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# Error Bounds for a Numerical Solution for Dynamic Economic Models

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**Abstract**—In this paper, we analyze a discretized version of the dynamic programming algorithm for a parameterized family of infinite-horizon economic models, and derive error bounds for the approximate value and policy functions. If $h$ is the mesh size of the discretization, then the approximation error for the value function is bounded by $Mh^2$, and the approximation error for the policy function is bounded by $Nh$, where the constants $M$ and $N$ can be estimated from primitive data of the model.

**Keywords**—Dynamic programming, Value and policy functions, Error bounds, Numerical solutions.

## 1. INTRODUCTION

In this paper, we consider a family of infinite-horizon models of economic growth. Via a dynamic programming algorithm, we analyze a numerical discretization procedure to compute the value and policy functions. We show that under the proposed scheme the value function converges quadratically to the true value function and the policy function converges linearly, as the mesh size of the discretization goes to zero. Furthermore, the constants involved in the orders of convergence can be computed from primitive data of the model.

Our orders of convergence are stronger than those typically found in related control literature (e.g., see [1–3], and references therein). We should note, however, that these higher order estimates are obtained at the expense of further concavity and interiority assumptions embedded in our optimization problem. These later assumptions are commonplace in economic models but are generally restrictive in some other areas.

Our results are based upon differentiability properties of the value function. It is known from the analysis in [4,5] that under strong concavity and interiority assumptions (and an appropriate smoothness hypothesis), the value function is a $\mathcal{C}^2$ mapping. It is also known [6,7] that under such regularity conditions the value function may fail to be differentiable of class $\mathcal{C}^3$. Hence, differentiability analysis suggests that without further specific restrictions, higher orders of convergence for the computed value function beyond the quadratic one may not be available.

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## 2. THE MODEL AND PRELIMINARY RESULTS

Consider the following family of Ramsey-type models of capital accumulation

$$
\begin{aligned}
W(k_0,\theta) = & \sup_{\{k_t\}_{t\geq 0}} \sum_{t=0}^{\infty} \beta^t v(k_t, k_{t+1}, \theta) \\
\text{s. t. } & k_{t+1} \in \Omega(k_t,\theta), \quad k_t \in K \subset \mathcal{R}^n, \quad \theta \in \Theta \subset \mathcal{R}^m, \\
& k_0 \text{ fixed}, \quad t = 0,1,\ldots, \quad \text{and} \quad 0 < \beta < 1.
\end{aligned} \tag{2.1}
$$

ASSUMPTION A. *The sets $K$ and $\Theta$ have nonempty interiors. The correspondence $\Omega : K \times \Theta \to K$ is continuous; moreover, for each fixed $\theta$ in $\Theta$, the relation $\Omega(\cdot,\theta)$ in $K$ has convex graph with nonempty interior.*

ASSUMPTION B. *Let $X = \operatorname{graph}(\Omega)$. The mapping $v : X \to \mathcal{R}$ is bounded, continuous, and on the interior of the domain, it is differentiable of class $C^2$ with bounded first- and second-order partial derivatives. Moreover, for all fixed $\theta$, there exists some constant $\alpha > 0$ such that $v(k,k',\theta) + (\alpha/2)\|k'\|^2$ is concave as a function on $(k,k')$.*

ASSUMPTION C. *For each $(k_0,\theta)$ in $K \times \Theta$, there exists an optimal solution $\{k_t\}_{t\geq 0}$ to problem (2.1) with the property that $k_{t+1} \in \operatorname{int}[\Omega(k_t,\theta)]$ for every $t \geq 0$.*

The foregoing assumptions are entirely standard in the economic literature and are usually presumed to hold over a certain compact domain which comprises the asymptotic or recurrent dynamics of the optimal law of motion (e.g., see [4,8]). Optimization problem (2.1) corresponds to a standard planning problem written in "reduced form." This framework is suitable to perform the typical exercises on sensitivity analysis over a relevant parameter's space $\Theta$ (cf. [5]). In Assumption B, $\|\cdot\|$ denotes the Euclidean norm. Hence, such an assumption imposes a strong form of concavity on the second component of the function $v$, and over compact sets, the assumption is weaker than the more conventional form of strong concavity. The interiority requirement in Assumption C is indispensable for our results below (e.g., see the example in [4]).

Under the above hypotheses, the *value function* $W$ given in (2.1) is well defined and jointly continuous (cf. [8]). Moreover, for fixed $\theta$, the mapping $W(\cdot,\theta)$ is concave on $K$, and satisfies the so-called Bellman equation

$$
\begin{aligned}
W(k_0,\theta) = & \sup_{k_1} v(k_0,k_1,\theta) + \beta W(k_1,\theta) \\
\text{s. t. } & k_1 \in \Omega(k_0,\theta).
\end{aligned} \tag{2.2}
$$

The optimal value is attained at a unique point given by the *policy function* $k_1 = g(k_0,\theta)$. The policy function is also continuous. Moreover, it follows from these definitions that $\{k_t\}_{t\geq 0}$ is an optimal solution to (2.1) if and only if it satisfies equation (2.2) at all times.

We recall that the value function $W$ may be obtained as the unique fixed point of the following dynamic programming algorithm. Let $\mathcal{W}$ be the space of bounded continuous functions $V$ on $K \times \Theta$ with the norm $\|V\| = \sup_{(k,\theta)\in K\times\Theta} |V(k,\theta)|$. Define the nonlinear operator $T : \mathcal{W} \to \mathcal{W}$ by

$$
\begin{aligned}
T(V)(k_0,\theta) = & \sup_{k_1} v(k_0,k_1,\theta) + \beta V(k_1,\theta) \\
\text{s. t. } & k_1 \in \Omega(k_0,\theta)
\end{aligned} \tag{2.3}
$$

for $V \in \mathcal{W}$. It is a well-established fact (e.g., see [8]) that $T$ is a contractive mapping on $\mathcal{W}$ with modulus $0 < \beta < 1$; i.e., $\|TV_0 - TV_1\| \leq \beta\|V_0 - V_1\|$ for $V_0, V_1 \in \mathcal{W}$. It follows that $W$ is the unique fixed point under $T$, and $\|W - V_n\| \leq \beta^n\|W - V_0\|$ for $V_n = T^n V_0$, where $T^n$ is the $n$-times composition of $T$.

It is well known that in general, the value function $W$ may fail to be differentiable (cf. [2]). Under the asserted assumptions, however, a simple extension of the analysis of Santos [4,5] suffices to validate in the present setting the differentiability of class $\mathcal{C}^2$ of the value function $W$ and the differentiability of class $\mathcal{C}^1$ of the policy function $g$ at every interior point $(k,\theta)$ in $K\times\Theta$. Moreover, Araujo [6] and Santos [7] provide some simple counterexamples in which, under the above conditions, the value function fails to be a $\mathcal{C}^3$ mapping even if $v$ is infinitely differentiable.

What has been neglected in the differentiability analysis, and it is generally useful for the design of computational procedures, is that these derivatives of $W$ may be bounded in terms of defining data of the model.[1] More specifically, regarding second-order differentiability, it follows from [4] that $D_{11}W(k_0,\theta)$ can be characterized as a solution to the following quadratic optimization problem:

$$x_0\cdot D_{11}W(k_0,\theta)\cdot x_0=\max_{\{x_t\}_{t\geq 0}}\sum_{t=0}^{\infty}\beta^t\,(x_t,x_{t+1})\cdot D^2v_\theta(k_t,k_{t+1})\cdot(x_t,x_{t+1}) \qquad \text{s. t. } x_0 \text{ fixed.} \tag{2.4}$$

Here the maximization proceeds over all vector sequences $\{x_t\}_{t\geq 0}$ with fixed $x_0$, the one-period objective $D^2v_\theta(\cdot,\cdot)$ is the Hessian matrix of the mapping $v(\cdot,\cdot,\theta)$ for $\theta$ fixed, and $\{k_t\}_{t\geq0}$ is the optimal solution to maximization problem (2.1) for $k_0$ given.

From this characterization, one can also show that the optimal plan $\{x_t^*\}_{t\geq0}$ to optimization problem (2.4) determines the derivative of the policy function $g$ with respect to $k_0$. That is, $x_t^*=D_1g^t(k_0,\theta)\cdot x_0$ for $t\geq1$, where $D_1g^t(k_0,\theta)$ denotes the derivative of the $t$-times composite $g(g(\dots g(k_0,\theta),\dots,\theta),\theta)$ with respect to $k_0$. Given that $(x_0,0,0,0,\dots)$ is a feasible solution to maximization problem (2.4), we must then have

$$\|D_{11}W(k_0,\theta)\|\leq\|D_{11}v(k_0,k_1,\theta)\|\leq L, \tag{2.5}$$

where $L=\|D_{11}v\|=\sup_{(k_0,k_1,\theta)\in X}\|D_{11}v(k_0,k_1,\theta)\|$. (In these calculations, for a matrix of derivatives $D_{11}v(k_0,k_1,\theta)$, we have used the notation $\|D_{11}v(k_0,k_1,\theta)\|=\max_{\eta\in R^n,\eta\neq0}(\|D_{11}v(k_0,k_1,\theta)\cdot\eta\|/\|\eta\|)$.) Moreover, if $\{x_t^*\}_{t\geq0}$ is an optimal solution to (2.4) with $\|x_0^*\|=1$, then by virtue of the asserted concavity of $v$ (Assumption B), we obtain that

$$\sum_{t=0}^{\infty}\beta^t\left(x_{t+1}^*\cdot x_{t+1}^*\right)\leq\frac{L}{\alpha}, \tag{2.6}$$

where $x_{t+1}^*\cdot x_{t+1}^*$ denotes inner vector multiplication. Note that (2.6) places an upper bound on the growth factor of the derivative of the policy function $g$. Indeed, for any $t$, it must hold that

$$\beta^t\left\|D_1g^{t+1}(k_0,\theta)\right\|^2\leq\frac{L}{\alpha}. \tag{2.7}$$

On the other hand, as shown in [5], the cross-partial derivatives $D_{12}W(k_0,\theta)$ and $D_{21}W(k_0,\theta)$ can be determined by the following computations:

$$\begin{aligned}D_{12}W(k_0,\theta)^T&=D_{21}W(k_0,\theta)\\&=\sum_{t=0}^{\infty}\beta^t\left[D_{31}v(k_t,k_{t+1},\theta)\cdot D_1g^t(k_0,\theta)+D_{32}v(k_t,k_{t+1},\theta)\cdot D_1g^{t+1}(k_0,\theta)\right].\end{aligned}$$

Now, making use of (2.7), and after some simple calculations, we obtain that

$$\|D_{12}W\|=\|D_{21}W\|\leq\left[1+\frac{2(L/\alpha)^{1/2}}{1-\beta^{1/2}}\right]G, \tag{2.8}$$

where $G=\max\{\|D_{31}v\|,\|D_{32}v\|\}$. Similar upper bounds can be found for $\|D_2g\|$ and $\|D_{22}W\|$.

[1]For functions $v: R^n\times R^n\times R^m\to R$, $Dv(k_0,k_1,\theta)$ will denote the derivative of $v$ evaluated at the point $(k_0,k_1,\theta)$, and $D_iv(k_0,k_1,\theta)$, $i=1,2,3$, will denote the (first-order) partial derivative of $v$ with respect to $i^{\text{th}}$ component variable. Similarly, $D_{ij}v(k_0,k_1,\theta)$ will denote a second-order partial derivative of $v$ with respect to the $i^{\text{th}}$ and $j^{\text{th}}$ components.

# 3. A DISCRETIZATION OF THE DYNAMIC PROGRAMMING ALGORITHM

In most applications, one must face the problem of computing the value function $W$. Several computational procedures are available in the existing literature (e.g., [1–3,9]). We consider here a discretized version of the dynamic programming algorithm as outlined in [10]. Our main goal is to establish the quadratic convergence of the sequence of approximate value functions as the mesh size of the discretization goes to zero. The implementational aspects of this numerical approach as well as an extension to stochastic models are dealt with in [11].

Let us assume that the state space $K \times \Theta$ is a polyhedron. This does not entail much loss of generality for most economic applications. Let $\{S^j\}$ be a family of simplices which conform a triangulation of $K \times \Theta$ (i.e., $\cup_j S^j = K \times \Theta$ and $\operatorname{int}(S^i) \cap \operatorname{int}(S^j) \neq \emptyset$ for every pair of simplices $S^i, S^j$).[2] Let

$$h = \sup_j \operatorname{diam}\{S^j\} < +\infty.$$

Let $(k^j, \theta^j)$ be a generic vertex of the triangulation. Consider the space of piecewise affine functions

$$\mathcal{W}^h = \left\{ \begin{array}{l} V^h : K \times \Theta \to \mathcal{R} \mid V^h \text{ is bounded, continuous,} \\ \text{and } DV^h \text{ is constant in int}\left(S^j\right) \text{ for each } S^j \end{array} \right\}.$$

Observe that $\mathcal{W}^h$ is a closed subspace of $\mathcal{W}$, equipped with the norm $\|V^h\| = \sup_{(k,\theta)\in K\times\Theta} |V^h(k,\theta)|$. Define the mapping $T^h : \mathcal{W} \to \mathcal{W}^h$, given by

$$\begin{gathered} T^h(V)\left(k_0^j, \theta^j\right) = \sup_{k_1} v\left(k_0^j, k_1, \theta^j\right) + \beta V\left(k_1, \theta^j\right) \\ \text{s. t. } k_1 \in \Omega\left(k_0^j, \theta^j\right) \\ \text{for each vertex point } \left(k_0^j, \theta^j\right) \text{ and } V \in \mathcal{W}. \end{gathered} \tag{3.1}$$

Note that the maximization operation on the right-hand side of (3.1) must be performed exactly. Also, nodal values $T^h(V)(k_0^j, \theta^j)$ for all vertex points $(k_0^j, \theta^j)$ yield a unique functional extension to the whole domain $K \times \Theta$ over the space of piecewise linear functions compatible with the given triangulation $\{S^j\}$.

LEMMA 2.1. *Under Assumptions (A)–(C), equation (3.1) has a unique fixed point $W^h$ in $\mathcal{W}^h$.*

The proof is the standard one (cf. [10]). One immediately checks that $T^h$ is a contraction mapping with modulus $0 < \beta < 1$. By a well-known fixed-point theorem, equation (3.1) has a unique fixed point $W^h$ in $\mathcal{W}^h$.

LEMMA 2.2. *Let $W$ be the value function defined in (2.2). Let $\gamma = (n+m)\|D^2W\|_{L_\infty(K\times\Theta)}$. Then under Assumptions (A)–(C), we have $\|TW - T^hW\| \le (\gamma/2)h^2$.*

The proof of this result is also standard, and it amounts to an application of Taylor's theorem (cf. [11]). Observe that by the definitions of $T$ and $T^h$, we have $TW(k^j,\theta^j) = W(k^j,\theta^j) = T^hW(k^j,\theta^j)$ for every vertex point $(k^j,\theta^j)$, and that the function $T^hW$ is piecewise affine. Hence the lemma follows from well-known results on piecewise affine approximations (cf. [12, Theorem 2.1.4.1]).[3]

[2]This kind of subdivision is not necessary for our results. For instance, rectangular subdivisions may sometimes be more suitable to certain applications.

[3]Note that the result in [12] applies only to univariate functions $f$; in the univariate case, the constant involved in the approximation is $\gamma/8$ (i.e., $|f(x) - f(x^j)| \le Lh^2$ for $L = \gamma/8$). One can show that a multivariate extension of this result is available for $L = \gamma/2$, where $\gamma = (n+m)\|D^2W\|_{L_\infty(K\times\Theta)}$, and $\|D^2W\|_{L_\infty(K\times\Theta)} = \sup_{(k_0,\theta)\in K\times\Theta} \|D^2W(k_0,\theta)\|_{\max}$, for $\|\cdot\|_{\max}$ the matrix max norm.

THEOREM 2.3. *Let $W$ be the fixed point of (2.3) and $W^h$ be the fixed point of (3.1). Then under Assumptions (A)–(C), we have $\|W - W^h\| \le (\gamma/2(1-\beta))h^2$.*

PROOF. Let $T$ and $T^h$ be as defined previously from (2.3) and (3.1), respectively. Then

$$\begin{aligned}\|W - W^h\| &= \|TW - T^hW^h\| \le \|TW - T^hW\| + \|T^hW - T^hW^h\| \\ &\le \|TW - T^hW\| + \beta\,\|W - W^h\|,\end{aligned}$$

where use is made in these computations of the triangle inequality and of Lemma 2.1. Therefore,

$$\|W - W^h\| \le \frac{1}{(1-\beta)}\,\|TW - T^hW\|.$$

Theorem 2.3 is now a direct consequence of Lemma 2.2.

COROLLARY 2.4. *Let $g(k^j, \theta^j)$ be the optimal policy for the original value function $W$ at a vertex point $(k^j, \theta^j)$, and let $g^h(k^j, \theta^j)$ be the optimal policy for the approximate value function $W^h$ at vertex point $(k^j, \theta^j)$. Then $\|g(k^j, \theta^j) - g^h(k^j, \theta^j)\| \le (2\gamma/\alpha(1-\beta))^{1/2}h$, for every vertex point $(k^j, \theta^j)$.*

This result follows essentially from the asserted concavity of the instantaneous return function $v$ (Assumption B); for a related proof, see [11]. The linear convergence in Corollary 2.4 can readily be extended to any arbitrary point $(k, \theta)$ in $K \times \Theta$.

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