{r} T_i^{\mu_i}) \to \text{Hom}(T, \tilde{Y}) \) is an isomorphism. This characterizes \( tr\tilde{Y} \) as \( \bigoplus_{i=1}^{r} T_i^{\mu_i} \). Therefore we have \( G(\tilde{Y}) = \tilde{Y}/tr\tilde{Y} \cong Y \) and the functor \( G \) is dense. \( \square \)

**Lemma 3.10.** Let \( A, B \in E_T \) be indecomposable representations of \( Q \) such that \( G(A) \cong G(B) \neq 0 \). Then we have \( A \cong B \).

**Proof.** If \( G(A) \) and \( G(B) \) are isomorphic in the category \( T^\perp \), then \( A \) and \( B \) are isomorphic in the quotient category \( E_T/I \). It follows that there are morphisms \( f \in \text{Hom}(A, B) \) and \( g \in \text{Hom}(B, A) \) with \( g \circ f - 1_A \in I(A, A) \). Since \( A \) is indecomposable, any endomorphism \( h \in \text{Hom}(A, A) \) is either an isomorphism or else nilpotent. But the vector space \( I(A, A) \) does not contain any isomorphism, otherwise the identity \( 1_A \) would factor through some \( Y \in \text{add}(Z) \), and therefore we would have \( G(A) = 0 \). Thus the map \( g \circ f - 1_A \) is nilpotent, and this implies that \( g \circ f \) is an isomorphism. Replacing \( A \) by \( B \) and repeating the same argument, we show that \( f \circ g \) is an isomorphism as well. \( \square \)

**Remark 3.11.** The map \( A \mapsto G(A) \) induces a bijective map \( \tilde{G} \) from the set of isoclasses of indecomposable representations in \( E_T \) which are not isomorphic to some \( T_i, i = 1, \ldots, r \), or to some \( Z_j, j = r + 1, \ldots, n \), to the set of isoclasses of indecomposable representations in \( T^\perp \). For an indecomposable representation \( Y \in T^\perp \), we choose an indecomposable representation \( F(Y) \in E_T \) such that \( G(F(Y)) \cong Y \). For a direct sum of indecomposables \( Y = \bigoplus_i Y_i \), we put \( F(Y) := \bigoplus_i F(Y_i) \).

**Remark 3.12.** We have \( F(T_j) \cong T_j \), for all \( j = r + 1, \ldots, n \).

**Remark 3.13.** Since the representation \( Z = T_1 \oplus \cdots \oplus T_r \oplus Z_{r+1} \oplus \cdots \oplus Z_n \) is a tilting module, the set \( \{ \text{dim} T_1, \ldots, \text{dim} T_r, \text{dim} Z_{r+1}, \ldots, \text{dim} Z_n \} \) is a basis of \( \mathbb{Q}^n \). (Indeed, the condition \( \text{Ext}(Z, Z) = 0 \) implies that these \( n \) dimension vectors are linearly independent in \( \mathbb{Z}^n \).) Let \( X, Y \) be in \( E_T \) with \( \text{dim} X = \text{dim} Y \). Then it is easy to prove that \( X \cong Y \) iff \( G(X) \cong G(Y) \).

As a consequence, for any \( X \in E_T \), we can write

\[
X \cong F(G(X)) \oplus \bigoplus_{i=1}^{r} T_i^{l_i} \oplus \bigoplus_{j=r+1}^{n} Z_j^{l_j},
\]

and the representation \( F(G(X)) \) has no direct summand in \( \text{add}(Z) \). Moreover, we have

\[
\text{dim} F(G(X)) = \text{dim} G(X) + \sum_{i=1}^{r} l_i \text{dim} T_i
\]

for some non-negative integers \( l_i \leq \text{dim} \text{Ext}(G(X), T_i), i = 1, \ldots, r \). Indeed, replace \( Y \) by \( G(X) \) in the part d) of the proof of Proposition 3.9. We have \( \tilde{Y} \cong F(Y) \oplus B \) for some \( B \in \text{add}(Z) \). Since \( \text{Hom}(Z_j, Y) = 0 \), \( Z_j \) cannot be a direct summand of \( B \), otherwise \( Z_j \) would be a direct summand of \( \bigoplus_{i=1}^{r} T_i^{\mu_i} \). Therefore, \( B \) lies in \( \text{add}(T) \). In particular, using Remark 3.12, we have

\[
\text{dim} T_j = \text{dim} \tilde{T}_j + \sum_{i=1}^{r} l_{j,i} \text{dim} T_i
\]

for some non-negative integers \( l_{j,i} \).

The following proposition shows that if \( X \in E_T \) satisfies \( \text{dim} X = d \), where \( d = \sum_{i=1}^{r} \lambda_i \text{dim} T_i \) is the dimension vector defined in Section 2, the integers \( a_{r+1}, \ldots, a_n \) are determined by the dimension vector of \( G(X) \):
Proposition 3.14. Let \( X \) be in \( ET \) with \( \text{dim} \, X = d \). Let \( a_j \) be the multiplicity of the direct summand \( Z_j \) of \( X \), \( j = r + 1, \ldots, n \), and let the exact sequence

\[
0 \to \bigoplus_{j=r+1}^n T^c_j \to \bigoplus_{j=r+1}^n T^b_j \to G(X) \to 0
\]

be a minimal projective presentation of \( G(X) \) in the category \( T^\perp \) (we set \( b_j = c_j = 0 \) if \( G(X) = 0 \)). Then we have \( a_j = b_j - c_j \) for all \( j = r + 1, \ldots, n \).

Proof. The set \( \{ \text{dim} \, T_1, \ldots, \text{dim} \, T_r, \text{dim} \, Z_{r+1}, \ldots, \text{dim} \, Z_n \} \) is a basis of \( \mathbb{Z}^n \). We will compute the dimension vector of \( X \) in \( \mathbb{Z}^n \) modulo \( W \), where \( W \) is the free group of rank \( r \) generated by \( \text{dim} \, T_1, \ldots, \text{dim} \, T_r \). We have \( \text{dim} \, T_j + \text{dim} \, Z_j = \text{dim} \, T_j^+ \in W \) and \( \text{dim} \, \overline{T}_j \equiv \text{dim} \, T_j \) modulo \( W \), for all \( j = r + 1, \ldots, n \). Therefore we have

\[
\text{dim} \, F(G(X)) \equiv \text{dim} \, G(X) = \sum_{j=r+1}^n (b_j - c_j) \text{dim} \, \overline{T}_j
\]

\[
\equiv \sum_{j=r+1}^n (b_j - c_j) \text{dim} \, T_j
\]

\[
\equiv - \sum_{j=r+1}^n (b_j - c_j) \text{dim} \, Z_j \bmod W
\]

and

\[
0 \equiv \text{dim} \, X \equiv \text{dim} \, F(G(X)) + \sum_{j=r+1}^n a_j \text{dim} \, Z_j \bmod W.
\]

But the set \( \{ \text{dim} \, Z_j : j = r + 1, \ldots, n \} \) induces a basis of \( \mathbb{Z}^n / W \). \( \square \)

Proposition 3.15. Let \( X \) be in \( ET \), and let

\[
0 \to \bigoplus_{j=r+1}^n T^c_j \xrightarrow{\varphi} \bigoplus_{j=r+1}^n T^b_j \to G(X) \to 0
\]

be a minimal projective presentation of \( G(X) \) in the category \( T^\perp \). Then, for all \( j = r + 1, \ldots, n \), we have

\[
\text{dim} \, \text{Ext}(Z_j, X) = \text{dim} \, \text{Hom}(T_j, G(X)) = \sum_{l=r+1}^n (b_l - c_l) \text{dim} \, \text{Hom}(\overline{T}_j, \overline{T}_l)
\]

and

\[
\text{dim} \, \text{Hom}(X, S'_j) = 1 \iff b_j = 1.
\]
**Proof.** By Lemma 3.1, for \( j = r + 1, \ldots, n \), we have

\[
\text{Ext}(Z_j, X) \cong \text{Ext}(Z_j, G(X)),
\]
\[
\text{Hom}(T_j, G(X)) \cong \text{Hom}(T_j, G(X)).
\]

In a similar way we show that

\[
\text{Hom}(X, S'_j) \cong \text{Hom}(G(X), S'_j).
\]

Applying \( \text{Hom}(\cdot, G(X)) \) to

\[
0 \rightarrow T_j \rightarrow T_j^{++} \rightarrow Z_j \rightarrow 0,
\]

we obtain the exact sequence

\[
0 = \text{Hom}(T_j^{++}, G(X)) \rightarrow \text{Hom}(T_j, G(X))
\]
\[
\rightarrow \text{Ext}(Z_j, G(X)) \rightarrow \text{Ext}(T_j^{++}, G(X)) = 0,
\]

and thus we have \( \text{Hom}(T_j, G(X)) \cong \text{Ext}(Z_j, G(X)) \). This proves the first equality.

The second equality follows from the exact sequence

\[
0 \rightarrow \text{Hom}(G(X), S'_j) \rightarrow \text{Hom}(\bigoplus_{l=r+1}^n T_l^{c_l}, S'_j) \xrightarrow{\phi^*} \text{Hom}(\bigoplus_{l=r+1}^n T_l^{b_l}, S'_j) \rightarrow \cdots.
\]

Since \( \phi \) is radical, \( \phi^* \) is the zero map and we have

\[
\text{Hom}(G(X), S'_j) \cong \text{Hom}(\bigoplus_{l=r+1}^n T_l^{c_l}, S'_j) \cong k^{b_j}. \qedhere
\]

**Definition 3.16.** We denote by \( S \) the set of sources of \( Q^\perp \) and we put

\[
w_j := \sum_{l \in S} \dim \text{Hom}(T_j, T_l),\]

\( j = r + 1, \ldots, n \). We define the dimension vector \( \mathbf{e} \) for \( Q \) by:

\[
\mathbf{e} = \sum_{j \in S} \dim T_j.
\]
We obtain the following result:

**Proposition 3.17.** Let $X$ be in $ET$ such that $\dim X = d$, $\dim \text{Hom}(X, S'_l) = 1$ for all $l = r + 1, \ldots, n$. Then the following conditions are equivalent:

1. $\dim G(X) = e$,
2. $\dim \text{Ext}(Z_j, X) = w_j$ for all $j \in Q^+_0 \setminus S$.

**Proof.** Let $X$ be in $ET$, and let

\[ 0 \to \bigoplus_{j=r+1}^n T^c_j \xrightarrow{\varphi} \bigoplus_{j=r+1}^n T^{b_j}_j \to G(X) \to 0 \]

be a minimal projective presentation of $G(X)$. The map $\varphi$ is radical, thus $c_j = 0$ for all $j \in S$. Assume that the representation $X$ satisfies $\dim X = d$ and $\dim \text{Hom}(X, S'_l) = 1$, $l = r + 1, \ldots, n$. By Proposition 3.15, we have $b_j = 1$, $j = r + 1, \ldots, n$, and by Proposition 3.14, we have $0 \leq c_j \leq 1$ for all $j \notin Q^+_0 \setminus S$. Using Proposition 3.15, with $j \notin Q^+_0 \setminus S$, we obtain

\[
\dim \text{Ext}(Z_j, X) = w_j + \sum_{l \in Q^+_0 \setminus S} (1 - c_l) \dim \text{Hom}(T_j, T_l).
\]

If $j \in S$, then $\dim \text{Hom}(T_j, T_l) = \delta_{j,l}$, where $\delta$ is the Kronecker delta. We have proved:

(a) $\dim \text{Ext}(Z_j, X) = w_j = 1$, if $j \in S$,
(b) $\dim \text{Ext}(Z_j, X) \geq w_j + 1 - c_j$, if $j \notin Q^+_0 \setminus S$,
(c) $\dim \text{Ext}(Z_j, X) = w_j$ for all $j \in Q^+_0 \setminus S$ \iff $c_j = 1$ for all $j \notin Q^+_0 \setminus S$,

and Proposition 3.17 follows. \qed

**4. A Dense Family in $Z_{Q,d}$**

We keep the same notations and assumptions. We have a sincere prehomogeneous dimension vector $d = \sum_{i=1}^r \lambda_i \dim T_i$, and we recall that there is a positive integer $N$ such that $Z_{Q,d}$ is an irreducible variety of codimension $n - r$ in $\text{rep}(Q, d)$, provided $\lambda_i \geq N$ for all $i = 1, \ldots, r$. From now on, we assume $\lambda_i \geq N$. We need the following well-known corollaries of Chevalley's theorem [7].

**Lemma 4.1.** Let $A$ be a representation of $Q$ and $v \in N^{Q_0}$ be a dimension vector for $Q$. Then the maps $\text{rep}(Q, v) \to N$ defined by:

\[
X \mapsto \dim \text{Hom}(A, X),
\]

\[
X \mapsto \dim \text{Ext}(A, X),
\]

...
\[ X \mapsto \dim \text{Hom}(X, A), \]
\[ X \mapsto \dim \text{Ext}(X, A) \]

are upper semicontinuous.

Recall that if \( V \) is a variety, then a function \( f : V \to \mathbb{Z} \) is upper semicontinuous if the set \( \{ x \in V : f(x) \leq m \} \) is open in \( V \) for all \( m \in \mathbb{Z} \).

**Definition 4.2.** We denote by \( F_{Q, d} \) the set of representations \( X \) in \( \text{rep}(Q, d) \) satisfying the following conditions:

1. \( \text{Ext}(T, X) = 0 = \text{Ext}(X, T) \),
2. \( \dim \text{Hom}(X, S_j) = 1, \ j \in Q_0^\perp \),
3. \( \dim G(X) = e \).

We recall that \( e \) is the dimension vector for \( Q \) defined by \( e = \sum_{j \in S} \dim T_j \), where \( S \) is the set of sources of \( Q \).

**Proposition 4.3.** The set \( F_{Q, d} \) is open in \( Z_{Q, d} \).

**Proof.** It is clear that the first condition is an open condition on \( \text{rep}(Q, d) \), and thus an open condition on \( Z_{Q, d} \) too. Since we have \( \text{Hom}(X, S_j') \neq 0 \) for all \( X \in Z_{Q, d} \) and all \( j \in Q_0^\perp \), the second condition is an open condition on \( Z_{Q, d} \). Therefore, by Proposition 3.17 and its proof, the three conditions together give an open condition on \( Z_{Q, d} \).

Of course, the set \( F_{Q, d} \) may be empty. We will prove that there are integers \( v_1, \ldots, v_r \) such that \( F_{Q, d} \) is not empty if \( \lambda_i \geq v_i \) for all \( i = 1, \ldots, r \).

**Definition 4.4.** The dimension vector

\[ e^\perp := \sum_{j \in S} \dim Q^\perp P_j \in \mathbb{N}^{n-r}, \]

which is the sum of the dimension vectors of the projective indecomposables corresponding to the sources of \( Q^\perp \), will be called the source-vector associated to \( Q^\perp \). We denote by \( s = \#S \) the number of sources of \( Q^\perp \). We define the varieties

\[ M_{Q^\perp, e^\perp} = \{ Y \in \text{rep}(Q^\perp, e^\perp) : \text{Hom}(Y, S_j^\perp) \neq 0, \ j \in Q_0^\perp \} \]

and

\[ M'_{Q^\perp, e^\perp} = \{ Y \in \text{rep}(Q^\perp, e^\perp) : \dim \text{Hom}(Y, S_j^\perp) = 1, \ j \in Q_0^\perp \}. \]

In Section 5 we will prove that the varieties \( M_{Q^\perp, e^\perp} \) and \( M'_{Q^\perp, e^\perp} \) are non-empty. We recall that the functor \( \text{rep}(Q^\perp) \xrightarrow{\sim} T^\perp \) is an equivalence of categories such that \( H(P_j^\perp) \cong T_j \) for all \( j \in Q_0^\perp \).

**Remark 4.5.** Let \( Y \) be a representation in \( M'_{Q^\perp, e^\perp} \). It follows from the proof of Proposition 3.14 that there are integers \( v_1(Y), \ldots, v_r(Y) \) such that

\[ \dim \left( F(H(Y)) \bigoplus \bigoplus_{j \in S} Z_j \right) = \sum_{i=1}^r v_i(Y) \dim T_i. \]
If we assume \( \lambda_i \geq v_i(Y) \) for all \( i = 1, \ldots, r \), then, by Proposition 3.15, the representation

\[
X_Y := F(H(Y)) \oplus \bigoplus_{j \in S} Z_j \oplus \bigoplus_{i=1}^r T_i^{\lambda_i - v_i(Y)}
\]

lies in \( \mathcal{F}_{\mathbb{Q},d} \).

**Definition 4.6.** We denote by \( S \) the semi-simple representation of \( \mathbb{Q} \) with \( \dim S = e \), and we put \( h_i := \dim \text{Ext}(S, T_i), \ i = 1, \ldots, r \). We denote by \( t_1, \ldots, t_r \) the integers satisfying the following linear equation in \( \mathbb{Z}^r \),

\[
\sum_{i=1}^r t_i \dim T_i = \sum_{j \in S} (\dim T_j^++ - \dim \text{tr} T_j),
\]

and we define

\[
v_i := \max\{N, h_i + t_i\}
\]

for all \( i = 1, \ldots, r \).

**Proposition 4.7.** Assume \( \lambda_i \geq v_i \) for all \( i = 1, \ldots, r \), and let \( Y \) be a representation in \( \mathcal{M}'_{\mathbb{Q}^+,e+} \). Then we have \( \lambda_i \geq v_i(Y) \) for all \( i = 1, \ldots, r \), and the representation

\[
X_Y := F(H(Y)) \oplus \bigoplus_{j \in S} Z_j \oplus \bigoplus_{i=1}^r T_i^{\lambda_i - v_i(Y)}
\]

lies in \( \mathcal{F}_{\mathbb{Q},d} \).

**Proof.** The semi-simple representation \( S \in \text{rep}(\mathbb{Q},e) \) lies in the closure of every \( \text{GL}(e) \)-orbit of \( \text{rep}(\mathbb{Q},e) \), thus we have \( \dim \text{Ext}(A, T_i) \leq \dim \text{Ext}(S, T_i) \) for all \( A \in \text{rep}(\mathbb{Q},e) \) [14]. Let \( Y \) be in \( \mathcal{M}'_{\mathbb{Q}^+,e+} \). We have \( \dim H(Y) = \sum_{j \in S} \dim T_j = e \), and thus, by Remark 3.13,

\[
\dim F(H(Y)) = \dim H(Y) + \sum_{i=1}^r l_i(Y) \dim T_i,
\]

for some integers \( l_i(Y) \leq \dim \text{Ext}(H(Y), T_i) \leq \dim \text{Ext}(S, T_i) = h_i, \ i = 1, \ldots, r \). We compute

\[
\sum_{i=1}^r v_i(Y) \dim T_i = \dim \left( F(H(Y)) \oplus \bigoplus_{j \in S} Z_j \right)
\]

\[
= \sum_{j \in S} \dim T_j + \sum_{i=1}^r l_i(Y) \dim T_i + \sum_{j \in S} \dim Z_j
\]

\[
= \sum_{j \in S} \dim T_j - \sum_{j \in S} \dim \text{tr} T_j + \sum_{i=1}^r l_i(Y) \dim T_i + \sum_{j \in S} \dim Z_j
\]
\[
\begin{align*}
\dim T_j &= \sum_{j \in \mathcal{S}} \dim T_j^+ - \sum_{j \in \mathcal{S}} \dim \text{tr} T_j + \sum_{i=1}^r l_i(Y) \dim T_i \\
&= \sum_{i=1}^r t_i \dim T_i + \sum_{i=1}^r l_i(Y) \dim T_i \\
&= \sum_{i=1}^r (t_i + l_i(Y)) \dim T_i,
\end{align*}
\]
and thus we obtain \( v_i(Y) = t_i + l_i(Y) \leq v_i \) for all \( i = 1, \ldots, r \). ~\( \square \)

**Corollary 4.8.** Assume \( \lambda_i \geq v_i \) for all \( i = 1, \ldots, r \). Then \( \mathcal{F}_{Q, \mathbf{d}} \) is dense in \( Z_{Q, \mathbf{d}} \) and we have

\[
Z_{Q, \mathbf{d}} = \text{Gl}(\mathbf{d}) \ast \{ X_Y : Y \in \mathcal{M}_{Q, \mathbf{d}}' \}.
\]

**Proof.** Since we have \( \lambda_i \geq v_i \geq N \) for all \( i = 1, \ldots, r \), the variety \( Z_{Q, \mathbf{d}} \) is irreducible. The set \( \mathcal{F}_{Q, \mathbf{d}} \) is non-empty and open in \( Z_{Q, \mathbf{d}} \), hence it is dense. Moreover, for any \( X \) in \( \mathcal{F}_{Q, \mathbf{d}} \), the representation \( Y = H^{-1}(G(X)) \) lies in \( \mathcal{M}_{Q, \mathbf{d}}' \), where the functor \( H^{-1} \) is a quasi-inverse of \( H \), and we have \( X \cong X_Y \).

Therefore, by Proposition 4.7,

\[
\mathcal{F}_{Q, \mathbf{d}} = \text{Gl}(\mathbf{d}) \ast \{ X_Y : Y \in \mathcal{M}_{Q, \mathbf{d}}' \}. \quad \square
\]

**Proposition 4.9.** We assume \( \lambda_i \geq v_i \) for all \( i = 1, \ldots, r \). Let \( Y \) be in \( \mathcal{M}_{Q, \mathbf{d}}' \) and consider the representation

\[
X_Y = F(H(Y)) \oplus \bigoplus_{j \in \mathcal{S}} Z_j \oplus \bigoplus_{i=1}^r T_i^{\lambda_i - v_i(Y)}.
\]

Then we have

\[
\dim \text{Ext}(X_Y, X_Y) = \dim \text{Ext}(Y, Y) + s,
\]

where \( s = \# \mathcal{S} \).

**Proof.** Combining Lemma 3.1 and the fact that \( \bar{Z}_j = 0 \), \( j = r + 1, \ldots, n \), we get \( \text{Ext}(F(H(Y)), \bigoplus_{j \in \mathcal{S}} Z_j) = 0 \). Applying Proposition 3.15 and Lemma 3.1 again, we compute

\[
\dim \text{Ext}(X_Y, X_Y) = \dim \text{Ext}(F(H(Y)), F(H(Y))) + \dim \text{Ext}(\bigoplus_{j \in \mathcal{S}} Z_j, F(H(Y)))
\]

\[
= \dim \text{Ext}(H(Y), H(Y)) + \dim \text{Ext}(\bigoplus_{j \in \mathcal{S}} Z_j, H(Y))
\]

\[
= \dim \text{Ext}(Y, Y) + \dim \text{Hom}(\bigoplus_{j \in \mathcal{S}} \bar{T}_j, H(Y))
\]

\[
= \dim \text{Ext}(Y, Y) + s. \quad \square
\]
Corollary 4.10. Suppose \( \lambda_i \geq v_i \) for all \( i = 1, \ldots, r \). Then the following conditions are equivalent:

1. The null cone \( Z_{\mathcal{Q}, \mathcal{d}} \) is the closure of one stable \( \text{Gl}(\mathcal{d}) \)-orbit.
2. There is a representation \( X \) of \( \mathcal{Q} \) with an open \( \text{Gl}(\mathcal{d}) \)-orbit in \( Z_{\mathcal{Q}, \mathcal{d}} \).
3. There is a representation \( Y \) of \( \mathcal{Q}^\perp \) with an open \( \text{Gl}(\mathcal{e}) \)-orbit in \( \mathcal{M}_{\mathcal{Q}^\perp, \mathcal{e}^\perp} \).

Proof. By Lemma 2.2, a representation \( X \in Z_{\mathcal{Q}, \mathcal{d}} \) is stable iff \( X \) lies in \( \mathcal{E}_T \) and satisfies \( \text{dimExt}(X, X) = n - r \). If there is a representation \( X \in Z_{\mathcal{Q}, \mathcal{d}} \) such that the orbit \( \text{Gl}(\mathcal{d}) \cdot X \) is open in \( Z_{\mathcal{Q}, \mathcal{d}} \), then \( X \) lies in \( \mathcal{F}_{\mathcal{Q}, \mathcal{d}} \), and we have \( X \cong X_Y \) for some \( Y \in \mathcal{M}_{\mathcal{Q}^\perp, \mathcal{e}^\perp} \). By Proposition 5.1, the varieties \( \mathcal{M}_{\mathcal{Q}^\perp, \mathcal{e}^\perp} \) and \( \mathcal{M}_{\mathcal{Q}^\perp, \mathcal{e}^\perp} \) are irreducible of codimension \( n - r - s \) in \( \text{rep}(\mathcal{Q}^\perp, \mathcal{e}^\perp) \). Therefore, by Remark 2.1, for any representation \( Y \in \mathcal{M}_{\mathcal{Q}^\perp, \mathcal{e}^\perp} \), we have: \( \mathcal{M}_{\mathcal{Q}^\perp, \mathcal{e}^\perp} = \text{Gl}(\mathcal{e}) \cdot Y \) iff \( \text{dimExt}(Y, Y) = n - r - s \) iff \( \text{dimExt}(X_Y, X_Y) = n - r \) iff \( Z_{\mathcal{Q}, \mathcal{d}} = \text{Gl}(\mathcal{d}) \cdot X_Y \). □

5. The variety \( \mathcal{M}_{\mathcal{Q}^\perp, \mathcal{e}^\perp} \)

To simplify the notations, we will write in this section \( \mathcal{Q} \) instead of \( \mathcal{Q}^\perp \), and \( \mathcal{e} \) will denote the source-vector associated to \( \mathcal{Q} \). Therefore we have \( \mathcal{e} = \sum_{i \in S} \text{dim} P_i \) and \( \mathcal{M}_{\mathcal{Q}, \mathcal{e}} = \{ X \in \text{rep}(\mathcal{Q}, \mathcal{e}) : \text{Hom}(X, S_i) \neq 0, i \in \mathcal{Q}_0 \} \), where \( S \) is the set of sources of \( \mathcal{Q} \), \( P_i \) is the projective indecomposable and \( S_i \) is the simple representation of \( \mathcal{Q} \) attached to the vertex \( i \in \mathcal{Q}_0 \), respectively. Recall that \( \mathcal{Q} \) is a finite quiver containing no oriented cycles. We will assume that \( \mathcal{Q} \) is connected with \( \# \mathcal{Q}_0 \geq 2 \).

Definition 5.1. We put \( P := \bigoplus_{i \in \mathcal{Q}_0} P_i \), \( R := \bigoplus_{i \in \mathcal{Q}_0 \setminus S} P_i \), and we define the morphism \( \psi \in \text{Hom}(R, P) \) by

\[
\psi := \sum_{\alpha \in \mathcal{Q}_1} \theta_{\alpha}.
\]

(The morphism \( \theta_{\alpha} : P_{h_{\alpha}} \to P_{t_{\alpha}}, \alpha \in \mathcal{Q}_1 \), is defined in Section 2.) We put

\[
X := \text{coker} \psi.
\]

Remark 5.2. It is clear that \( \psi \) is injective and radical, thus the representation \( X \) lies in \( \mathcal{M}_{\mathcal{Q}, \mathcal{e}}^\prime = \{ X \in \text{rep}(\mathcal{Q}, \mathcal{e}) : \text{dim} \text{Hom}(Y, S_i) = 1, i \in \mathcal{Q}_0 \} \). Therefore, the sets \( \mathcal{M}_{\mathcal{Q}, \mathcal{e}}^\prime \) and \( \mathcal{M}_{\mathcal{Q}, \mathcal{e}} \) are not empty.

Proposition 5.3. The variety \( \mathcal{M}_{\mathcal{Q}, \mathcal{e}} \) is irreducible and its codimension in \( \text{rep}(\mathcal{Q}, \mathcal{e}) \) is equal to \( \# \mathcal{Q}_0 - \# S \).

Proof. If \( Y \) lies in \( \mathcal{M}_{\mathcal{Q}, \mathcal{e}} \), for every \( i \in \mathcal{Q}_0 \) there is a surjective morphism \( Y \to S_i \), and it is easy to see that there is an epimorphism \( Y \to S \), where \( S := \bigoplus_{i \in \mathcal{Q}_0 \setminus S} S_i \). By [16], the variety

\[
\mathcal{M}_{\mathcal{Q}, \mathcal{e}} = \{ A \in \text{rep}(\mathcal{Q}, \mathcal{e}) : \exists \text{ epimorphism } A \to S \}
\]

is irreducible.

Let \( Y \) be in \( \text{rep}(\mathcal{Q}, \mathcal{e}) \). If \( i \in S \), the condition \( \text{Hom}(Y, S_i) \neq 0 \) is always satisfied. If \( i \in \mathcal{Q}_0 \setminus S \), we denote by \( j_1 \xrightarrow{\alpha_1} i, \ldots, j_s \xrightarrow{\alpha_s} i \) all arrows of \( \mathcal{Q} \) ending at the vertex \( i \). (The vertices \( j_1, \ldots, j_s \) need not be distinct.) The matrix \( (Y(\alpha_1)| \cdots |Y(\alpha_s)) \) is a square matrix and we have \( \text{Hom}(Y, S_i) \neq 0 \) iff its determinant is zero. If \( i' \neq i \) is another vertex in \( \mathcal{Q}_0 \setminus S \), the two equations \( \det(Y(\alpha_1)| \cdots |Y(\alpha_s)) = 0 \) and \( \det(Y(\alpha'_1)| \cdots |Y(\alpha'_s)) = 0 \) have no common variables, thus the codimension of \( \mathcal{M}_{\mathcal{Q}, \mathcal{e}} \) in \( \text{rep}(\mathcal{Q}, \mathcal{e}) \) is given by \( \# \mathcal{Q}_0 - \# S \). □

Corollary 5.4. Let \( \mathcal{Q} \) be a quiver, \( \mathcal{e} \) the source vector associated to \( \mathcal{Q} \) and \( Y \in \mathcal{M}_{\mathcal{Q}, \mathcal{e}} \). Then the following conditions are equivalent:
(1) The $\text{Gl}(e)$-orbit of $Y$ is dense in $\mathcal{M}_{Q,e}$.
(2) $\dim \text{Ext}(Y, Y) = #Q_0 - #S$,
(3) $\dim \text{End}(Y) = #Q_0$.

**Proof.** We only have to show that the last two conditions are equivalent. We assume that $Y$ lies in $\mathcal{M}'_{Q,e}$, and using a minimal projective presentation of $Y$, we obtain an exact sequence

$$0 \to \text{Hom}(Y, Y) \to \text{Hom}(P, Y) \to \text{Hom}(R, Y) \to \text{Ext}(Y, Y) \to 0.$$ 

Thus we have

$$\dim \text{End}(Y) - \dim \text{Ext}(Y, Y) = \dim \text{Hom}(P, Y) - \dim \text{Hom}(R, Y) = \sum_{j \in S} \dim \text{Hom}(P_j, Y) = #S. \quad \square$$

We recall from Section 1 that an admissible tree is a tree which has no subquiver of type $\tilde{D}_m$, $m \geq 4$, with the following orientation

$$a_1 \quad a_2 \quad a_3 \quad a_4 \quad \ldots \quad a_{m-1} \quad a_m \quad a_{m+1}$$

This orientation for $\tilde{D}_m$ will be called the **critical orientation**.

From now on, we assume that $Q$ is an admissible tree. In particular, $Q$ is connected and does not contain any cycle. Thus we may associate a level to every vertex of $Q$ in the following way.

We fix a vertex $i_0$. For $i \in Q_0$, there is only one (non-oriented) path from $i$ to $i_0$, and we define the height of $i$ relative to $i_0$ to be the number of arrows contained in this path in the direction from $i$ to $i_0$ minus the number of arrows in the opposite direction. Let $b_0$ be a vertex such that its height relative to $i_0$ is minimal. For each $i \in Q_0$ we define the level of $i$ to be the height of $i$ relative to $b_0$. It is a non-negative integer and it will be denoted by $L(i)$. We have $L(b_0) = 0$. For every $m \in \mathcal{N}$ we put $L_m = \{i \in Q_0: L(i) = m\}$ and $p = p(Q) = \max\{m \in \mathcal{N}: L_m \neq \emptyset\}$. We define the sets $L_m^S := \{i \in L_m: i \notin S\}$ and $L_m^B := \{i \in L_m: i \notin B\}$, where $B$ is the set of sinks of $Q$.

There is a first obvious decomposition of $X$:

$$X = \bigoplus_{m=1}^{p} X_m \bigoplus \bigoplus_{b \in B} P_b,$$

where the representation $X_m$ is given by the exact sequence

$$0 \to V_m \xrightarrow{\psi_m} W_m \to X_m \to 0,$$

where $V_m := \bigoplus_{i \in L_m^S} P_i$, $W_m := \bigoplus_{i \in L_m^B} P_i$, and where $\psi_m$ is the restriction of $\psi$ to $V_m$. 

**Proposition 5.5.** Assume that the quiver $Q$ is an admissible tree. Then we have:

1. $\dim \text{Ext}(X_{n+1}, X_n) = \#(L^S_n \cap L^B_n)$ for all $n = 1, \ldots, p - 1$,
2. $\text{Ext}(X_n, X_m) = 0$ for all $n, m \in \{1, \ldots, p\}$ with $m \neq n + 1$.

**Proof.** Applying $\text{Hom}(\cdot, X_n)$ to

$$0 \to V_m \xrightarrow{\psi_m} W_m \to X_m \to 0,$$

we obtain the long exact sequence

$$0 \to \text{Hom}(X_m, X_n) \to \text{Hom}(W_m, X_n) \xrightarrow{\psi_m^n} \text{Hom}(V_m, X_n)$$

$$\to \text{Ext}(X_m, X_n) \to 0,$$

where $\psi_m^n$ is the linear map

$$(X_n(\alpha))_{\alpha \in Q_1, t_\alpha \in L_m} : \bigoplus_{i \in L^B_m} X_n(i) \to \bigoplus_{j \in L^S_{m-1}} X_n(j).$$

We have

$$\dim \text{Ext}(X_m, X_n) = \dim \text{Hom}(V_m, X_n) - \text{rank} \psi_m^n,$$

$$\dim \text{Hom}(V_m, X_n) = \dim \text{Hom}(V_m, W_n) - \dim \text{Hom}(V_m, V_n).$$

If $m > n + 1$, we have $\text{Hom}(V_m, X_n) = 0$, and thus $\text{Ext}(X_m, X_n) = 0$.

If $m = n + 1$, then $\psi_m^n = 0$ and we obtain:

$$\dim \text{Ext}(X_{n+1}, X_n) = \dim \text{Hom}(V_{n+1}, X_n)$$

$$= \dim \text{Hom}(V_{n+1}, W_n)$$

$$= \sum_{i \in L^S_n} \sum_{j \in L^B_n} \dim \text{Hom}(P_i, P_j)$$

$$= \#(L^S_n \cap L^B_n).$$

For $m \leq n$, we will show that the map $\psi_m^n$ is surjective.

Assume $m = n$. Since $X_n$ is given by the exact sequence

$$0 \to \bigoplus_{i \in L^S_{n-1}} P_i \xrightarrow{\psi_n} \bigoplus_{j \in L^B_n} P_j \to X_n \to 0,$$

where $\psi_n = \sum_{\alpha \in Q_1, t_\alpha \in L_n} \theta_\alpha$, the rank of the linear map

$$\psi_n^n = (X_n(\alpha))_{\alpha \in Q_1, t_\alpha \in L_n} : \bigoplus_{i \in L^B_n} X_n(i) \to \bigoplus_{j \in L^S_{n-1}} X_n(j)$$
is given by rank $\psi_n^m = \sum_{i \in L_{n-1}^S} (H_i - 1)$, where $H_i$ denotes the number of arrows $\alpha \in Q_1$ with $h_\alpha = i$. On the other hand, we see that

$$\dim \text{Hom}(V_n, X_n) = \dim \text{Hom}(V_n, W_n) - \dim \text{Hom}(V_n, V_n)$$

$$= \sum_{i \in L_{n-1}^S} \dim \text{Hom}(P_i, W_n) - \sum_{i \in L_{n-1}^S} \dim \text{Hom}(P_i, P_i)$$

$$= \sum_{i \in L_{n-1}^S} H_i - \#L_{n-1}^S,$$

thus the map $\psi_n^m$ is surjective.

Now assume $m < n$. Let $l$ be in $L_{n-1}^S$, and denote by $\beta_1 : b_1 \to l$, $\ldots$, $\beta_v : b_v \to l$ all arrows in $Q_1$ ending at $l$. Since $m < n$, the linear map

$$X_n^l := (X_n(\beta_1), \ldots, X_n(\beta_v)) : \bigoplus_{i = 1}^v X_n(b_i) \to X_n(l)$$

is an isomorphism. Therefore, the map

$$\left( X_n^l \right)_{l \in L_{m-1}^S} : \bigoplus_{l \in L_{m-1}^S} \bigoplus_{\beta \in Q_1, h_\beta = l} X_n(t_\beta) \to \bigoplus_{l \in L_{m-1}^S} X_n(l)$$

is an isomorphism as well. If $X_n(b) \neq 0$, $b \in L_m^S$, there is at most one arrow starting at $b$, otherwise the quiver

$$L_n : \bullet \quad \bullet$$

$$L_{n-1} : \bullet$$

$$\vdots$$

$$L_m : b$$

$$L_{m-1} : l \quad l'$$

is a subquiver of $Q$, and this contradicts our assumptions. We may assume $X_n(t_\beta) \neq 0$ for all $\beta \in Q_1$ with $h_\beta \in L_{m-1}^S$, therefore the vector spaces $\bigoplus_{l \in L_m^S} X_n(l)$ and $\bigoplus_{l \in L_{m-1}^S} \bigoplus_{\beta \in Q_1, h_\beta = l} X_n(t_\beta)$ are equal, and $\psi_m^n$ is the isomorphism $\left( X_n^l \right)_{l \in L_{m-1}^S}$. □

**Corollary 5.6.** Assume that $Q$ is an admissible tree. Then the $\text{Gl}(e)$-orbit of $X$ is open in $\mathcal{M}_{Q,e}$.

**Proof.** Using Corollary 5.4, it is enough to show that $\dim \text{Ext}(X, X) = \#Q_0 - \#S$. From

$$0 \to R \to P \to X \to 0$$
we obtain the long exact sequence

\[ 0 \to \text{Hom}(X, \bigoplus_{b \in B} P_b) \to \text{Hom}(P, \bigoplus_{b \in B} P_b) \to \text{Hom}(R, \bigoplus_{b \in B} P_b) \]

\[ \to \text{Ext}(X, \bigoplus_{b \in B} P_b) \to 0 \]

and we have \( \text{Hom}(X, \bigoplus_{b \in B} P_b) \cong \text{End}(\bigoplus_{b \in B} P_b) \cong \text{Hom}(P, \bigoplus_{b \in B} P_b) \), thus \( \dim \text{Ext}(X, \bigoplus_{b \in B} P_b) = \dim \text{Hom}(R, \bigoplus_{b \in B} P_b) = \sum_{b \in B} \dim \text{Hom}(P_b, P_b) = \# B \). (We use that \( P_b \) is simple projective.)

Using the last proposition, for \( X = \bigoplus_{m=1}^p X_m \oplus \bigoplus_{b \in B} P_b \), we obtain

\[
\dim \text{Ext}(X, X) = \dim \text{Ext}(X, \bigoplus_{b \in B} P_b) + \sum_{m=1}^{p-1} \#(L_m^S \cap L_m^B)
\]

\[ = \# B + \#(Q_0 \setminus (S \cup B)) \]

\[ = \# Q_0 - \# S. \quad \square \]

**Remark 5.7.** Theorem 1.1 follows from Corollary 5.6 and Corollary 4.10.

**Remark 5.8.** If \( Q \) if one of the following quivers, then \( \mathcal{M}_{Q, e} \) is not the closure of one \( \text{Gl}(e) \)-orbit.

\[
\begin{array}{ccc}
a_1 & \xrightarrow{a_2} & \\
\downarrow & & \downarrow \\
a_3 & \xrightarrow{a_4} & \\
\end{array}
\quad
\begin{array}{ccc}
b_1 & \xrightarrow{b_2} & \\
\downarrow & & \downarrow \\
b_3 & \xrightarrow{b_4} & \\
\end{array}
\quad
\begin{array}{ccc}
c_1 & \xrightarrow{c_2} & \\
\downarrow & & \downarrow \\
c_3 & \xrightarrow{c_4} & \\
\end{array}
\quad
\begin{array}{ccc}
c_5 & \xrightarrow{c_6} & \\
\downarrow & & \downarrow \\
c_5 & \xrightarrow{c_6} & \\
\end{array}
\]

Indeed, for the first quiver, a representation \( Y \in \text{rep}(Q, e) \) lies in \( \mathcal{M}_{Q, e} \) iff \( Y \cong P_{a_1} \oplus P_{a_4} \oplus C \) for some representation \( C \) with dimension vector \( (1, 1, 1) \). We have \( \dim \text{End}(Y) = \dim \text{Hom}(P_{a_1} \oplus P_{a_4}, Y) + \dim \text{End}(C) = 4 + \dim \text{End}(C) > 4 \).

For the second quiver, if a representation \( Y \in \text{rep}(Q, e) \) lies in \( \mathcal{M}_{Q, e} \), then \( Y \cong P_{b_4} \oplus P_{b_5} \oplus P_{b_6} \oplus D \) for some representation \( D \) with dimension vector \( (1, 2, 1, 1) \). We have \( \dim \text{End}(Y) = \dim \text{Hom}(P_{b_4} \oplus P_{b_5} \oplus P_{b_6}, Y) + \dim \text{End}(D) = 6 + \dim \text{End}(D) > 6 \).

A similar argument holds for the third quiver.

**Remark 5.9.** If \( Q \) contains a quiver \( \overleftrightarrow{D} \) with critical orientation as a subquiver, we have \( \text{Ext}(X_m, X_n) \neq 0 \) for some \( m \leq n \) and the \( \text{Gl}(e) \)-orbit of \( X \) is not open in \( \mathcal{M}_{Q, e} \).

Indeed, consider the quiver

\[
\begin{array}{c}
L_2 : \quad 1 \quad 2 \\
L_1 : \quad \alpha \quad \beta \\
L_0 : \quad 4 \quad 5 \\
\end{array}
\]

and choose $n = 2, m = 0$. However, there still exists a representation $Y = \text{Coker} \phi$ having an open orbit in $\mathcal{M}_{Q,e}$, but it is given by the morphism

$$\phi = \theta_\alpha + \theta_\beta + \theta_\gamma + \theta_\delta + \theta_\delta \alpha : \bigoplus_{i=3}^{5} P_i \to \bigoplus_{i=1}^{5} P_i.$$ 

The next step in this section is to decompose $X$ into a direct sum of indecomposables, since our aim is also to give a construction, as a direct sum of indecomposables, for a stable representation in $\mathcal{Z}_{Q,d}$.

For any level $m \in \{1, \ldots, p\}$, we define the following relation: for a pair of vertices $a_1, a_2 \in L_m$, we write $a_1 \lor a_2$ iff there is a vertex $b \in L_{m-1}$ and arrows $\alpha_1, \alpha_2 \in Q_1$ such that $t_\alpha = a_i, h_\alpha = b, i = 1, 2$. Of course $\lor$ is symmetric and reflexive. We extend $\lor$ by transitivity to an equivalence relation on $L_m$. We denote by $\text{these to equivalence classes } [l] in L_m$. We put

$$\nabla[l] := \{ i \in L_{m-1}: \exists \alpha \in Q_1, t_\alpha \in [l], h_\alpha = i \}.$$ 

Note that $\nabla[l]$ is non-empty, $l \in L_m$. We obtain the following decomposition of the representation $X_m$:

$$X_m = \bigoplus_{[l] \in \text{Lie}_m} X_m,[l],$$

where the representation $X_m,[l]$ is the quotient

$$X_m,[l] = \left( \bigoplus_{j \in [l]} P_j \right) / \left( \psi_m,[l] \left( \bigoplus_{i \in \nabla[l]} P_i \right) \right).$$

The map $\psi_m,[l] = \sum_{\alpha \in Q_1, t_\alpha \in [l]} \theta_\alpha$ is the restriction of $\psi$ to $\bigoplus_{i \in \nabla[l]} P_i$.

**Example 5.10.** Consider the following admissible tree:

$$L_2 : a_1 \quad a_2 \quad a_3$$

$$L_1 : \quad b_1 \quad b_2$$

$$L_0 : c_1$$

We have $X = X_{1,[b_1]} \oplus X_{2,[a_1]} \oplus P_{b_2} \oplus P_{c_1}$, where $[a_1] = \{a_1, a_2, a_3\}, [b_1] = \{b_1\}, X_{1,[b_1]} = S_{b_1}$, and

$$X_{2,[a_1]} = \begin{pmatrix} k & k & k \\ k^2 & 0 & k^2 \end{pmatrix}$$

**Proposition 5.11.** Let $m \in \{1, \ldots, p\}$ and $[l] \in \text{Lie}_m$. Then the representation $X_m,[l]$ is indecomposable and non-projective.
Proof. Consider the exact sequence

\[ 0 \to \bigoplus_{i \in \mathcal{V}[l]} P_i \xrightarrow{\psi_m[l]} \bigoplus_{j \in [l]} P_j \to X_{m,[l]} \to 0. \]

By Proposition 5.5, we have \( \text{Ext}(X_{m,[l]}, X_{m,[l]}) = 0 \), and we obtain the exact sequence

\[ 0 \to \text{Hom}(X_{m,[l]}, X_{m,[l]}) \to \text{Hom} \left( \bigoplus_{j \in [l]} P_j, X_{m,[l]} \right) \]

\[ \to \text{Hom} \left( \bigoplus_{i \in \mathcal{V}[l]} P_i, X_{m,[l]} \right) \to \text{Ext}(X_{m,[l]}, X_{m,[l]}) = 0. \]

We compute

\[
\dim \text{Hom}(X_{m,[l]}, X_{m,[l]}) = \dim \text{Hom} \left( \bigoplus_{j \in [l]} P_j, X_{m,[l]} \right) - \dim \text{Hom} \left( \bigoplus_{i \in \mathcal{V}[l]} P_i, X_{m,[l]} \right) \\
= \dim \text{Hom} \left( \bigoplus_{j \in [l]} P_j, \bigoplus_{h \in [l]} P_h \right) - \dim \text{Hom} \left( \bigoplus_{j \in [l]} P_j, \bigoplus_{h \in [l]} P_h \right) \\
- \dim \text{Hom} \left( \bigoplus_{i \in \mathcal{V}[l]} P_i, \bigoplus_{h \in [l]} P_h \right) + \dim \text{Hom} \left( \bigoplus_{i \in \mathcal{V}[l]} P_i, \bigoplus_{h \in [l]} P_h \right) \\
= #[l] - 0 - A[l] + \mathcal{V}[l] = 1,
\]

where \( A[l] = \{ \alpha \in Q_1 : t_{\alpha} \in [l] \} \). The last equality follows from the fact that the subquiver of \( Q \) consisting of the vertices \( j \in [l] \cup \mathcal{V}[l] \) and the arrows \( \alpha \in A[l] \) is a connected tree. Therefore \( X_{m,[l]} \) is indecomposable.

Moreover, since the map \( \psi_m[l] \) is radical, it is not a section and the exact sequence

\[ 0 \to \bigoplus_{i \in \mathcal{V}[l]} P_i \xrightarrow{\psi_m[l]} \bigoplus_{j \in [l]} P_j \to X_{m,[l]} \to 0 \]

does not split. Therefore, the representation \( X_{m,[l]} \) is non-projective. \( \square \)

6. A sharp bound on the integers \( v_1, \ldots, v_r \)

In this section we prove Theorem 1.4.

We assume that \( Q \) is a connected Dynkin quiver, and \( \lambda_i \geq 3 \) for all \( i = 1, \ldots, r \). Therefore, the variety \( Z_{Q,d} \) is irreducible [17]. (Note that in the particular case where \( Q \) is a Dynkin quiver of type \( A_n \), we only need the condition \( \lambda_i \geq 2 \) for all \( i = 1, \ldots, r \), if we want to guarantee that \( Z_{Q,d} \) is irreducible.) Since \( Q \) is of finite representation type [10], there exists a representation \( X \in Z_{Q,d} \) such that its \( \text{GL}(d) \)-orbit is dense in \( Z_{Q,d} \). Moreover, every dimension vector for \( Q \) is prehomogeneous. We will prove that the set

\[ \mathcal{W} := \{ A \in Z_{Q,d} : \text{Ext}(A, T) = 0 = \text{Ext}(T, A) \} \]
is not empty. As a consequence, since $W$ is open in $Z_{Q,d}$, the generic representation $X$ in $Z_{Q,d}$ lies in $W$, thus its $Gl(d)$-orbit is stable by Lemma 2.2. We define the sets

$$W_1 := \{ A \in Z_{Q,d}: \text{Ext}(T, A) = 0 \}$$

and

$$W_2 := \{ A \in Z_{Q,d}: \text{Ext}(A, T) = 0 \}.$$ 

Since $Z_{Q,d}$ is irreducible, it is enough to prove that the open sets $W_1$ and $W_2$ are both non-empty. We will use the following lemma.

**Lemma 6.1.** Let $i \in \{1, \ldots, r\}$ and $j \in S$. Assume that any non-zero morphism $T_j \to T_i$ factors through $T_\ell$, for some $\ell \in S$, $\ell \neq j$. Then $T_i$ is not a direct summand of $T_j^{++}$.

**Proof.** Let $i \in \{1, \ldots, r\}$, $j \in S$, and $f : T_j \to T_i$ be a non-zero morphism. By assumption, there is some $\ell \in S$, $\ell \neq j$, such that $f$ factors through $T_\ell$. Since $j$ and $\ell$ are two different sources of $Q^+$, we have $\text{Hom}(T_j, T_\ell) = 0$, thus any morphism $T_j \to T_i$ factors through $trT_\ell \in \text{add}(T)$. Moreover, since $\text{Hom}(T_i, T_\ell) \neq 0$, we have $\text{Hom}(T_i, T_\ell) = 0$ [1, IX.1.1], and thus $T_i$ cannot be a direct summand of $trT_\ell$.

This proves that any non-zero morphism $T_j \to T_i$ factors through some representation in $\text{add}(T_1 \oplus \cdots T_{i-1} \oplus T_{i+1} \oplus \cdots T_r)$. Therefore, there exists a morphism $T_j \to T_i^c \oplus \cdots \oplus T_{r-c}^c \oplus T_{i-1}^c \oplus \cdots \oplus T_{r-c}^c$ which satisfies property (1) of a source map from $T_j$ to $\text{add}(T)$ (see Section 3). Thus, by property (2), $T_i$ cannot be a direct summand of $T_j^{++}$.  

**Proposition 6.2.** The set $W_1$ is not empty.

**Proof.** Using Remark 4.5, we choose a representation $Y \in M'_{Q^+,d^+}$ and we obtain a representation

$$X = H(Y) \oplus \bigoplus_{j \in S} Z_j \oplus \bigoplus_{i=1}^r T_i^{\lambda_i - u_i(Y)}$$

in $W_1$, provided $u_i(Y) \leq \lambda_i$ for all $i = 1, \ldots, r$. (We consider here the representation $H(Y)$ instead of $F(H(Y))$, since we do not need the condition $\text{Ext}(X, T) = 0$.) The dimension vector of the representation $H(Y) \oplus \bigoplus_{j \in S} Z_j$ is the following vector

$$\sum_{i=1}^r u_i(Y) \dim T_i = \sum_{j \in S} \dim T_j^{++} - \sum_{j \in S} \dim trT_j.$$ 

We have $\bigoplus_{j \in S} T_j^{++} \cong \bigoplus_{i=1}^r T_i^{b_i}$ for some integers $b_1, \ldots, b_r$, and it suffices to show $b_i \leq 3$ for all $i = 1, \ldots, r$.

a) First we assume that the Dynkin quiver $Q$ is of type $A_n$. We will show that in this case we have $b_i \leq 2$ for all $i = 1, \ldots, r$.

A path $U_0 \to U_1 \to \cdots \to U_m$ in the Auslander–Reiten quiver $\Gamma_Q$ is called **sectional** if $U_{l-1}$ is not the Auslander–Reiten translate of $U_{l+1}$, for $l = 1, \ldots, m - 1$. Let $i \in \{1, \ldots, r\}$. Since $Q$ is a Dynkin quiver of type $A_n$, there are at most two maximal sectional paths in $\Gamma_Q$ ending at the vertex $T_i$. Let $j \in S$. If $\text{Hom}(T_j, T_i) = 0$, the representation $T_i$ is not a direct summand of $T_j^{++}$. Assume $\text{Hom}(T_j, T_i) \neq 0$. Since $\text{Ext}(T_i, T_j) = 0$ and $Q$ is of type $A_n$, there is a unique path in $\Gamma_Q$ from $T_j$ to $T_i$, and this path is sectional [1, IX.6.4]. By Lemma 6.1, this sectional path contains at most one vertex $T_i$, $i \in S$, such that $T_i$ is a direct summand of $T_j^{++}$. This shows $b_i \leq 2$. 


b) Now we assume that $Q$ is a Dynkin quiver (of type $D_n$, $E_6$, $E_7$ or $E_8$). Let $i \in \{1, \ldots, r\}$ and assume, for a contradiction, that $b_i \geq 4$. Since $Q$ is of finite representation type, we have $\dim \text{Hom}(T_j, T_i) \leq 1$ for all $j = r + 1, \ldots, n$ (This is a straightforward computation using the Tits form associated to the quiver $Q$, compare with [17], 3.3.) As a consequence, if $T_i$ is a direct summand of $T_i^{++}$, for some $j \in S$, its multiplicity is only 1. Therefore, there are at least 4 distinct vertices $j_1, \ldots, j_4 \in S$ such that $T_i$ is a direct summand of $T_{jl}^{++}$.

By Lemma 6.1, if $l \in \{1, \ldots, 4\}$, there is a non-zero morphism $T_{jl} \to T_i$ which does not factor through $T_j$, for any $j \in S \setminus \{j_l\}$. Recall that the direct sum $\bigoplus_{h=1}^{n} T_h$ is a tilting module. Let us consider the tilted algebra

$$B = \text{End} \left( \bigoplus_{h=1}^{n} T_h \right)$$

and its associated bound quiver $(Q_B, I)$. The vertices $h$ of $Q_B$ correspond bijectively to the indecomposable representations $T_h$, $h = 1, \ldots, n$, and there is an arrow $\beta : h_1 \to h_2$ in $Q_B$ iff $h_1 \neq h_2$ and there is a non-zero morphism $T_{h_2} \to T_{h_1}$ which does not factor through $T_h_{h_3}$, for any $h_3 \in \{1, \ldots, n\} \setminus \{h_1, h_2\}$ (see [1, II.3 and VI.3] for more details). It follows that for each $l = 1, \ldots, 4$, there exists a path $i \to \cdots \to j_l$ in $Q_B$, which does not go through any vertex $j_{l'}$, $l' \in \{1, \ldots, 4\}$, $l' \neq l$. Therefore, $Q_B$ has to contain a quiver of the following type as a subquiver:

We see that in each case, the quiver $Q_B$ contains an Euclidean quiver of type $\tilde{D}_m$ as a subquiver. Moreover, by construction, there is no zero relation on this subquiver. Therefore, the algebra $B$ is of infinite representation type by [12]. (Indeed, comparing with [3, 4.1], we can see the algebra $B$ as a $k$-category containing a tame concealed algebra of type $\tilde{D}_m$ as a subcategory.) But this is a contradiction, since any tilted algebra of Dynkin type is of finite representation type [1, VIII.3.2]. □

**Proposition 6.3.** The set $W_2$ is not empty.

**Proof.** There is a duality $D : \text{rep}(Q) \to \text{rep}(Q^{\text{op}})$ between the category of representations of $Q$ and the category of representations of the opposite quiver $Q^{\text{op}}$ [1], and we have $\text{dim} DT = \text{dim} T = d$. Since $Q^{\text{op}}$ is a connected Dynkin quiver (of the same type as $Q$), Proposition 6.2 holds in $\text{rep}(Q^{\text{op}})$ and we obtain a representation $X$ in $W_1^{\text{op}} := \{Y \in Z_{Q^{\text{op}}, d} : \text{Ext}_{Q^{\text{op}}}(DT, Y) = 0\}$. We have $\text{Ext}(DX, T) = 0$, and using a dual description of $Z_{Q, d}$ [19,17], it is easy to see that the representation $DX$ lies in $W_2$. □
7. Construction of a stable representation $X$ in $\mathcal{Z}_{Q,d}$

In this section, we give a method for constructing a stable representation $X$ in $\mathcal{Z}_{Q,d}$ as a direct sum of indecomposables.

Let $Q$ be a quiver and $d$ be a prehomogeneous dimension vector for $Q$. We determine indecomposable pairwise non-isomorphic representations $T_1, \ldots, T_r$ of $Q$ such that $\text{Ext}(T_i, T_j) = 0$ for all $i, j \in \{1, \ldots, r\}$, and $d = \sum_{i=1}^r \lambda_i \dim T_i$, for some positive integers $\lambda_1, \ldots, \lambda_r$. We assume that all hypotheses of Theorem 1.1 are satisfied. Then, we do the following steps:

1. Determine the category $T^\perp$.
2. Determine the quiver $Q^\perp$ and the set $S$ of its sources.
3. Construct a generic representation $Y$ in $\mathcal{M}_{Q^\perp, e^\perp}$ (i.e. the $\text{Gl}(e^\perp)$-orbit of $Y$ is dense in $\mathcal{M}_{Q^\perp, e^\perp}$), as a direct sum of indecomposables $Y = \bigoplus_{l \in L} Y_l$.
4. For each $l \in L$, determine the indecomposable representation $H(Y_l)$ of $Q$ corresponding to the representation $Y_l$ of $Q^\perp$, via the equivalence of categories $H : \text{rep}(Q^\perp) \sim \to T^\perp$.
5. For each $l \in L$, construct the indecomposable representation $F(H(Y_l))$.
6. Construct the representations $Z_j, j \in S$.
7. Compute the integers $v_i(Y), i = 1, \ldots, r$.
8. Put

$$X = \bigoplus_{l \in L} F(H(Y_l)) \oplus \bigoplus_{j \in S} Z_j \oplus \bigoplus_{i=1}^r T_i^{\lambda_i - v_i(Y)}.$$

Example 7.1. Let $Q$ be the quiver

```
  1   2
 / \ / \n3   \   4
|   \ |   |
|    v |
```

and $d = \lambda \left(\begin{array}{cc}1 & 1 \\ 2 & 1 \end{array}\right) = \dim T_1$ with $\lambda \geq 3$. We denote by $\tau$ the Auslander–Reiten translate. The Auslander–Reiten quiver $\Gamma_Q$ is the following quiver:

```
  P_1  \tau^{-1} P_1  \tau^{-1} P_3  I_1
     \     \     \     
  P_2  \tau^{-1} P_2  \tau^{-1} P_3  I_2
     \     \     \     
P_3  \tau^{-1} P_3  \tau^{-1} P_3  I_3
     \     \     \     
P_4  \tau^{-1} P_3  \tau^{-1} P_3  I_4
     \     \     \     
S_3  \tau^{-1} P_3  \tau^{-1} P_3  I_4
```
We label each vertex of $\Gamma_Q$ with the dimension vector of a corresponding indecomposable:

The double frame corresponds to the representation $T_1 = \tau^* P_3$, and we have $T = T_1^\perp$. The non-dotted frames correspond to the indecomposable representations in $T^\perp$: $P_1$, $P_2$ and $S_3$. Since $n - r = 4 - 1 = 3$, the quiver $Q^\perp$ has three vertices. There are only three indecomposable representations in $T^\perp$, up to isomorphism, thus $Q^\perp$ has no arrow. Since $n - r = 4 - 1 = 3$, the quiver $Q^\perp$ has three vertices. There are only three indecomposable representations in $T^\perp$, up to isomorphism, thus $Q^\perp$ has no arrow. We can write $Q^\perp_3 = S = \{2, 3, 4\}$. Since $\lambda = 1$, we have $F(H(Y_l)) = H(Y_l)$, $l = 1, 2, 3$. The representations $Z_j$, $j \in S$, are given by the exact sequences

$$\begin{align*}
0 \rightarrow P_1 \rightarrow T_1 \rightarrow Z_2 \rightarrow 0, \\
0 \rightarrow P_2 \rightarrow T_1 \rightarrow Z_3 \rightarrow 0, \\
0 \rightarrow S \rightarrow T_1 \rightarrow Z_4 \rightarrow 0,
\end{align*}$$

thus we have $Z_2 = \tau P_1$, $Z_3 = \tau P_2$, $Z_4 = I_4$. We have used three copies of $T_1$ to construct $\bigoplus_{j \in S} Z_j$, and none to construct $F(H(Y_l))$, thus we have $v_1(Y_l) = 3$. Finally, we put

$$X = P_1 \oplus P_2 \oplus S_3 \oplus \tau P_1 \oplus \tau P_2 \oplus I_4 \oplus T_1^\perp,$$

and we obtain $Z_{Q, d} = \text{Gl}(d) \star X$.

If $\lambda < 3$, we do not have enough copies of $T_1$ to construct the representation $X$. In fact, if $\lambda = 2$, the variety $Z_{Q, d}$ has two irreducible components of codimension 3 in $\text{rep}(Q, d)$:

$$\text{Gl}(d) \star (P_1 \oplus P_2 \oplus S_3 \oplus I_3)$$

and

$$\text{Gl}(d) \star (P_3 \oplus \tau P_1 \oplus \tau P_2 \oplus I_4).$$

If $\lambda = 1$, the variety $Z_{Q, d}$ is irreducible of codimension 2. It is the closure of the $\text{Gl}(d)$-orbit of $P_3 \oplus I_3$, but this orbit is not stable, since the representation $P_3 \oplus I_3$ is not in $E_T$. 
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References