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Derivations on symmetric quasi-Banach ideals of compact operators

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ABSTRACT

Let $\mathcal{I}, \mathcal{J}$ be symmetric quasi-Banach ideals of compact operators on an infinite-dimensional complex Hilbert space H, let $\mathcal{J}:\mathcal{I}$ be the space of multipliers from $\mathcal{I}$ to $\mathcal{J}$. Obviously, ideals $\mathcal{I}$ and $\mathcal{J}$ are quasi-Banach algebras and it is clear that ideal $\mathcal{J}$ is a bimodule for $\mathcal{I}$. We study the set of all derivations from $\mathcal{I}$ into $\mathcal{J}$. We show that any such derivation is automatically continuous and there exists an operator $a \in \mathcal{J}:\mathcal{I}$ such that $\delta(\cdot) = [a, \cdot]$, moreover $\|a + \alpha\mathbb{1}\|_{\mathcal{B}(H)} \le \|\delta\|_{\mathcal{I}\to\mathcal{J}} \le 2C\|a\|_{\mathcal{J}:\mathcal{I}}$ for some complex number α, where C is the modulus of concavity of the quasi-norm $\|\cdot\|_{\mathcal{J}}$ and $\mathbb{1}$ is the identity operator on H. In the special case, when $\mathcal{I} = \mathcal{J} = \mathcal{K}(H)$ is a symmetric Banach ideal of compact operators on H our result yields the classical fact that any derivation δ on $\mathcal{K}(H)$ may be written as $\delta(\cdot) = [a, \cdot]$, where a is some bounded operator on H and $\|a\|_{\mathcal{B}(H)} \le \|\delta\|_{\mathcal{I}\to\mathcal{I}} \le 2\|a\|_{\mathcal{B}(H)}$.

1. Introduction

Let $\mathcal{I}, \mathcal{J}$ be ideals of compact operators on an infinite-dimensional complex Hilbert space H. Obviously, $\mathcal{J}$ is an $\mathcal{I}$-module and we can consider the set $\mathrm{Der}(\mathcal{I}, \mathcal{J})$ of all derivations $\delta: \mathcal{I} \to \mathcal{J}$. Consider two closely related questions (here, $\mathcal{B}(H)$ is the set of all bounded linear operators on H):

Question 1.1. *Let $\delta \in \mathrm{Der}(\mathcal{I}, \mathcal{J})$. Does there exist a bounded operator $a \in \mathcal{B}(H)$ such that $\delta(x) = [a, x]$ for every $x \in \mathcal{I}$?*

Question 1.2. *What is the set $D(\mathcal{I}, \mathcal{J}) = \{a \in \mathcal{B}(H) : [a, x] \in \mathcal{J}, \ \forall x \in \mathcal{I}\}$?*

The second question was completely answered by Hoffman in [1], who also coined the term $\mathcal{J}$-essential commutant of $\mathcal{I}$ for the set $D(\mathcal{I}, \mathcal{J})$. We completely answer the first question in the setting when the ideals $\mathcal{I}, \mathcal{J}$ are symmetric quasi-Banach (see precise definition in the next section). In this setting, it is also natural to ask.

Question 1.3. *Let $\delta \in \mathrm{Der}(\mathcal{I}, \mathcal{J})$. Is it continuous?*

Of course, if $\delta \in \mathrm{Der}(\mathcal{I}, \mathcal{J})$ is such that $\delta(x) = [a, x]$ for some $a \in \mathcal{B}(H)$ (that is when δ is implemented by the operator a), then δ is a continuous mapping from $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ to $(\mathcal{J}, \|\cdot\|_{\mathcal{J}})$, that is a positive answer to Question 1.1 implies also a positive answer to Question 1.3. However, in this paper, we are establishing a positive answer to Question 1.1 via firstly answering Question 1.3 in positive. Both these results (Theorems 3.1 and 3.2) are proven in Section 3. We also provide a detailed discussion of the $\mathcal{J}$-essential commutant of $\mathcal{I}$ in Section 4.

It is also instructive to outline a connection between Questions 1.1 and 1.3 with some classical results. It is well known [2, Lemma 4.1.3] that every derivation on a C^*-algebra is norm continuous. In fact, this also easily follows from the following

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well-known fact [2, Corollary 4.1.7] that every derivation on a C^*-algebra $\mathcal{M} \subset \mathcal{B}(H)$ is given by a reduction of an inner derivation on a von Neumann algebra $\overline{\mathcal{M}}^{wo}$ (the weak closure of $\mathcal{M}$ in the C^*-algebra $\mathcal{B}(H)$). The latter result [2, Lemma 4.1.4 and Theorem 4.1.6], in the setting when $\mathcal{M}$ is a C^*-algebra $\mathcal{K}(H)$ of all compact operators on H states that for every derivation δ on $\mathcal{M}$ there exists an operator $a \in \mathcal{B}(H)$ such that $\delta(x) = [a, x]$ for every $x \in \mathcal{K}(H)$, in addition, $\|a\|_{\mathcal{B}(H)} \leqslant \|\delta\|_{\mathcal{M}\to\mathcal{M}}$. The ideal $\mathcal{K}(H)$ equipped with the uniform norm is an element from the class of so-called symmetric Banach operator ideals in $\mathcal{B}(H)$ and evidently this example also suggests the statements of Questions 1.1 and 1.3. In the case of Schatten ideals $C_p(H) = \{x \in \mathcal{K}(H) : \|x\|_p = \mathrm{tr}(|x|^p)^{\frac{1}{p}} < \infty\}$, where $|x| = (x^*x)^{\frac{1}{2}}$, $1 \leqslant p < \infty$, somewhat similar problems concerning derivations from $C_p(H)$ into $C_r(H)$ were also considered in the work by Kissin and Shulman [3]. In particular, it is shown in [3] that every closed $*$-derivation δ from $C_p(H)$ into $C_r(H)$ is implemented by a symmetric operator S, in addition the domain $D(\delta)$ of δ is dense $*$-subalgebra in $C_p(H)$. In our case, we have $D(\delta) = C_p$ and it follows from our results that the derivation δ is necessarily continuous and implemented by an operator $a \in \mathcal{B}(H)$.

It is also worth to mention that Hoffman's results in [1] were an extension of earlier results by Calkin [4] who considered the case when $\mathcal{I} = \mathcal{B}(H)$. Recently, Calkin's and Hoffman's results were extended to the setting of general von Neumann algebras in [5,6] and, in the special setting when $\mathcal{I} = \mathcal{J}$, Questions 1.1 and 1.3 were also discussed in [7]. However, our methods in this paper are quite different from all the approaches applied in [1,3–6].

As a corollary of solving Questions 1.1 and 1.3, in Theorem 3.6 we present a description of all derivations δ acting from a symmetric quasi-Banach ideal $\mathcal{I}$ into a symmetric quasi-Banach ideal $\mathcal{J}$. Indeed, every such derivation δ is an inner derivation $\delta(\cdot) = \delta_a(\cdot) = [a, \cdot]$, where a is some operator from $\mathcal{J}$-dual space $\mathcal{J} : \mathcal{I}$ of $\mathcal{I}$. Recall that $D(\mathcal{I}, \mathcal{J}) = \mathcal{J} : \mathcal{I} + \mathbb{C}\mathbb{1}$ [1], where $\mathbb{1}$ is the identity operator in $\mathcal{B}(H)$. Theorem 3.6 gives a complete answer to Question 1.2. In particular, using the equality $C_r : C_p = C_q$, $0 < r < p < \infty$, $\frac{1}{q} = \frac{1}{r} - \frac{1}{p}$, we recover Hoffman's result that any derivation $\delta \colon C_p \to C_r$ has a form $\delta = \delta_a$ for some $a \in C_q$. If $0 < p \leqslant r < \infty$, then $D(C_p, C_r) = \mathcal{B}(H)$.

When $\mathcal{I}, \mathcal{J}$ are arbitrary symmetric quasi-Banach ideals of compact operators and $\mathcal{I} \subseteq \mathcal{J}$, then $\mathcal{J} : \mathcal{I} = \mathcal{B}(H)$, and, in this case, a linear operator $\delta \colon \mathcal{I} \to \mathcal{J}$ is a derivation if and only if $\delta = \delta_a$ for some $a \in \mathcal{B}(H)$. However, if $\mathcal{I} \not\subseteq \mathcal{J}$, then to obtain a complete description of $\mathcal{J}$-essential commutant of $\mathcal{I}$ we need a procedure of finding $\mathcal{J} : \mathcal{I}$.

To this end, we use the classical Calkin's correspondence between two-sided ideals $\mathcal{I}$ of compact operators and rearrangement invariant solid sequence subspaces $E_{\mathcal{I}}$ of the space c_0 of null sequences. The meaning of this correspondence is the following. Take a compact operator $x \in \mathcal{I}$ and consider a sequence of eigenvalues $\{\lambda_n(x)\}_{n=1}^{\infty} \in c_0$. For each sequence $\xi = \{\xi_n\} \in c_0$, let $\xi^* = \{\xi_n^*\}_{n=1}^{\infty}$ denote a decreasing rearrangement of the sequence $|\xi| = \{|\xi_n|\}_{n=1}^{\infty}$. The set

$$E_{\mathcal{I}} := \{\{\xi_n\}_{n=1}^{\infty} \in c_0 : \{\xi_n^*\}_{n=1}^{\infty} = \{\lambda_n^*(|x|)\}_{n=1}^{\infty} \text{ for some } x \in \mathcal{I}\},$$

is a solid linear subspace in the Banach lattice c_0. In addition, the space $E_{\mathcal{I}}$ is rearrangement invariant, that is if $\eta \in c_0$, $\xi \in E_{\mathcal{I}}$, $\eta^* = \xi^*$, then $\eta \in E_{\mathcal{I}}$. Conversely, if E is a rearrangement invariant solid sequence subspace in c_0, then

$$C_E = \{x \in \mathcal{K}(H) : \{\lambda_n(|x|)\}_{n=1}^{\infty} \in E\}$$

is a two-sided ideal of compact operators from $\mathcal{B}(H)$.

For the proof of the following theorem we refer to Calkin's original paper, [4], and to Simon's book, [8, Theorem 2.5].

Theorem 1.4. *The correspondence $\mathcal{I} \leftrightarrow E_{\mathcal{I}}$ is a bijection between rearrangement invariant solid spaces in c_0 and two-sided ideals of compact operators.*

In the recent paper [9] this correspondence has been extended to symmetric quasi-Banach (Banach) ideals and p-convex symmetric quasi-Banach (Banach) sequence spaces. We use the notation $\|\cdot\|_{\mathcal{B}(H)}$ and $\|\cdot\|_{\infty}$ to denote the uniform norm on $\mathcal{B}(H)$ and on l_{∞} respectively.

Recall, that a two-sided ideal $\mathcal{I}$ of compact operators from $B(H)$ is said to be symmetric quasi-Banach (Banach) ideal if it is equipped with a complete quasi-norm (respectively, norm) $\|\cdot\|_{\mathcal{I}}$ such that

$$\|axb\|_{\mathcal{I}} \leqslant \|a\|_{\mathcal{B}(H)}\|x\|_{\mathcal{I}}\|b\|_{\mathcal{B}(H)}, \quad x \in \mathcal{I}, a, b \in \mathcal{B}(H).$$

A symmetric quasi-Banach (Banach) sequence space $E \subset c_0$ is a rearrangement invariant solid sequence space equipped with a complete quasi-norm (respectively, norm) $\|\cdot\|_E$ such that $\|\eta\|_E \leq \|\xi\|_E$ for every $\xi \in E$ and $\eta \in c_0$ such that $\eta^* \leqslant \xi^*$.

It is clear that if $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ is a symmetric quasi-Banach ideal of compact operators, $x \in \mathcal{I}$ and $y \in \mathcal{K}(H)$ is such that $\{\lambda_n^*(|y|)\}_{n=1}^{\infty} \leqslant \{\lambda_n^*(|x|)\}_{n=1}^{\infty}$, then $y \in \mathcal{I}$ and $\|y\|_{\mathcal{I}} \leq \|x\|_{\mathcal{I}}$. In Theorem 4.4 we show that if $E_{\mathcal{I}}$ is a rearrangement invariant solid space in c_0 corresponding to symmetric quasi-Banach ideal $\mathcal{I}$, then setting $\|\xi\|_{E_{\mathcal{I}}} := \|x\|_{\mathcal{I}}$ (where $x \in \mathcal{I}$ is such that $\xi^* = \{\lambda_n^*(|x|)\}_{n=1}^{\infty}$) we obtain that $(E_{\mathcal{I}}, \|\cdot\|_{E_{\mathcal{I}}})$ is a symmetric quasi-Banach sequence space. The converse implication is much harder [9].

Theorem 1.5. *If $(E, \|\cdot\|_E)$ is a symmetric Banach (respectively, p-convex symmetric quasi-Banach) sequence space in c_0, then C_E equipped with the norm*

$$\|x\|_{C_E} := \|\{\lambda_n^*(|x|)\}_{n=1}^{\infty}\|_E$$

is a symmetric Banach (respectively, p-convex quasi-Banach) ideal of compact operators from $\mathcal{B}(H)$.

In [10] it was shown that for $\mathcal{J} = C_1$ is the trace class and an arbitrary two-sided ideal $\mathcal{I}$ with $C_1 \subset \mathcal{I} \subset \mathcal{K}(H)$ the C_1-dual space (also sometimes called the Köthe dual) $\mathcal{I}^\times := C_1 : \mathcal{I}$ of $\mathcal{I}$ is precisely an ideal corresponding to symmetric sequence space $l_1 : E_\mathcal{I}$, where $l_1 : E_\mathcal{I}$ is l_1-dual space of $E_\mathcal{I}$ (see precise definitions in Section 4). If $\mathcal{I}$ is a symmetric Banach ideal of compact operators, then C_1-dual space $\mathcal{I}^\times$ is symmetric Banach ideal of compact operator and norms on $C_1 : \mathcal{I}$ and $C_{l_1:E_\mathcal{I}}$ are equal [11]. We extend these results to arbitrary symmetric quasi-Banach ideals $\mathcal{I}, \mathcal{J}$ of compact operators with $\mathcal{I} \not\subseteq \mathcal{J}$, that allows to describe completely all derivations from one symmetric quasi-Banach ideal to another. In addition, we use the technique of $\mathcal{J}$-dual spaces in order to obtain the estimation $\|\delta_a\|_{\mathcal{I}\to\mathcal{J}} \leqslant 2\|a\|_{\mathcal{J}:\mathcal{I}}$ for an arbitrary derivation $\delta = \delta_a \colon \mathcal{I} \to \mathcal{J}$, $a \in \mathcal{J} : \mathcal{I}$.

2. Preliminaries

Let H be an infinite-dimensional Hilbert space over the field $\mathbb{C}$ of complex numbers and $\mathcal{B}(H)$ be the C^*-algebra of all bounded linear operators on H. Set

$$\mathcal{B}_h(H) = \{x \in \mathcal{B}(H) : x^* = x\},$$
$$\mathcal{B}_+(H) = \{x \in \mathcal{B}_h(H) : \forall \varphi \in H\ (x(\varphi), \varphi) \geqslant 0\},$$
$$\mathcal{P}(H) = \{p \in \mathcal{B}(H) : p = p^2 = p^*\}.$$

It is well known [12, Chapter 2, Section 4] that $\mathcal{B}_+(H)$ is a proper cone in $\mathcal{B}_h(H)$ and with the partial order given by $x \leqslant y \Leftrightarrow y - x \in \mathcal{B}_+(H)$ the set $\mathcal{B}_h(H)$ is a partially ordered vector space over the field $\mathbb{R}$ of real numbers, satisfying $y^*xy \geqslant 0$ for all $y \in \mathcal{B}(H)$, $x \in \mathcal{B}_+(H)$. Note, that $-\|x\|_{\mathcal{B}(H)}\mathbb{1} \leqslant x \leqslant \|x\|_{\mathcal{B}(H)}\mathbb{1}$ for all $x \in \mathcal{B}_h(H)$, where $\mathbb{1}$ is the identity operator on H. It is known (see e.g. [12, Chapter 4, Section 2, Proposition 4.2.3]) that every operator x in $\mathcal{B}_h(H)$ can be uniquely written as follows: $x = x_+ - x_-$, where $x_+, x_- \in \mathcal{B}_+(H)$ and $x_+x_- = 0$. In addition, every operator $x \in \mathcal{B}(H)$ can be represented as $x = u|x|$ (the polar decomposition of the operator x), where $|x| = (x^*x)^{\frac{1}{2}}$ and u is a partial isometry in $\mathcal{B}(H)$ such that u^*u is the right support of x [13, Chapter VI, Section 5, Theorem VI.10].

We need the following useful proposition.

Proposition 2.1 *([14, Chapter 2, Section 4, Proposition 2.4.3]). If $x, y \in \mathcal{B}_+(H)$, $x \leqslant y$, then there exists an operator $a \in \mathcal{B}(H)$ such that $\|a\|_{\mathcal{B}(H)} \leqslant 1$ and $x = a^*ya$.*

Let $\mathcal{K}(H)$ be a two-sided ideal in $\mathcal{B}(H)$ of all compact operators and $x \in \mathcal{K}(H)$. The eigenvalues $\{\lambda_n(|x|)\}_{n=1}^\infty$ of the operator $|x|$ arranged in decreasing order and repeated according to algebraic multiplicity are called singular values of the operator x, i.e. $s_n(x) = \lambda_n(|x|)$, $n \in \mathbb{N}$, where $\lambda_1(|x|) \geqslant \lambda_2(|x|) \geqslant \cdots$ and $\mathbb{N}$ is the set of all natural numbers. We need the following properties of singular values.

Proposition 2.2 *([15, Chapter II]).*

(a) $s_n(x) = s_n(x^*)$, $s_n(\alpha x) = |\alpha| s_n(x)$ *for all* $x \in \mathcal{K}(H)$, $\alpha \in \mathbb{C}$;
(b) $s_n(xb) \leqslant s_n(x)\|b\|_{\mathcal{B}(H)}$, $s_n(bx) \leqslant s_n(x)\|b\|_{\mathcal{B}(H)}$ *for all* $x \in \mathcal{K}(H)$, $b \in \mathcal{B}(H)$.

Let $\mathcal{F}(H)$ be a two-sided ideal in $\mathcal{B}(H)$ of all operators with finite range and let $\mathcal{I}$ be an arbitrary proper two-sided ideal in $\mathcal{B}(H)$. Then $\mathcal{I}$ is a $*$-ideal [12, Chapter 6, Section 8, Proposition 6.8.9] and the following inclusion holds: $\mathcal{F}(H) \subseteq \mathcal{I}$ [12, Chapter 6, Section 8, Theorem 6.8.3], in particular, $\mathcal{I}$ contains all finite-dimensional projections from $\mathcal{P}(H)$. If H is a separable Hilbert space, then the inclusion $\mathcal{I} \subseteq \mathcal{K}(H)$ also holds [4, Theorem 1.4]. If, however, H is not separable, then for proper two-sided ideals in $\mathcal{B}(H)$ we have the following proposition.

Proposition 2.3 *([10, Proposition 1]).*

(i) *$\mathcal{D} = \{x \in \mathcal{B}(H) : x(H)$ is separable$\}$ is a proper two-sided ideal in $\mathcal{B}(H)$, in addition $\mathcal{K}(H) \subset \mathcal{D}$;*
(ii) *If $\mathcal{I}$ is an ideal in $\mathcal{B}(H)$, then either $\mathcal{I} \subseteq \mathcal{K}(H)$ or $\mathcal{D} \subseteq \mathcal{I}$.*

Let X be a linear space over the field $\mathbb{C}$. A function $\|\cdot\|$ from X to $\mathbb{R}$ is a quasi-norm, if for all $x, y \in X$, $\alpha \in \mathbb{C}$ the following properties hold:

(1) $\|x\| \geqslant 0$, $\|x\| = 0 \Leftrightarrow x = 0$;
(2) $\|\alpha x\| = |\alpha|\|x\|$;
(3) $\|x + y\| \leqslant C(\|x\| + \|y\|)$, $C \geqslant 1$.

The couple $(X, \|\cdot\|)$ is called a quasi-normed space and the least of all constants C satisfying the inequality (3) above is called the modulus of concavity of the quasi-norm $\|\cdot\|$.

It is known (see e.g. [16, Section 1]) that for each quasi-norm $\|\cdot\|$ on X there exists an equivalent p-additive quasi-norm $|||\cdot|||$, that is a quasi-norm $|||\cdot|||$ on X satisfying the following property of p-additivity: $|||x + y|||^p \leqslant |||x|||^p + |||y|||^p$, where p is such that $C = 2^{\frac{1}{p}-1}$, in particular, $0 < p \leqslant 1$ since $C \geqslant 1$. In this case, the function $d : X^2 \to \mathbb{R}$ defined by $d(x, y) := |||x - y|||^p$, $x, y \in X$ is an invariant metric on X, and in the topology τ_d, generated by the metric d, the linear space X is a topological vector space. If (X, d) is a complete metric space, then $(X, \|\cdot\|)$ is called a quasi-Banach space and the quasi-norm $\|\cdot\|$ is a complete quasi-norm; in this case, (X, τ_d) is an F-space.

Proposition 2.4. *Let $(X, \|\cdot\|)$ be a quasi-Banach space with the modulus of concavity C, let $|||\cdot|||$ be a p-additive quasi-norm equivalent to the quasi-norm $\|\cdot\|$, $C = 2^{\frac{1}{p}-1}$. If $x_n \in X, n \geq 1$ and $\sum_{n=1}^{\infty} |||x_n|||^p < \infty$, then the series $\sum_{n=1}^{\infty} x_n$ converges in $(X, \|\cdot\|)$, i.e. there exists $x \in X$ such that $\left\|x - \sum_{n=1}^{k} x_n\right\| \to 0$ for $k \to \infty$.*

Proof. For partial sums $S_k = \sum_{n=1}^{k} x_n$ we have

$$d(S_{k+l}, S_k) = |||S_{k+l} - S_k|||^p = ||| \sum_{n=l+1}^{k+l} x_n |||^p \leqslant \sum_{n=l+1}^{k+l} |||x_n|||^p \to 0 \quad \text{for } k, l \to \infty,$$

i.e. $\{S_k\}_{k=1}^{\infty}$ is a Cauchy sequence in (X, d). Since the metric space (X, d) is complete, there exists $x \in X$ such that $d(S_k, x) = |||S_k - x|||^p \to 0$ for $k \to \infty$. Since quasi-norms $\|\cdot\|$ and $|||\cdot|||$ are equivalent we have that $\|S_k - x\| \to 0$ for $k \to \infty$. □

Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be quasi-normed spaces and let $\mathcal{B}(X, Y)$ be the linear space of all bounded linear mappings $T : X \to Y$. For each $T \in \mathcal{B}(X, Y)$ set $\|T\|_{\mathcal{B}(X,Y)} = \sup\{\|Tx\|_Y : \|x\| \leqslant 1\}$. As in the case of normed spaces, the set $\mathcal{B}(X, Y)$ coincides with the set of all continuous linear mappings from X into Y, moreover, the function $\|\cdot\|_{\mathcal{B}(X,Y)} : \mathcal{B}(X, Y) \to \mathbb{R}$ is a quasi-norm on $\mathcal{B}(X, Y)$ whose modulus of concavity, does not exceed the modulus of concavity of the quasi-norm $\|\cdot\|_Y$ [16, Section 1]. Furthermore, $\|Tx\|_Y \leqslant \|T\|_{\mathcal{B}(X,Y)}\|x\|_X$ for all $T \in \mathcal{B}(X, Y)$ and $x \in X$.

Proposition 2.5. *If $(Y, \|\cdot\|_Y)$ is a quasi-Banach space, then $(\mathcal{B}(X, Y), \|\cdot\|_{\mathcal{B}(X,Y)})$ is a quasi-Banach space too.*

Proof. Since $\|\cdot\|_Y$ is a quasi-norm on Y, there exists a p-additive quasi-norm $|||\cdot|||_Y$ equivalent to $\|\cdot\|_Y$, i.e. $\alpha_1|||y|||_Y \leqslant \|y\|_Y \leqslant \beta_1|||y|||_Y$ for all $y \in Y$ and some constants $\alpha_1, \beta_1 > 0$. Similarly, there exists a q-additive quasi-norm $|||\cdot|||_{\mathcal{B}(X,Y)}$ equivalent to the quasi-norm $\|\cdot\|_{\mathcal{B}(X,Y)}$, i.e. $\alpha_2|||T|||_{\mathcal{B}(X,Y)} \leqslant \|T\|_{\mathcal{B}(X,Y)} \leqslant \beta_2|||T|||_{\mathcal{B}(X,Y)}$ for all $T \in \mathcal{B}(X, Y)$ and some $\alpha_2, \beta_2 > 0$, $0 < p, q \leqslant 1$.

Let $\{T_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $(\mathcal{B}(X, Y), d)$, where $d(T, S) = |||T - S|||^q_{\mathcal{B}(X,Y)}$, $T, S \in \mathcal{B}(X, Y)$. Fix $\varepsilon > 0$ and select a positive integer $n(\varepsilon)$ such that $|||T_n - T_m|||^q_{\mathcal{B}(X,Y)} < \varepsilon^q$ for all $n, m \geqslant n(\varepsilon)$. For every $x \in X$ we have

$$\begin{aligned} |||T_nx - T_mx|||_Y^p &\leqslant \frac{1}{\alpha_1^p}\|T_nx - T_mx\|_Y^p \leqslant \frac{1}{\alpha_1^p}\|T_n - T_m\|^p_{\mathcal{B}(X,Y)}\|x\|_X^p \\ &\leqslant \left(\frac{\beta_2}{\alpha_1}\right)^p |||T_n - T_m|||^p_{\mathcal{B}(X,Y)}\|x\|_X^p < \left(\frac{\beta_2}{\alpha_1}\right)^p \|x\|_X^p\varepsilon^p \quad \text{for } n, m \geqslant n(\varepsilon). \end{aligned}$$

Thus, $\{T_nx\}_{n=1}^{\infty}$ is a Cauchy sequence in (Y, d_Y), where $d_Y(x, y) = |||x - y|||^p_Y$. Since the metric space (Y, d_Y) is complete, there exists $T(x) \in Y$ such that $|||T_n(x) - T(x)|||^p_Y \to 0$ for $n \to \infty$. The verification that $T \in \mathcal{B}(X, Y)$ and $|||T_n - T|||^q_{\mathcal{B}(X,Y)} \to 0$ for $n \to \infty$ is routine and is therefore omitted. □

Let $\mathcal{I}$ be a nonzero two-sided ideal in $\mathcal{B}(H)$.

A quasi-norm $\|\cdot\|_{\mathcal{I}} : \mathcal{I} \to \mathbb{R}$ is called symmetric quasi-norm if

(1) $\|axb\|_{\mathcal{I}} \leqslant \|a\|_{\mathcal{B}(H)}\|x\|_{\mathcal{I}}\|b\|_{\mathcal{B}(H)}$ for all $x \in \mathcal{I}$, $a, b \in \mathcal{B}(H)$;
(2) $\|p\|_{\mathcal{I}} = 1$ for any one-dimensional projection $p \in \mathcal{I}$.

Proposition 2.6 (*Compare [15, Chapter III, Section 2]*). *Let $\|\cdot\|_{\mathcal{I}}$ be a symmetric quasi-norm on a two-sided ideal $\mathcal{I}$. Then*

(a) $\|x\|_{\mathcal{I}} = \|x^*\|_{\mathcal{I}} = \left\||x|\right\|_{\mathcal{I}}$ *for all* $x \in \mathcal{I}$;
(b) *If* $x \in \mathcal{I} \subset \mathcal{K}(H)$, $y \in \mathcal{K}(H)$, $s_n(y) \leqslant s_n(x)$, $n = 1, 2, \ldots$, *then* $y \in \mathcal{I}$ *and* $\|y\|_{\mathcal{I}} \leqslant \|x\|_{\mathcal{I}}$;
(c) *If* $\mathcal{I} \subset \mathcal{K}(H)$, *then* $\|x\|_{\mathcal{B}(H)} \leqslant \|x\|_{\mathcal{I}}$ *for all* $x \in \mathcal{I}$.

Proof. (a) Let $x = u|x|$ be the polar decomposition of the operator x. Then $\|x\|_{\mathcal{I}} = \|u|x|\|_{\mathcal{I}} \leqslant \left\||x|\right\|_{\mathcal{I}}$. Since $u^*x = |x|$, the inequality $\left\||x|\right\|_{\mathcal{I}} \leqslant \|x\|_{\mathcal{I}}$ holds and so $\left\||x|\right\|_{\mathcal{I}} = \|x\|_{\mathcal{I}}$. Using the equalities $x^* = |x|u^*$, $x^*u = |x|$ in the same manner, we obtain that $\left\||x|\right\|_{\mathcal{I}} = \|x^*\|_{\mathcal{I}}$.

(b) Since x, y are compact operators and $s_n(y) \leqslant s_n(x)$ we have $s_n(y) = \alpha_n s_n(x)$, where $0 \leqslant \alpha_n \leqslant 1$, $n \in \mathbb{N}$. By the Hilbert–Schmidt theorem, there exists an orthogonal system of eigenvectors $\{\varphi_n\}_{n=1}^{\infty}$ for the operator $|y|$ such that $|y|(\varphi) = \sum_{n=1}^{\infty} s_n(y)c_n\varphi_n$, where $c_n = (\varphi, \varphi_n)$, $\varphi \in H$. Since $s_n(y) = \alpha_n s_n(x)$, it follows that card$\{\varphi_n\} \leqslant$ card$\{\psi_n\}$, where $\{\psi_n\}_{n=1}^{\infty}$ is an orthogonal system of eigenvectors for the operator $|x|$. Thus, there exists a unitary operator $u \in \mathcal{B}(H)$ such that $u(\psi_n) = \varphi_n$, in addition, $u|x|u^{-1} \geqslant |y|$.

By Proposition 2.1, there exists an operator $a \in \mathcal{B}(H)$ with $\|a\|_{\mathcal{B}(H)} \leqslant 1$ such that $|y| = a^*u|x|u^{-1}a$. Consequently, $|y| \in \mathcal{I}$ and $\left\||y|\right\|_{\mathcal{I}} \leqslant \left\||x|\right\|_{\mathcal{I}}$, thus $y \in \mathcal{I}$ and $\|y\|_{\mathcal{I}} \leqslant \|x\|_{\mathcal{I}}$.

(c) Let $y(\cdot) = s_1(x)(\cdot, \varphi)\varphi$, where φ is an arbitrary vector in H with $\|\varphi\|_H = 1$. Whereas $s_n(y) \leqslant s_n(x)$, we have $\|x\|_{\mathcal{B}(H)} = s_1(x) = \|y\|_{\mathcal{B}(H)} = \|y\|_{\mathcal{I}} \leqslant \|x\|_{\mathcal{I}}$ (see (b)). □

A two-sided ideal $\mathcal{I}$ of compact operators from $\mathcal{B}(H)$ is called a symmetric quasi-Banach (respectively, Banach) ideal, if $\mathcal{I}$ is equipped with a complete symmetric quasi-norm (respectively, norm).

Let $\mathcal{I}$, $\mathcal{J}$ be two-sided ideals of compact operators from $\mathcal{B}(H)$. A linear mapping $\delta : \mathcal{I} \to \mathcal{J}$ is called a derivation, if $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in \mathcal{I}$. If, in addition, $\delta(x^*) = (\delta(x))^*$ for all $x \in \mathcal{I}$, then δ is called a $*$-derivation. Denote by $\mathrm{Der}(\mathcal{I}, \mathcal{J})$ the linear space of all derivations from $\mathcal{I}$ into $\mathcal{J}$.

For each derivation $\delta : \mathcal{I} \to \mathcal{J}$ define the mappings $\delta_{\mathrm{Re}}(x) := \frac{\delta(x)+\delta(x^*)^*}{2}$ and $\delta_{\mathrm{Im}}(x) := \frac{\delta(x)-\delta(x^*)^*}{2i}$, $x \in \mathcal{I}$. It is easy to see that δ_{Re} and δ_{Im} are $*$-derivations from $\mathcal{I}$ into $\mathcal{J}$, moreover $\delta = \delta_{\mathrm{Re}} + i\delta_{\mathrm{Im}}$.

If $a \in \mathcal{B}(H)$, then the mapping $\delta_a : \mathcal{B}(H) \to \mathcal{B}(H)$ given by $\delta_a(x) := [a, x] = ax - xa$, $x \in \mathcal{B}(H)$, is a derivation. Derivations of this type are called inner. When $\mathcal{I}$ is a two-sided ideal in $\mathcal{B}(H)$, then $\delta_a(\mathcal{I}) \subset \mathcal{I}$ for all $a \in \mathcal{B}(H)$. If $\mathcal{J}$ is also a two-sided ideal in $\mathcal{B}(H)$ and $a \in \mathcal{J}$, then $\delta_a(\mathcal{I}) \subset \mathcal{I} \cap \mathcal{J}$.

3. The set Der($\mathcal{I}$, $\mathcal{J}$) for symmetric quasi-Banach ideals $\mathcal{I}$ and $\mathcal{J}$

The following theorem gives a positive answer to Question 1.3.

Theorem 3.1. *Let $\mathcal{I}$, $\mathcal{J}$ be symmetric quasi-Banach ideals of compact operators from $\mathcal{B}(H)$ and δ is a derivation from $\mathcal{I}$ into $\mathcal{J}$. Then δ is a continuous mapping from $\mathcal{I}$ into $\mathcal{J}$, i.e. $\delta \in \mathcal{B}(\mathcal{I}, \mathcal{J})$.*

Proof. Without loss of generality we may assume that δ is a $*$-derivation. The spaces $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$, $(\mathcal{J}, \|\cdot\|_{\mathcal{J}})$ are F-spaces, and therefore it is sufficient to prove that the graph of δ is closed. Suppose a contrary, that is there exists a sequence $\{x_n\}_{n=1}^{\infty} \subset \mathcal{I}$ such that $\|\cdot\|_{\mathcal{I}} - \lim_{n\to\infty} x_n = 0$ and $\|\cdot\|_{\mathcal{J}} - \lim_{n\to\infty} \delta(x_n) = x \neq 0$.

Since $x_n = \mathrm{Re}x_n + i\mathrm{Im}x_n$ for all $n \in \mathbb{N}$, where $\mathrm{Re}x_n = \frac{x_n+x_n^*}{2}$, $\mathrm{Im}x_n = \frac{x_n-x_n^*}{2}$, and $\|x_n\|_{\mathcal{I}} \to 0$, $\|x_n^*\|_{\mathcal{I}} = \|x_n\|_{\mathcal{I}} \to 0$, we have

$$\|\mathrm{Re}x_n\|_{\mathcal{I}} = \left\|\frac{x_n + x_n^*}{2}\right\|_{\mathcal{I}} \leqslant \frac{C(\|x_n\|_{\mathcal{I}} + \|x_n^*\|_{\mathcal{I}})}{2} \to 0$$

and

$$\|\mathrm{Im}x_n\|_{\mathcal{I}} = \left\|\frac{x_n - x_n^*}{2}\right\|_{\mathcal{I}} \leqslant \frac{C(\|x_n\|_{\mathcal{I}} + \|x_n^*\|_{\mathcal{I}})}{2} \to 0,$$

where C is the modulus of concavity of the quasi-norm $\|\cdot\|_{\mathcal{I}}$. Consequently, we may assume that $x_n^* = x_n$ for all $n \in \mathbb{N}$. In this case, from the relationships

$$x \xleftarrow{\|\cdot\|_{\mathcal{J}}} \delta(x_n) = \delta(x_n^*) = \delta(x_n)^* \xrightarrow{\|\cdot\|_{\mathcal{J}}} x^*,$$

we obtain $x = x^*$.

Writing $x = x_+ - x_-$, where $x_+, x_- \geqslant 0$ and $x_+x_- = 0$, we may assume that $x_+ \neq 0$, otherwise we consider the sequence $\{-x_n\}_{n=1}^{\infty}$. Since x_+ is a nonzero positive compact operator, $\lambda = \|x_+\|_{\mathcal{B}(H)}$ is an eigenvalue of x_+ corresponding to a finite-dimensional eigensubspace. Let q be a projection onto this subspace.

Fix an arbitrary non-zero vector $\varphi \in q(H)$ and consider the projection p onto the one-dimensional subspace spanned by φ. Combining the inequality $p \leqslant q$ with the equality $qx_+q = \lambda q$, we obtain $pxp = pqxqp = \lambda pqp = \lambda p$. Replacing, if necessary, the sequence $\{x_n\}_{n=1}^{\infty}$ with the sequence $\{\frac{x_n}{\lambda}\}_{n=1}^{\infty}$, we may assume

$$pxp = p. \tag{1}$$

Since p is one-dimensional, it follows that $pap = \alpha p$, $\alpha \in \mathbb{C}$ for any operator $a \in \mathcal{B}(H)$, in particular, $px_np = \alpha_n p$, therefore $|\alpha_n| = \|px_np\|_{\mathcal{I}} \to 0$ for $n \to \infty$. Writing

$$\|\delta(p)x_np\|_{\mathcal{J}} \leqslant \|\delta(p)\|_{\mathcal{J}}\|x_np\|_{\mathcal{B}(H)} \leqslant \|\delta(p)\|_{\mathcal{J}}\|x_n\|_{\mathcal{B}(H)} \leqslant \|\delta(p)\|_{\mathcal{J}}\|x_n\|_{\mathcal{I}},$$

we infer $\|\delta(p)x_np\|_{\mathcal{J}} \to 0$ and $\|px_n\delta(p)\|_{\mathcal{J}} = \|(\delta(p)x_np)^*\|_{\mathcal{J}} \to 0$.

Since $pxp \overset{(1)}{=} p \in \mathcal{J}$, we have

$$\begin{aligned}\|\delta(px_np) - pxp\|_{\mathcal{J}} &= \|\delta(p)x_np + p\delta(x_n)p + px_n\delta(p) - pxp\|_{\mathcal{J}} \\ &\leqslant C_1\|\delta(p)x_np + px_n\delta(p)\|_{\mathcal{J}} + C_1\|p\delta(x_n)p - pxp\|_{\mathcal{J}} \\ &\leqslant C_1^2\|\delta(p)x_np\|_{\mathcal{J}} + C_1^2\|px_n\delta(p)\|_{\mathcal{J}} + C_1\|p\delta(x_n)p - pxp\|_{\mathcal{J}} \to 0,\end{aligned}$$

where C_1 is the modulus of concavity of the quasi-norm $\|\cdot\|_{\mathcal{J}}$, i.e. $\delta(px_np) \xrightarrow{\|\cdot\|_{\mathcal{J}}} pxp$. Hence

$$p \overset{(1)}{=} pxp = \|\cdot\|_{\mathcal{J}} - \lim_{n\to\infty} \delta(px_np) = \|\cdot\|_{\mathcal{J}} - \lim_{n\to\infty} \delta(\alpha_np) = \|\cdot\|_{\mathcal{J}} - \lim_{n\to\infty} \alpha_n\delta(p) = 0,$$

which is a contradiction, since $p \neq 0$.

Consequently, δ is a continuous mapping from $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ into $(\mathcal{J}, \|\cdot\|_{\mathcal{J}})$. $\square$

Note, that in [7, Theorem 8] a version of Theorem 3.1 is obtained for the case of an arbitrary symmetric Banach ideal $\mathcal{I} = \mathcal{J}$ of τ-compact operators in a von Neumann algebra $\mathcal{M}$ equipped with a semi-finite normal faithful trace τ.

The following theorem gives a positive answer to Question 1.1.

Theorem 3.2. *If $\mathcal{I}, \mathcal{J}$ are symmetric quasi-Banach ideals of compact operators from $\mathcal{B}(H)$, then for every derivation $\delta : \mathcal{I} \to \mathcal{J}$ there exists an operator $a \in \mathcal{B}(H)$ such that $\delta(\cdot) = \delta_a(\cdot) = [a, \cdot]$, in addition, $\|a\|_{\mathcal{B}(H)} \leqslant \|\delta\|_{\mathcal{B}(\mathcal{I},\mathcal{J})}$.*

Proof. Fix an arbitrary vector $\varphi_0 \in H$ with $\|\varphi_0\|_H = 1$ and consider projection $p_0(\cdot) := (\cdot, \varphi_0)\varphi_0$ onto one-dimensional subspace spanned by φ_0. Obviously, $p_0 \in \mathcal{I} \cap \mathcal{J}$.

Let $x \in \mathcal{I}$, $x(\varphi_0) = 0$ and $\varphi \in H$. Since

$$xp_0(\varphi) = x(p_0(\varphi)) = x\big((\varphi, \varphi_0)\varphi_0\big) = (\varphi, \varphi_0)x(\varphi_0) = 0,$$

it follows that $xp_0 = 0$, and so $\delta(xp_0)(\varphi_0) = 0$. Consequently, the linear operator $a(z(\varphi_0)) = \delta(zp_0)(\varphi_0)$ is correctly defined on the linear subspace $L := \{z(\varphi_0) : z \in \mathcal{I}\} \subset H$. If $\varphi \in H$, $z(\cdot) = (\cdot, \varphi_0)\varphi$, then $z \in \mathcal{I}$ and $z(\varphi_0) = \varphi$, which implies $L = H$.

For arbitrary $z \in \mathcal{B}(H)$, $\varphi \in H$, we have

$$\begin{aligned}|zp_0|^2(\varphi) &= (p_0z^*zp_0)(\varphi) = (p_0z^*z)((\varphi, \varphi_0)\varphi_0) = (\varphi, \varphi_0)p_0(z^*z(\varphi_0)) \\ &= (z\varphi_0, z\varphi_0)(\varphi, \varphi_0)\varphi_0 = (z\varphi_0, z\varphi_0)p_0(\varphi) = \|z(\varphi_0)\|_H^2 p_0(\varphi),\end{aligned}$$

in particular, $\|zp_0\|_{\mathcal{B}(H)} = \big\||zp_0|\big\|_{\mathcal{B}(H)} = \big\|\,\|z(\varphi_0)\|_H\, p_0\big\|_{\mathcal{B}(H)} = \|z(\varphi_0)\|_H$. Applying this observation together with Theorem 3.1 guaranteeing $\|\delta(x)\|_{\mathcal{J}} \leqslant \|\delta\|_{\mathcal{B}(\mathcal{I},\mathcal{J})}\|x\|_{\mathcal{I}}$ for all $x \in \mathcal{I}$, we have

$$\begin{aligned}\|a(x(\varphi_0))\|_H &= \|\delta(xp_0)(\varphi_0)\|_H = \|\delta(xp_0)p_0\|_{\mathcal{B}(H)} \leqslant \|\delta(xp_0)\|_{\mathcal{B}(H)}\|p_0\|_{\mathcal{B}(H)} \\ &\leqslant \|\delta(xp_0)\|_{\mathcal{J}} \leqslant \|\delta\|_{\mathcal{B}(\mathcal{I},\mathcal{J})}\|xp_0\|_{\mathcal{I}} \\ &\leqslant \|\delta\|_{\mathcal{B}(\mathcal{I},\mathcal{J})}\|p_0\|_{\mathcal{I}}\|xp_0\|_{\mathcal{B}(H)} = \|\delta\|_{\mathcal{B}(\mathcal{I},\mathcal{J})}\|x(\varphi_0)\|_H.\end{aligned}$$

This shows that a is a bounded operator on H and $\|a\|_{\mathcal{B}(H)} \leqslant \|\delta\|_{\mathcal{B}(\mathcal{I},\mathcal{J})}$.

Finally, for all $x, z \in \mathcal{I}$ we have

$$\begin{aligned}[a, x](z(\varphi_0)) &= ax(z(\varphi_0)) - xa(z(\varphi_0)) = a(xz(\varphi_0)) - xa(z(\varphi_0)) \\ &= \delta(xzp_0)(\varphi_0) - x\delta(zp_0)(\varphi_0) = \delta(x)zp_0(\varphi_0) = \delta(x)z(\varphi_0)\end{aligned}$$

and since $L = H$, it follows $\delta(\cdot) = [a, \cdot] = \delta_a(\cdot)$. □

Let $\mathcal{I}, \mathcal{J}$ be arbitrary two-sided ideals in $\mathcal{B}(H)$. The set

$$D(\mathcal{I}, \mathcal{J}) = \{a \in \mathcal{B}(H) : \ ax - xa \in \mathcal{J}, \ \forall x \in \mathcal{I}\}$$

is called the $\mathcal{J}$-essential commutant of $\mathcal{I}$, and the set

$$\mathcal{J} : \mathcal{I} = \{a \in \mathcal{B}(H) : \ ax \in \mathcal{J}, \ \forall x \in \mathcal{I}\}$$

is called the $\mathcal{J}$-dual space of $\mathcal{I}$. It is clear that $\mathcal{J} : \mathcal{I}$ is a two-sided ideal in $\mathcal{B}(H)$. Hence $\mathcal{J} : \mathcal{I}$ is a $*$-ideal, and therefore $xa \in \mathcal{J}$ for all $x \in \mathcal{I}, a \in \mathcal{J} : \mathcal{I}$. If $\mathcal{I} \not\subseteq \mathcal{J}$, then $\mathbb{1} \notin \mathcal{J} : \mathcal{I}$, i.e. $\mathcal{J} : \mathcal{I} \neq \mathcal{B}(H)$, and so $\mathcal{J} : \mathcal{I}$ is a proper ideal in $\mathcal{B}(H)$. However, in case when $\mathcal{I} \subseteq \mathcal{J}$ we have $\mathcal{J} : \mathcal{I} = \mathcal{B}(H)$, in particular, $C_r : C_p = \mathcal{B}(H)$ for all $0 < p \leqslant r$, where $C_p = \{x \in \mathcal{K}(H) : \|x\|_p = \big(tr(|x|^p)\big)^{\frac{1}{p}} < \infty\}$ is the Schatten ideal of compact operators from $\mathcal{B}(H)$, $0 < p < \infty$, tr is the standard trace on $\mathcal{B}_+(H)$.

Proposition 3.3. *If $\mathcal{I}, \mathcal{J}$ are proper two-sided ideals of compact operators in $\mathcal{B}(H)$ and $\mathcal{I} \not\subseteq \mathcal{J}$, then $\mathcal{J} : \mathcal{I} \subset \mathcal{K}(H)$.*

Proof. Since $\mathcal{I} \not\subseteq \mathcal{J}$, $\mathcal{J} : \mathcal{I}$ is a proper two-sided ideal in $\mathcal{B}(H)$. If H is a separable Hilbert space, then $\mathcal{J} : \mathcal{I} \subset \mathcal{K}(H)$ [4, Theorem 1.4]. Suppose that H is not separable and $\mathcal{J} : \mathcal{I} \not\subseteq \mathcal{K}(H)$. By Proposition 2.3, the proper two-sided ideal $\mathcal{D} = \{x \in \mathcal{B}(H) : x(H) \text{ is separable}\} \subset \mathcal{J} : \mathcal{I}$. Since $\mathcal{I} \not\subseteq \mathcal{J}$ there exists a positive compact operator $a \in \mathcal{I} \setminus \mathcal{J}$. Since $a \in \mathcal{D}$, we have that $L := \overline{a(H)}$ is separable. Let $p \in \mathcal{P}(H)$ be the orthogonal projection onto L. Since $a \notin \mathcal{J}$, it follows that L is infinite-dimensional subspace. Indeed, if it were not the case, then a would be a finite rank operator and automatically belonging to $a \in \mathcal{J}$. Therefore $p \in \mathcal{D} \setminus \mathcal{K}(H) \subset \mathcal{J} : \mathcal{I}$, in addition, $0 \neq a = pap \in (p\mathcal{I}p) \setminus (p\mathcal{J}p)$, i.e. $p\mathcal{I}p \not\subseteq p\mathcal{J}p$. Since L is a separable Hilbert space, we have $(p\mathcal{J}p) : (p\mathcal{I}p) \subset \mathcal{K}(L)$.

Let $y \in p\mathcal{I}p$, i.e. $y = py'p$ for some $y' \in \mathcal{I}$. Since $p \in \mathcal{D} \subset \mathcal{J} : \mathcal{I}$ we have $py' \in \mathcal{J}$, hence, $p(py')p \in p\mathcal{J}p$. Consequently, $p \in (p\mathcal{J}p) : (p\mathcal{I}p)$, i.e. p is a compact operator in L, which is a contradiction. Thus, $\mathcal{J} : \mathcal{I} \subset \mathcal{K}(H)$. □

For arbitrary two-sided ideals $\mathcal{I}, \mathcal{J}$ in $\mathcal{B}(H)$ we denote by $d(\mathcal{I}, \mathcal{J})$ the set of all derivations δ from $\mathcal{B}(H)$ into $\mathcal{B}(H)$ such that $\delta(\mathcal{I}) \subset \mathcal{J}$. To characterize the set $d(\mathcal{I}, \mathcal{J})$ we need the following theorem.

Theorem 3.4 *([1, Theorem 1.1]).* $D(\mathcal{I}, \mathcal{J}) = \mathcal{J} : \mathcal{I} + \mathbb{C}\mathbb{1}$.

It should be noted that Theorem 3.4 holds for arbitrary von Neumann algebras, i.e. for any two-sided ideals $\mathcal{I}$, $\mathcal{J}$ in von Neumann algebra $\mathcal{M}$ we have $D(\mathcal{I},\mathcal{J}) = \mathcal{J} : \mathcal{I} + Z(\mathcal{M})$, where $Z(\mathcal{M})$ is the center of $\mathcal{M}$ [5, Corollary 5].

Proposition 3.5. $d(\mathcal{I},\mathcal{J}) = \{\delta_a : a \in D(\mathcal{I},\mathcal{J})\} = \{\delta_a : a \in \mathcal{J} : \mathcal{I}\}$.

Proof. Let $\delta \in d(\mathcal{I},\mathcal{J})$. Since δ is a derivation from $\mathcal{B}(H)$ into $\mathcal{B}(H)$ there exists an operator $a \in \mathcal{B}(H)$ such that $\delta = \delta_a$. If $x \in \mathcal{I}$, then $[a, x] = \delta(x) \in \mathcal{J}$, i.e. $a \in D(\mathcal{I},\mathcal{J})$. Using Theorem 3.4, we have that $a = b + \alpha\mathbb{1}$, where $b \in \mathcal{J} : \mathcal{I}$, $\alpha \in \mathbb{C}$, and therefore $\delta = \delta_a = \delta_b$.

Further, let $\delta_a(\cdot) = [a, \cdot]$ be the inner derivation on $\mathcal{B}(H)$ generated by an operator $a \in \mathcal{J} : \mathcal{I}$. For all $x \in \mathcal{I}$ we have $\delta_a(x) = [a, x] = ax - xa \in \mathcal{J}$. Consequently, $\delta_a \in d(\mathcal{I},\mathcal{J})$. □

Now, let $\mathcal{I}$, $\mathcal{J}$ be arbitrary symmetric quasi-Banach ideals of compact operators from $\mathcal{B}(H)$. According to Theorem 3.2, for each derivation $\delta \in \mathrm{Der}(\mathcal{I},\mathcal{J})$ there exists an operator $a \in \mathcal{B}(H)$ such that $\delta(x) = \delta_a(x) = [a, x]$ for all $x \in \mathcal{I}$. Since $\delta(\mathcal{I}) \subset \mathcal{J}$ we have $[a, x] \in \mathcal{J}$ for all $x \in \mathcal{I}$, i.e. $a \in D(\mathcal{I},\mathcal{J})$. Hence, $\delta_a \in d(\mathcal{I},\mathcal{J})$ (see Proposition 3.5). On the other hand, if $a \in \mathcal{J} : \mathcal{I}$, then $\delta_a \in d(\mathcal{I},\mathcal{J})$ (see Proposition 3.5), in particular, $\delta_a(\mathcal{I}) \subset \mathcal{J}$.

Hence, in view of Proposition 3.5 and Theorem 3.2, the following theorem holds.

Theorem 3.6. *For arbitrary symmetric quasi-Banach ideals $\mathcal{I}$, $\mathcal{J}$ of compact operators in $\mathcal{B}(H)$ each derivation $\delta : \mathcal{I} \to \mathcal{J}$ has a form $\delta = \delta_a$ for some $a \in \mathcal{J} : \mathcal{I}$, in addition $\|a + \alpha\mathbb{1}\|_{\mathcal{B}(H)} \leqslant \|\delta_a\|_{\mathcal{B}(\mathcal{I},\mathcal{J})}$ for some $\alpha \in \mathbb{C}$. Conversely, if $a \in \mathcal{J} : \mathcal{I}$ then the restriction of the derivation δ_a on $\mathcal{I}$ is a derivation from $\mathcal{I}$ into $\mathcal{J}$.*

If $0 < r < p < \infty$, then we have $C_r : C_p = C_q$, where $\frac{1}{q} = \frac{1}{r} - \frac{1}{p}$ [1, Proposition 5.6]. Therefore, the following corollary follows immediately from Theorem 3.6.

Corollary 3.7. *If $0 < p \leqslant r < \infty$, then the mapping $\delta : C_p \to C_r$ is a derivation if and only if $\delta = \delta_a$ for some $a \in \mathcal{B}(H)$. If $0 < r < p < \infty$, then the mapping $\delta : C_p \to C_r$ is a derivation if and only if $\delta = \delta_a$ for some $a \in C_q$, where $\frac{1}{q} = \frac{1}{r} - \frac{1}{p}$.*

4. The $\mathcal{J}$-dual space of $\mathcal{I}$ for symmetric quasi-Banach ideals $\mathcal{I}$ and $\mathcal{J}$

In this section we show that any symmetric quasi-Banach ideal $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ of compact operators from $\mathcal{B}(H)$ has a form of $\mathcal{I} = C_{E_{\mathcal{I}}}$ with the quasi-norm $\|\cdot\|_{\mathcal{I}} = \|\cdot\|_{C_{E_{\mathcal{I}}}}$ for a special symmetric quasi-Banach sequence space $(E_{\mathcal{I}}, \|\cdot\|_{E_{\mathcal{I}}})$ in c_0 constructed by $\mathcal{I}$ with the help of Calkin correspondence. The equality $\mathcal{J} : \mathcal{I} = C_{E_{\mathcal{J}}:E_{\mathcal{I}}}$ established in this section provides a full description of all derivations $\delta \in \mathrm{Der}(\mathcal{I},\mathcal{J})$ in terms of $E_{\mathcal{J}}$-dual space $E_{\mathcal{J}} : E_{\mathcal{I}}$ of $E_{\mathcal{I}}$ of symmetric quasi-Banach sequence spaces $E_{\mathcal{I}}$ and $E_{\mathcal{J}}$ in c_0.

A quasi-Banach lattice E is a vector lattice with a complete quasi-norm $\|\cdot\|_E$, such that $\|a\|_E \leqslant \|b\|_E$ whenever $a, b \in E$ and $|a| \leqslant |b|$. In this case, $\big\||a|\big\|_E = \|a\|_E$ for all $a \in E$ and the lattice operations $a \vee b$ and $a \wedge b$ are continuous in the topology τ_d, generated by the metric $d(a, b) = \|\!|a - b|\!\|_E^p$, where $\|\!|\cdot|\!\|_E$ is a p-additive quasi-norm equivalent to the quasi-norm $\|\cdot\|_E$. Consequently, the set $E_+ = \{a \in E : a \geqslant 0\}$ is closed in (E, τ_d). Thus, for any increasing sequence $\{a_k\}_{k=1}^\infty \subset E$ converging in the topology τ_d to some $a \in E$, we have $a = \sup_{k\geq 1} a_k$ [17, Chapter V, Section 4].

A sequence $\{a_n\}_{n=1}^\infty$ from a vector lattice E is said to be (r)-convergent to $a \in E$ (notation: $a_n \xrightarrow{(r)} a$) with the regulator $b \in E_+$, if and only if there exists a sequence of positive numbers $\varepsilon_n \downarrow 0$ such that $|a_n - a| \leqslant \varepsilon_n b$ for all $n \in \mathbb{N}$ (see e.g. [18, Chapter III, Section 11].

Observe, that in any quasi-Banach lattice $(E, \|\cdot\|_E)$ it follows from $a_n \xrightarrow{(r)} a$, $a_n, a \in E$ that $\|a_n - a\|_E \to 0$.

The following proposition is a quasi-Banach version of the well-known criterion of sequential convergence in Banach lattices.

Proposition 4.1 (*Compare [18, Chapter VII, Theorem VII.2.1]*). *Let $(E, \|\cdot\|_E)$ be a quasi-Banach lattice, $a, a_n \in E$. The following conditions are equivalent:*

(i) $\|a_n - a\|_E \to 0$ *for* $n \to \infty$*;*

(ii) *for any subsequence a_{n_k} there exists a subsequence $a_{n_{k_s}}$ such that $a_{n_{k_s}} \xrightarrow{(r)} a$.*

Proof. Without loss of generality we may assume that $a = 0$.

(i) $\Rightarrow$ (ii) For an equivalent p-additive quasi-norm $\|\!|\cdot|\!\|_E$ we have $\|\!||a_n||\!\|_E \to 0$ for $n \to \infty$. Hence, we may choose an increasing sequence of positive integers $n_1 < n_2 < \cdots < n_k < \cdots$ such that $\|\!||a_{n_k}||\!\|^p \leqslant \frac{1}{k^3}$. The estimate

$$\sum_{k=1}^{\infty} \|\!|k^{\frac{1}{p}}|a_{n_k}||\!\|^p = \sum_{k=1}^{\infty} k\|\!||a_{n_k}||\!\|^p \leqslant \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty,$$

shows that the series $\sum_{k=1}^{\infty} k^{\frac{1}{p}}|a_{n_k}|$ converges in $(E, \|\cdot\|_E)$ to some $b \in E_+$ (see Proposition 2.4) and therefore there exists $b = \sup_{n\geq 1} \sum_{k=1}^{n} k^{\frac{1}{p}}|a_{n_k}|$ such that we also have $k^{\frac{1}{p}}|a_{n_k}| \leqslant b$ for all $k \in \mathbb{N}$. In particular, $|a_{n_k}| \leqslant k^{-\frac{1}{p}}b$, which immediately implies $a_{n_k} \xrightarrow{(r)} 0$. The same reasoning may be repeated for any subsequence $\{a_{n_k}\}_{k=1}^{\infty}$.

The proof of the implication (ii) $\Rightarrow$ (i) is the verbatim repetition of the analogous result for Banach lattices [18, Chapter VII, Theorem VII.2.1]. □

Let m be the Lebesgue measure on the semi-axis $(0, \infty)$, let $L_1(0, \infty)$ be the Banach space of all integrable functions on $(0, \infty)$ with the norm $\|f\|_1 := \int_0^{\infty} |f|dm$ and let $L_\infty(0, \infty)$ be the Banach space of all essentially bounded measurable functions on $(0, \infty)$ with the norm $\|f\|_\infty := \operatorname{ess\,sup}\{|f(t)| : 0 < t < \infty\}$). For each $f \in L_1(0, \infty) + L_\infty(0, \infty)$ we define the decreasing rearrangement f^* of f by setting

$$f^*(t) := \inf\{s > 0 : m(\{|f| > s\}) \leqslant t\}, t > 0.$$

The function $f^*(t)$ is equimeasurable with $|f|$, in particular, $f^* \in L_1(0, \infty) + L_\infty(0, \infty)$ and $f^*(t)$ is non-increasing and right-continuous.

We need the following properties of decreasing rearrangements (see e.g. [19, Chapter II, Section 2]).

Proposition 4.2. *Let $f, g \in L_1(0, \infty) + L_\infty(0, \infty)$. We have*

(i) *if $|f| \leqslant |g|$, then $f^* \leqslant g^*$;*
(ii) *$(\alpha f)^* = |\alpha| f^*$ for all $\alpha \in \mathbb{R}$;*
(iii) *if $f \in L_\infty(0, \infty)$, then $(fg)^* \leqslant \|f\|_\infty g^*$;*
(iv) *$(f + g)^*(t + s) \leqslant f^*(t) + g^*(s)$;*
(v) *if $fg \in L_1(0, \infty) + L_\infty(0, \infty)$, then $(fg)^*(t + s) \leqslant f^*(t)g^*(s)$.*

Let l_∞ be the Banach lattice of all bounded real-valued sequences $\xi := \{\xi_n\}_{n=1}^{\infty}$ equipped with the norm $\|\xi\|_\infty = \sup_{n\geqslant 1} |\xi_n|$. For each $\xi = \{\xi_n\}_{n=1}^{\infty} \in l_\infty$ the function $f_\xi(t) := \sum_{n=1}^{\infty} \xi_n \chi_{[n-1,n)}(t), t > 0$ is contained in $L_\infty(0, \infty)$. For the decreasing rearrangement f_ξ^*, we obviously have $f_\xi^*(t) = \sum_{n=1}^{\infty} \xi_n^* \chi_{[n-1,n)}(t), t > 0$, where $\xi^* := \{\xi_n^*\}_{n=1}^{\infty}$ is a decreasing sequence of nonnegative numbers with $|\xi_1^*| = \sup_{n\geq 1} |\xi_n|$, which, in case when $\xi \in c_0$, coincides with the decreasing rearrangement of the sequence $\{|\xi_n|\}_{n=1}^{\infty}$. By Proposition 4.2(i), (ii) we have $\xi^* \leqslant \eta^*$ for $\xi, \eta \in l_\infty$ with $|\xi| \leqslant |\eta|$, and $(\alpha\xi)^* = |\alpha|\xi^*, \alpha \in \mathbb{R}$.

A linear subspace $\{0\} \neq E \subset l_\infty$ is said to be solid rearrangement-invariant, if for every $\eta \in E$ and every $\xi \in l_\infty$ the assumption $\xi^* \leqslant \eta^*$ implies that $\xi \in E$. Every solid rearrangement-invariant space E contains the space c_{00} of all finitely supported sequences from c_0. If E contains an element $\{\xi_n\}_{n=1}^{\infty} \notin c_0$, then $E = l_\infty$. Thus, for any solid rearrangement-invariant space $E \neq l_\infty$ the embeddings $c_{00} \subset E \subset c_0$ hold.

A solid rearrangement-invariant space E equipped with a complete quasi-norm (norm) $\|\cdot\|_E$ is called symmetric quasi-Banach (Banach) sequence space, if

(1) $\|\xi\|_E \leqslant \|\eta\|_E$, provided $\xi^* \leqslant \eta^*, \xi, \eta \in E$;
(2) $\|\{1, 0, 0, \ldots\}\|_E = 1$.

The inequality $\|a\xi\|_E \leqslant \|a\|_\infty \|\xi\|_E$ for all $a \in l_\infty, \xi \in E$ immediately follows from Proposition 4.2(iii). In particular, if $E = l_\infty$, then the norm $\|\cdot\|_E$ is equivalent to $\|\cdot\|_\infty$; for example, this is the case for any Lorentz space $(l_\psi, \|\cdot\|_\psi)$, where $\psi : [0, \infty) \to \mathbb{R}$ is an arbitrary nonnegative increasing concave function with the properties $\psi(0) = 0, \psi(+0) \neq 0, \lim_{t\to\infty} \psi(t) < \infty$ (see details in [19, Chapter II, Section 5]).

The spaces $(c_0, \|\cdot\|_\infty), (l_p, \|\cdot\|_p), 1 \leqslant p < \infty$ (respectively, $(l_p, \|\cdot\|_p)$ for $0 < p < 1$), where

$$l_p = \left\{\{\xi_n\}_{n=1}^{\infty} \in c_0 : \|\{\xi_n\}\|_p = \left(\sum_{n=1}^{\infty} |\xi_n|^p\right)^{\frac{1}{p}} < \infty\right\}$$

are examples of the classical symmetric Banach (respectively, quasi-Banach) sequence spaces in c_0.

Let $(E, \|\cdot\|_E)$ be a symmetric quasi-Banach sequence space. For every $\xi = \{\xi_n\}_{n=1}^{\infty} \in E, m \in \mathbb{N}$, we set

$$\sigma_m(\xi) = (\underbrace{\xi_1, \ldots, \xi_1}_{m \text{ times}}, \underbrace{\xi_2, \ldots, \xi_2}_{m \text{ times}}, \ldots),$$

$$\eta^{(1)} = (\xi_1, \underbrace{0, \ldots, 0}_{m-1 \text{ times}}, \xi_2, \underbrace{0, \ldots, 0}_{m-1 \text{ times}}, \ldots),$$

$$\eta^{(2)} = (0, \xi_1, \underbrace{0, \ldots, 0}_{m-2 \text{ times}}, 0, \xi_2, \underbrace{0, \ldots, 0}_{m-2 \text{ times}}, \ldots),$$

$$\ldots,$$

$$\eta^{(m)} = (\underbrace{0, \ldots, 0}_{m-1 \text{ times}}, \xi_1, \underbrace{0, \ldots, 0}_{m-1 \text{ times}}, \xi_2, \ldots).$$

Since $(\eta^{(1)})^* = (\eta^{(2)})^* = \cdots = (\eta^{(m)})^* = \xi^* \in E$, it follows $\eta^{(1)}, \ldots, \eta^{(m)} \in E$. Consequently, $\sigma_m(\xi) = \eta^{(1)} + \eta^{(2)} + \cdots + \eta^{(m)} \in E$, i.e. σ_m is a linear operator from E into E. In addition, we have

$$\begin{aligned}\|\sigma_m(\xi)\|_E &= \|\eta^{(1)} + \eta^{(2)} + \cdots + \eta^{(m)}\|_E \leqslant C(\|\eta^{(1)}\|_E + \|\eta^{(2)} + \eta^{(3)} + \cdots + \eta^{(m)}\|_E)\\ &\leqslant C(\|\eta^{(1)}\|_E + C(\|\eta^{(2)}\|_E + \|\eta^{(3)} + \cdots + \eta^{(m)}\|_E)) \leqslant (C + C^2 + \cdots + C^{m-1} + C^{m-1})\|\xi\|_E,\end{aligned}$$

where C is the modulus of concavity of the quasi-norm $\|\cdot\|_E$, in particular $\|\sigma_m\|_{\mathcal{B}(E,E)} \leqslant C + C^2 + \cdots + C^{m-2} + 2C^{m-1}$ for all $m \in \mathbb{N}$.

Proposition 4.3. *The inequalities*

$$(\xi + \eta)^* \leqslant \sigma_2(\xi^* + \eta^*), (\xi\eta)^* \leqslant \sigma_2(\xi^*\eta^*)$$

hold for all $\xi = \{\xi_n\}_{n=1}^\infty, \eta = \{\eta_n\}_{n=1}^\infty \in l_\infty$.

Proof. Since $f_{\xi+\eta}(t) = \sum_{n=1}^\infty (\xi_n + \eta_n)\chi_{[n-1,n)}(t) = f_\xi(t) + f_\eta(t)$, $t > 0$, we have by Proposition 4.2 (iv) that

$$\begin{aligned}\sum_{n=1}^\infty (\xi_n + \eta_n)^* \chi_{[n-1,n)}(2t) &= f^*_{\xi+\eta}(2t) = (f_\xi + f_\eta)^*(2t)\\ &\leqslant f^*_\xi(t) + f^*_\eta(t) = \sum_{n=1}^\infty (\xi^*_n + \eta^*_n)\chi_{[n-1,n)}(t) = \sum_{n=1}^\infty (\sigma_2(\xi^* + \eta^*))_n \chi_{[n-1,n)}(2t)\end{aligned}$$

for all $t > 0$, where $\{(\sigma_2(\xi^* + \eta^*))_n\}_{n=1}^\infty = \sigma_2(\xi^* + \eta^*)$. In other words, $(\xi + \eta)^* \leqslant \sigma_2(\xi^* + \eta^*)$. The proof of the inequality $(\xi\eta)^* \leqslant \sigma_2(\xi^*\eta^*)$ is very similar (one needs to use Proposition 4.2(v)) and is therefore omitted. □

For a symmetric quasi-Banach sequence space $(E, \|\cdot\|_E)$, we set

$$C_E := \{x \in \mathcal{K}(H) : \{s_n(x)\}_{n=1}^\infty \in E\}, \quad \|x\|_{C_E} := \|s_n(x)\|_E, x \in C_E.$$

If $E = l_p$ (respectively, $E = c_0$) then $C_{l_p} = C_p$, $\|\cdot\|_{C_{l_p}} = \|\cdot\|_{C_p}$, $0 < p < \infty$ (respectively, $C_{c_0} = \mathcal{K}(H)$, $\|\cdot\|_{C_{c_0}} = \|\cdot\|_{\mathcal{B}(H)}$).

A quasi-Banach vector sublattice $(E, \|\cdot\|_E)$ in l_∞ is said to be p-convex, $0 < p < \infty$, if there is a constant M, so that

$$\left\| \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} \right\|_E \leqslant M \left(\sum_{i=1}^n \|x_i\|_E^p \right)^{\frac{1}{p}} \tag{2}$$

for every finite collection $\{x_i\}_{i=1}^n \subset E$, $n \in \mathbb{N}$.

If the estimate (2) holds for elements from a symmetric quasi-Banach ideal $(\mathcal{I}, \|\cdot\|_\mathcal{I})$ of compact operators from $\mathcal{B}(H)$, then the ideal $(\mathcal{I}, \|\cdot\|_\mathcal{I})$ is said to be p-convex. As already stated in Theorem 1.5, for every symmetric Banach (respectively, symmetric p-convex quasi-Banach, $0 < p < \infty$) sequence space E in c_0 the couple $(C_E, \|\cdot\|_{C_E})$ is a symmetric Banach (respectively, p-convex symmetric quasi-Banach) ideal of compact operators in $\mathcal{B}(H)$.

Thus, for every symmetric Banach (p-convex quasi-Banach) sequence space $(E, \|\cdot\|_E)$ the corresponding symmetric Banach (p-convex quasi-Banach) ideal $(C_E, \|\cdot\|_{C_E})$ of compact operators from $\mathcal{B}(H)$ is naturally constructed. This extends the classical Calkin correspondence [4].

Conversely, if $(\mathcal{I}, \|\cdot\|_\mathcal{I})$ is a symmetric quasi-Banach ideal $(\mathcal{I}, \|\cdot\|_\mathcal{I})$ of compact operators from $\mathcal{B}(H)$, then it is of the form $C_{E_\mathcal{I}}$ with $\|\cdot\|_\mathcal{I} = \|\cdot\|_{C_{E_\mathcal{I}}}$ for the corresponding symmetric quasi-Banach sequence space $(E_\mathcal{I}, \|\xi\|_{E_\mathcal{I}})$. The definition of the latter space is given below.

Denote by $E_\mathcal{I}$ the set of all $\xi \in c_0$, for which there exists some $x \in \mathcal{I}$, such that $\xi^* = \{s_n(x)\}_{n=1}^\infty$. For $\xi \in E_\mathcal{I}$ with $\xi^* = \{s_n(x)\}_{n=1}^\infty$, $x \in \mathcal{I}$ set $\|\xi\|_{E_\mathcal{I}} = \|x\|_\mathcal{I}$.

Fix an orthonormal set $\{e_n\}_{n=1}^\infty$ in H and for every $\xi = \{\xi_n\}_{n=1}^\infty \in c_0$ consider the diagonal operator $x_\xi \in \mathcal{K}(H)$ defined as follows

$$x_\xi(\varphi) = \sum_{n=1}^\infty \xi_n c_n(\varphi) e_n,$$

where $c_n(\varphi) = (\varphi, e_n)$, $\varphi \in H$. If $\xi \in E_\mathcal{I}$, then $\xi^* = \{s_n(x)\}_{n=1}^\infty$ for some $x \in \mathcal{I}$, and due to equalities $\{s_n(x_{\xi^*})\}_{n=1}^\infty = \{\xi^*_n\}_{n=1}^\infty = \{s_n(x)\}_{n=1}^\infty$ we have $x_{\xi^*} \in \mathcal{I}$ and $\|x_{\xi^*}\|_\mathcal{I} = \|x\|_\mathcal{I} = \|\xi\|_{E_\mathcal{I}}$ (see Proposition 2.6(b)). Moreover, since $\{s_n(x_\xi)\}_{n=1}^\infty = \{s_n(x_{\xi^*})\}_{n=1}^\infty$ and $x_{\xi^*} \in \mathcal{I}$, it follows that $x_\xi \in \mathcal{I}$ and $\|\xi\|_{E_\mathcal{I}} = \|x_\xi\|_\mathcal{I}$. Thus, a sequence $\xi \in c_0$ is contained in $E_\mathcal{I}$, if and only if operators x_ξ and x_{ξ^*} are in $\mathcal{I}$, in addition, $\|\xi\|_{E_\mathcal{I}} = \|x_{\xi^*}\|_\mathcal{I} = \|x_\xi\|_\mathcal{I}$. In particular, if $\eta \in c_0$, $\xi \in E_\mathcal{I}$, $\eta^* \leqslant \xi^*$, then $\eta \in E_\mathcal{I}$ and $\|\eta\|_{E_\mathcal{I}} \leqslant \|\xi\|_{E_\mathcal{I}}$.

Theorem 4.4. *For any symmetric quasi-Banach ideal $\mathcal{I}$ of compact operators from $\mathcal{B}(H)$ the couple $(E_\mathcal{I}, \|\cdot\|_{E_\mathcal{I}})$ is a symmetric quasi-Banach sequence space in c_0 with the modulus of concavity which does not exceed the modulus of concavity of the quasi-norm $\|\cdot\|_\mathcal{I}$, in addition, $C_{E_\mathcal{I}} = \mathcal{I}$ and $\|\cdot\|_{C_{E_\mathcal{I}}} = \|\cdot\|_\mathcal{I}$.*

Proof. If $\xi, \eta \in E_{\mathcal{I}}$, then $x_\xi, x_\eta \in \mathcal{I}$, hence $x_\xi + x_\eta \in \mathcal{I}$. Since

$$(x_\xi + x_\eta)(\varphi) = \sum_{n=1}^{\infty} \xi_n c_n(\varphi) e_n + \sum_{n=1}^{\infty} \eta_n c_n(\varphi) e_n = \sum_{n=1}^{\infty} (\xi_n + \eta_n) c_n(\varphi) e_n = x_{\xi+\eta}(\varphi), \quad \varphi \in H,$$

we have $x_{\xi+\eta} \in \mathcal{I}$. Consequently, $\xi + \eta \in E_{\mathcal{I}}$, moreover,

$$\|\xi + \eta\|_{E_{\mathcal{I}}} = \|x_{\xi+\eta}\|_{\mathcal{I}} = \|x_\xi + x_\eta\|_{\mathcal{I}} \leqslant C(\|x_\xi\|_{\mathcal{I}} + \|x_\eta\|_{\mathcal{I}}) = C(\|\xi\|_{E_{\mathcal{I}}} + \|\eta\|_{E_{\mathcal{I}}}),$$

where C is the modulus of concavity of the quasi-norm $\|\cdot\|_{\mathcal{I}}$.

Now, let $\xi \in E_{\mathcal{I}}$, $\alpha \in \mathbb{R}$. Since

$$x_{\alpha\xi}(\varphi) = \sum_{n=1}^{\infty} \alpha \xi_n c_n(\varphi) e_n = \alpha x_\xi(\varphi), \quad \varphi \in H,$$

we have $\alpha\xi \in E_{\mathcal{I}}$ and $\|\alpha\xi\|_{E_{\mathcal{I}}} = \|x_{\alpha\xi}\|_{\mathcal{I}} = \|\alpha x_\xi\|_{\mathcal{I}} = |\alpha|\|x_\xi\|_{\mathcal{I}} = |\alpha|\|\xi\|_{E_{\mathcal{I}}}$.

It is easy to see that $\|\xi\|_{E_{\mathcal{I}}} \geqslant 0$ and $\|\xi\|_{E_{\mathcal{I}}} = 0 \Leftrightarrow \xi = 0$.

Hence, $E_{\mathcal{I}}$ is a solid rearrangement-invariant subspace in c_0 and $\|\cdot\|_{E_{\mathcal{I}}}$ is a quasi-norm on $E_{\mathcal{I}}$.

Let us show that $(E_{\mathcal{I}}, \|\cdot\|_{E_{\mathcal{I}}})$ is a quasi-Banach space. Let $|||\cdot|||_{\mathcal{I}}$ (respectively, $|||\cdot|||_{E_{\mathcal{I}}}$) be a p-additive (respectively, q-additive) quasi-norm equivalent to the quasi-norm $\|\cdot\|_{\mathcal{I}}$ (respectively, $\|\cdot\|_{E_{\mathcal{I}}}$), $0 < p, q \leqslant 1$.

Let $\xi^{(k)} = \{\xi_n^{(k)}\}_{n=1}^{\infty} \in E_{\mathcal{I}}$ and $|||\xi^{(k)} - \xi^{(m)}|||_{E_{\mathcal{I}}} \to 0$ for $k, m \to \infty$. Then $\|x_{\xi^{(k)}} - x_{\xi^{(m)}}\|_{\mathcal{I}} \to 0$ and $|||x_{\xi^{(k)}} - x_{\xi^{(m)}}|||_{\mathcal{I}}^p \to 0$ for $k, m \to \infty$, i.e. $x_{\xi^{(k)}}$ is a Cauchy sequence in $(\mathcal{I}, d_{\mathcal{I}})$, where $d_{\mathcal{I}}(x, y) = |||x - y|||_{\mathcal{I}}^p$. Since $(\mathcal{I}, d_{\mathcal{I}})$ is a complete metric space, there exists an operator $x \in \mathcal{I}$ such that $|||x_{\xi^{(k)}} - x|||_{\mathcal{I}}^p \to 0$ for $k \to \infty$. If p_n is the one-dimensional projection onto subspace spanned by e_n, then

$$\xi^{(k)} p_n = p_n x_{\xi_n^{(k)}} p_n \xrightarrow{\|\cdot\|_{\mathcal{I}}} p_n x p_n := \lambda_n p_n,$$
$$0 = p_n x_{\xi_n^{(k)}} p_m \to p_n x p_m, \quad n \neq m.$$

Hence, x is also a diagonal operator, i.e. $x = x_\xi$, where $\xi = \{\lambda_n\}_{n=1}^{\infty}$. Since $x \in \mathcal{I}$ we have $\xi \in E_{\mathcal{I}}$, moreover, $\|\xi^{(k)} - \xi\|_{E_{\mathcal{I}}} = \|x_{\xi^{(k)}} - x_\xi\|_{\mathcal{I}} \to 0$ for $k \to \infty$.

Consequently, $(E_{\mathcal{I}}, \|\cdot\|_{E_{\mathcal{I}}})$ is a symmetric quasi-Banach sequence space in c_0.

Now, let us show that $C_{E_{\mathcal{I}}} = \mathcal{I}$ and $\|x\|_{C_{E_{\mathcal{I}}}} = \|x\|_{\mathcal{I}}$ for all $x \in \mathcal{I}$. Let $x \in C_{E_{\mathcal{I}}}$, i.e. $\{s_n(x)\}_{n=1}^{\infty} \in E_{\mathcal{I}}$. Hence, there exists an operator $y \in \mathcal{I}$, such that $s_n(x) = s_n(y)$, $n \in \mathbb{N}$. Consequently, $x \in \mathcal{I}$, moreover, $\|x\|_{\mathcal{I}} = \|\{s_n(x)\}_{n=1}^{\infty}\|_{E_{\mathcal{I}}} = \|x\|_{C_{E_{\mathcal{I}}}}$. Conversely, if $x \in \mathcal{I}$, then $\{s_n(x)\}_{n=1}^{\infty} \in E_{\mathcal{I}}$ and therefore $x \in C_{E_{\mathcal{I}}}$. □

The definition of symmetric Banach (p-convex quasi-Banach) ideal $(C_E, \|\cdot\|_{C_E})$ of compact operators from $\mathcal{B}(H)$ jointly with Theorem 4.4 implies the following corollary:

Corollary 4.5. *Let $(E, \|\cdot\|_E)$ be a symmetric Banach (p-convex quasi-Banach) sequence space from c_0. Then $E_{C_E} = E$ and $\|\cdot\|_{E_{C_E}} = \|\cdot\|_E$.*

Proof. If $\xi \in E$, then $x_{\xi^*} \in C_E$, and due to the equality $\{s_n(x_{\xi^*})\}_{n=1}^{\infty} = \xi^*$, we have $\xi \in E_{C_E}$ and $\|\xi\|_{E_{C_E}} = \|x_{\xi^*}\|_{C_E} = \|\xi^*\|_E = \|\xi\|_E$. The converse inclusion $E_{C_E} \subset E$ may be proven similarly. □

Let G, F be solid rearrangement-invariant spaces in c_0. It is easy to see that G and F are ideals in the algebra l_∞, in particular, it follows from the assumptions $|\xi| \leqslant |\eta|$, $\xi \in l_\infty$, $\eta \in G$ that $\xi \in G$, i.e. G and F are solid linear subspaces in l_∞. We define F-dual space $F : G$ of G by setting

$$F : G = \{\xi \in l_\infty : \xi\eta \in F,\ \forall \eta \in G\}.$$

It is clear that $F : G$ is an ideal in l_∞ containing c_{00}. If $G \subset F$, then $F : G = l_\infty$, in particular, $l_\infty : G = l_\infty$ for any solid rearrangement-invariant space G. However, if $G \not\subseteq F$, then $F : G \neq l_\infty$.

Proposition 4.6. *If $F : G \neq l_\infty$, then $F : G \subset c_0$.*

Proof. Suppose that there exists $\xi = \{\xi_n\}_{n=1}^{\infty} \in (F : G)$, $\xi \notin c_0$. Let $\alpha_n = \text{sign}\xi_n$, $n \in \mathbb{N}$, $\eta = \{\eta_n\}_{n=1}^{\infty} \in G$. Obviously, $\{\alpha_n\eta_n\}_{n=1}^{\infty} \in G$ and hence, $|\xi|\eta = \{\xi_n\alpha_n\eta_n\}_{n=1}^{\infty} \in F$ for all $\eta \in G$, that is $|\xi| \in (F : G)$, and, in addition, $|\xi| \notin c_0$. This implies that there exists a subsequence $0 \neq |\xi_{n_k}| \to \alpha > 0$ for $k \to \infty$. Consider a sequence $\zeta = \{\zeta_k\}_{k=1}^{\infty}$ from $l_\infty \setminus c_0$ such that $\zeta_k = |\xi_{n_k}|$ and show that $\zeta \in F : G$.

For every $\eta = \{\eta_n\}_{n=1}^{\infty} \in G$ define the sequence $a_\eta = \{a_n\}_{n=1}^{\infty}$ such that $a_{n_k} = \eta_k$ and $a_n = 0$, if $n \neq n_k$, $k \in \mathbb{N}$. Since $a_\eta^* = \eta^*$, we have $a_\eta \in G$, and therefore $\zeta\eta = \{|\xi_{n_k}|\eta_k\}_{k=1}^{\infty} = \{|\xi_n|a_n\}_{n=1}^{\infty} = |\xi|a_\eta \in F$ for all $\eta \in G$. Consequently, $\zeta = \{\zeta_n\}_{n=1}^{\infty} \in F : G$, moreover, $\zeta_n \geqslant \beta$ for some $\beta > 0$ and all $n \in \mathbb{N}$. Since $F : G$ is an ideal in l_∞, it follows that $F : G$ is a solid linear subspace in l_∞, containing the sequence $\{\zeta_n\}_{n=1}^{\infty}$ with $\zeta_n \geqslant \beta > 0$, $n \in \mathbb{N}$, that implies $F : G = l_\infty$. □

Proposition 4.7. *If* $F : G \neq l_\infty$, *then* $F : G = \{\xi \in c_0 : \xi^*\eta^* \in F, \ \forall \eta \in G\}$.

Proof. By Proposition 4.6, we have that $F : G \subset c_0$. Let $\xi = \{\xi_n\}_{n=1}^\infty \in c_0$ and $\xi^*\eta^* \in F$ for all $\eta \in G$. Due to Proposition 4.3, we have $(\xi\eta)^* \leqslant \sigma_2(\xi^*\eta^*) \in F$, i.e. $(\xi\eta)^* \in F$. Since F is a symmetric sequence space, it follows that $\xi\eta \in F$ for all $\eta \in G$, i.e. $\xi \in F : G$.

Conversely, suppose that $\xi = \{\xi_n\}_{n=1}^\infty \in F : G$. Let $\alpha_n = \text{sign}\xi_n$, $\eta = \{\eta_n\}_{n=1}^\infty \in G$. Then $\{\alpha_n\eta_n\}_{n=1}^\infty \in G$, and therefore $|\xi|\eta = \{\xi_n\alpha_n\eta_n\}_{n=1}^\infty \in F$ for all $\eta \in G$, i.e. $|\xi| \in F : G \subset c_0$. Since $|\xi| = \{|\xi_n|\}_{n=1}^\infty \in c_0$, there exists a bijection of the set $\mathbb{N}$ of natural numbers, such that $\xi^* = |\xi_{\pi(n)}|$. For linear bijective mapping $U_\pi : l_\infty \to l_\infty$ defined by $U_\pi(\{\eta_n\}_{n=1}^\infty) = \{\eta_{\pi(n)}\}_{n=1}^\infty$ we have $U_\pi(\eta\zeta) = U_\pi(\eta)U_\pi(\zeta)$, $\big(U_\pi(\zeta)\big)^* = \zeta^*$, $\big(U_\pi^{-1}(\zeta)\big)^* = \zeta^*$ for all $\zeta \in l_\infty$, in particular, $U_\pi(E) = E$ for any solid rearrangement-invariant space $E \subset l_\infty$. Consequently, for all $\eta \in G$ we have $\xi^*\eta^* = U_\pi(|\xi|)U_\pi(U_\pi^{-1}(\eta^*)) = U_\pi(|\xi|U_\pi^{-1}(\eta^*)) \in F$. □

Propositions 4.6 and 4.7 imply the following corollary.

Corollary 4.8. $F : G$ *is a solid rearrangement-invariant space, moreover, if* $F : G \neq l_\infty$, *then* $c_{00} \subset F : G \subset c_0$.

Proof. The definition of $F : G$ immediately implies that $F : G$ is an ideal in l_∞ and $c_{00} \subset F : G$. If $F : G \neq l_\infty$, then, due to Proposition 4.6, we have $F : G \subset c_0$.

In the case when $F : G \neq l_\infty$, we have for any $\xi \in c_0$, $\eta \in F : G$, $\xi^* \leqslant \eta^*$, $\zeta \in G$ that $\xi^*\zeta^* \leqslant \eta^*\zeta^* \in F$ (see Proposition 4.7). Consequently, $\xi^*\zeta^* \in F$ for any $\zeta \in G$, which implies the inclusion $\xi \in F : G$. □

We need some complementary properties of singular values of compact operators. For every operator $x \in \mathcal{B}(H)$ define the decreasing rearrangement $\mu(x, t)$ of x by setting

$$\mu(x, t) = \inf\{s > 0 : \text{tr}(|x| > s) \leqslant t\}, \quad t > 0$$

(see e.g. [20]). If $x \in \mathcal{K}(H)$, then

$$\mu(x, t) = \sum_{n=1}^\infty s_n(x)\chi_{[n-1,n)}(t) = f^*_{\{s_n(x)\}_{n=1}^\infty}(t).$$

In [20, Lemma 2.5 (v),(vii)] it is established that for every $x, y \in \mathcal{B}(H)$ the inequalities

$$\mu(x + y, t + s) \leqslant \mu(x, t) + \mu(y, s),$$
$$\mu(xy, t + s) \leqslant \mu(x, t)\mu(y, s)$$

hold, in particular, if $x, y \in \mathcal{K}(H)$, then

$$\{s_n(x + y)\}_{n=1}^\infty \leqslant \sigma_2\big(\{s_n(x) + s_n(y)\}_{n=1}^\infty\big), \tag{3}$$
$$\{s_n(xy)\}_{n=1}^\infty \leqslant \sigma_2\big(\{s_n(x)s_n(y)\}_{n=1}^\infty\big). \tag{4}$$

Let $\mathcal{I}$, $\mathcal{J}$ be symmetric quasi-Banach ideals of compact operators from $\mathcal{B}(H)$ and $\mathcal{I} \not\subseteq \mathcal{J}$. In this case, $\mathcal{J} : \mathcal{I} \subset \mathcal{K}(H)$ (see Proposition 3.3) and $E_\mathcal{I} \not\subseteq E_\mathcal{J}$ (see Theorem 4.4), therefore $E_\mathcal{J} : E_\mathcal{I} \subset c_0$ (see Proposition 4.6). The following proposition establishes that the set of operators belonging to the $\mathcal{J}$-dual space $\mathcal{J} : \mathcal{I}$ of $\mathcal{I}$ coincides with the set

$$C_{E_\mathcal{J}:E_\mathcal{I}} = \{x \in \mathcal{K}(H) : \{s_n(x)\}_{n=1}^\infty \in E_\mathcal{J} : E_\mathcal{I}\}.$$

Proposition 4.9. $\mathcal{J} : \mathcal{I} = C_{E_\mathcal{J}:E_\mathcal{I}}$.

Proof. Let $a \in \mathcal{J} : \mathcal{I}$. We claim that $a \in C_{E_\mathcal{J}:E_\mathcal{I}}$, i.e. $\xi = \{s_n(a)\}_{n=1}^\infty \in E_\mathcal{J} : E_\mathcal{I}$. For any sequence $\eta \in E_\mathcal{I}$ consider operators x_ξ and x_{η^*}. Since $x_\xi \in \mathcal{J} : \mathcal{I}$, $x_{\eta^*} \in \mathcal{I}$, we have $x_\xi x_{\eta^*} \in \mathcal{J}$. On the other hand, $x_\xi x_{\eta^*}(\varphi) = \|\cdot\|_H - \lim_{n\to\infty}\big(\sum_{k=1}^n s_k(a)c_k(x_{\eta^*}(\varphi))e_k\big) = \sum_{n=1}^\infty s_n(a)\eta_n^* c_n(\varphi)e_n = x_{\xi\eta^*}(\varphi)$ for all $\varphi \in H$. Thus $x_{\xi\eta^*} \in \mathcal{J}$, i.e. $\xi\eta^* \in E_\mathcal{J}$. Consequently, $\{s_n(a)\}_{n=1}^\infty \in E_\mathcal{J} : E_\mathcal{I}$ (see Proposition 4.7) yielding our claim.

Conversely, let $a \in C_{E_\mathcal{J}:E_\mathcal{I}}$, i.e. $\{s_n(a)\}_{n=1}^\infty \in E_\mathcal{J} : E_\mathcal{I}$. Due to (4), for all $x \in \mathcal{I}$ we have $\{s_n(ax)\}_{n=1}^\infty \leqslant \sigma_2(\{s_n(a)s_n(x)\}_{n=1}^\infty)$. Since $\{s_n(a)s_n(x)\}_{n=1}^\infty \in E_\mathcal{J}$, it follows that $\sigma_2(\{s_n(a)s_n(x)\}_{n=1}^\infty) \in E_\mathcal{J}$, and therefore $\{s_n(ax)\}_{n=1}^\infty \in E_\mathcal{J}$, i.e. $ax \in \mathcal{J}$. Consequently, $a \in \mathcal{J} : \mathcal{I}$. □

Let $\mathcal{I}$, $\mathcal{J}$ be symmetric quasi-Banach ideals of compact operators from $\mathcal{B}(H)$, $\mathcal{I} \not\subseteq \mathcal{J}$ and $\mathcal{J} : \mathcal{I}$ be the $\mathcal{J}$-dual space of $\mathcal{I}$. For any $a \in \mathcal{J} : \mathcal{I}$ define a linear mapping $T_a : \mathcal{I} \to \mathcal{J}$ by setting $T_a(x) = ax$, $x \in \mathcal{I}$.

Proposition 4.10. T_a *is a continuous linear mapping from* $\mathcal{I}$ *into* $\mathcal{J}$ *for every* $a \in \mathcal{J} : \mathcal{I}$.

Proof. Let $a \in \mathcal{J} : \mathcal{I}$, $\xi = \{s_n(a)\}_{n=1}^{\infty}$, $x_k \in \mathcal{I}$ and $\|x_k\|_{\mathcal{I}} \to 0$ for $k \to \infty$. Then $\xi^{(k)} = \{s_n(x_k)\}_{n=1}^{\infty} \in E_{\mathcal{I}}$ and $\|\xi^{(k)}\|_{E_{\mathcal{I}}} \to 0$. By Proposition 4.1, for every subsequence $\{\xi^{(k_l)}\}_{l=1}^{\infty}$ there exists a subsequence $\{\xi^{(k_{l_s})}\}_{s=1}^{\infty}$ such that $\xi^{(k_{l_s})} \xrightarrow{(r)} 0$ for $s \to \infty$, i.e. there exist $0 \leqslant \eta \in E_{\mathcal{I}}$ and a sequence $\{\varepsilon_s\}_{s=1}^{\infty}$ of positive numbers decreasing to zero such that $|\xi^{(k_{l_s})}| \leqslant \varepsilon_s \eta$. Since $a \in \mathcal{J} : \mathcal{I}$, we have $\xi \in E_{\mathcal{J}} : E_{\mathcal{I}}$ (see Proposition 4.9), and therefore $\zeta = \xi\eta \in E_{\mathcal{J}}$, in addition, $\zeta \geqslant 0$. Since $|\xi\xi^{(k_{l_s})}| \leqslant \varepsilon_s \zeta$, it follows that $\xi\xi^{(k_{l_s})} \xrightarrow{(r)} 0$. By Proposition 4.1, we have $\|\xi\xi^{(k)}\|_{E_{\mathcal{J}}} \to 0$. Consequently,

$$\|ax_k\|_{\mathcal{J}} = \|\{s_n(ax_k)\}\|_{E_{\mathcal{J}}} \leqslant \|\sigma_2(\xi\xi^{(k)})\|_{E_{\mathcal{J}}} \leqslant 2C\|\xi\xi^{(k)}\|_{E_{\mathcal{J}}} \to 0 \quad \text{for } k \to \infty. \quad \square$$

By Proposition 4.10, T_a is a bounded linear operator from $\mathcal{I}$ into $\mathcal{J}$, therefore $\|T_a\|_{\mathcal{B}(\mathcal{I},\mathcal{J})} = \sup\{\|T_a(x)\|_{\mathcal{J}} : \|x\|_{\mathcal{I}} \leqslant 1\} = \sup\{\|ax\|_{\mathcal{J}} : \|x\|_{\mathcal{I}} \leqslant 1\} < \infty$, i.e. for all $a \in \mathcal{J} : \mathcal{I}$ the quantity

$$\|a\|_{\mathcal{J}:\mathcal{I}} := \sup\{\|ax\|_{\mathcal{J}} : x \in \mathcal{I},\ \|x\|_{\mathcal{I}} \leqslant 1\}$$

is well-defined.

Theorem 4.11. *Let $\mathcal{I}, \mathcal{J}$ be symmetric quasi-Banach ideals of compact operators in $\mathcal{B}(H)$ such that $\mathcal{I} \not\subseteq \mathcal{J}$. Then $(\mathcal{J} : \mathcal{I}, \|\cdot\|_{\mathcal{J}:\mathcal{I}})$ is a symmetric quasi-Banach ideal of compact operators whose modulus of concavity does not exceed the modulus of concavity of the quasi-norm $\|\cdot\|_{\mathcal{J}}$, in addition, $\|ax\|_{\mathcal{J}} \leqslant \|a\|_{\mathcal{J}:\mathcal{I}}\|x\|_{\mathcal{I}}$ for all $a \in \mathcal{J} : \mathcal{I}$, $x \in \mathcal{I}$.*

Proof. Since $\|\cdot\|_{\mathcal{B}(\mathcal{I},\mathcal{J})}$ is a quasi-norm with the modulus of concavity which does not exceed the modulus of concavity of the quasi-norm $\|\cdot\|_{\mathcal{J}}$, we see that $\|\cdot\|_{\mathcal{J}:\mathcal{I}}$ is a quasi-norm on $\mathcal{J} : \mathcal{I}$ with the modulus of concavity which does not exceed the modulus of concavity of the quasi-norm $\|\cdot\|_{\mathcal{J}}$.

If $y \in \mathcal{B}(H)$, $a \in \mathcal{J} : \mathcal{I}$, then

$$\begin{aligned}\|ya\|_{\mathcal{J}:\mathcal{I}} &= \sup\{\|(ya)x\|_{\mathcal{J}} : x \in \mathcal{I},\ \|x\|_{\mathcal{I}} \leqslant 1\}\\ &\leqslant \sup\{\|y\|_{\mathcal{B}(H)}\|ax\|_{\mathcal{J}} : x \in \mathcal{I},\ \|x\|_{\mathcal{I}} \leqslant 1\} = \|y\|_{\mathcal{B}(H)}\|a\|_{\mathcal{J}:\mathcal{I}}.\end{aligned}$$

Since $yx \in \mathcal{I}$ for all $x \in \mathcal{I}$ and $\|yx\|_{\mathcal{I}} \leqslant \|y\|_{\mathcal{B}(H)}\|x\|_{\mathcal{I}}$ then for $y \neq 0$ and $\|x\|_{\mathcal{I}} \leqslant 1$ we have $\|\frac{yx}{\|y\|_{\mathcal{B}(H)}}\|_{\mathcal{I}} \leqslant 1$. Hence,

$$\begin{aligned}\|ay\|_{\mathcal{J}:\mathcal{I}} &= \sup\{\|a(yx)\|_{\mathcal{J}} : x \in \mathcal{I},\ \|x\|_{\mathcal{I}} \leqslant 1\}\\ &= \|y\|_{\mathcal{B}(H)} \sup\left\{\left\|a\left(\frac{yx}{\|y\|_{\mathcal{B}(H)}}\right)\right\|_{\mathcal{J}} : x \in \mathcal{I},\ \|x\|_{\mathcal{I}} \leqslant 1\right\}\\ &\leqslant \|y\|_{\mathcal{B}(H)} \sup\{\|ax\|_{\mathcal{J}} : x \in \mathcal{I},\ \|x\|_{\mathcal{I}} \leqslant 1\} = \|y\|_{\mathcal{B}(H)}\|a\|_{\mathcal{I}:\mathcal{J}}.\end{aligned}$$

If p is a one-dimensional projection from $\mathcal{B}(H)$, then $p \in \mathcal{I}$, $\|p\|_{\mathcal{I}} = 1$, and so

$$\|p\|_{\mathcal{J}:\mathcal{I}} = \sup\{\|px\|_{\mathcal{J}} : x \in \mathcal{I},\ \|x\|_{\mathcal{I}} \leqslant 1\} \geqslant \|p\|_{\mathcal{J}} = 1.$$

On the other hand, for $x \in \mathcal{I}$ with $\|x\|_{\mathcal{I}} \leqslant 1$ we have $\|x\|_{\mathcal{B}(H)} \leqslant 1$ (see Proposition 2.6(c)), and therefore

$$\|px\|_{\mathcal{J}} = \|p(px)\|_{\mathcal{J}} \leqslant \|px\|_{\mathcal{B}(H)}\|p\|_{\mathcal{J}} \leqslant 1.$$

Consequently, $\|p\|_{\mathcal{J}:\mathcal{I}} = 1$.

Thus, $\|\cdot\|_{\mathcal{J}:\mathcal{I}}$ is a symmetric quasi-norm on the two-sided ideal $\mathcal{J} : \mathcal{I}$. The inequality $\|ax\|_{\mathcal{J}} \leqslant \|a\|_{\mathcal{J}:\mathcal{I}}\|x\|_{\mathcal{I}}$ immediately follows from the definition of $\|\cdot\|_{\mathcal{J}:\mathcal{I}}$.

Let us show that $(\mathcal{J} : \mathcal{I}, \|\cdot\|_{\mathcal{J}:\mathcal{I}})$ is a quasi-Banach space.

Denote by $|||\cdot|||_{\mathcal{J}}$ (respectively $|||\cdot|||_{\mathcal{J}:\mathcal{I}}$) a p-additive (respectively, q-additive) quasi-norm on $\mathcal{J}$ (respectively, on $\mathcal{J} : \mathcal{I}$) which is equivalent to the quasi-norm $\|\cdot\|_{\mathcal{J}}$ (respectively, $\|\cdot\|_{\mathcal{J}:\mathcal{I}}$), where $0 < p, q \leqslant 1$. In particular, we have $\alpha_1|||x|||_{\mathcal{J}} \leqslant \|x\|_{\mathcal{J}} \leqslant \beta_1|||x|||_{\mathcal{J}}$ and $\alpha_2|||a|||_{\mathcal{J}:\mathcal{I}} \leqslant \|a\|_{\mathcal{J}:\mathcal{I}} \leqslant \beta_2|||a|||_{\mathcal{J}:\mathcal{I}}$ for all $x \in \mathcal{J}$, $a \in \mathcal{J} : \mathcal{I}$ and some constants $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$. Let $d_{\mathcal{J}}(x, y) = |||x - y|||_{\mathcal{J}}^p$, $d_{\mathcal{J}:\mathcal{I}}(a, b) = |||a - b|||_{\mathcal{J}:\mathcal{I}}^q$ be metrics on $\mathcal{J}$ and $\mathcal{J} : \mathcal{I}$ respectively.

Let $\{a_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $(\mathcal{J} : \mathcal{I}, d_{\mathcal{J}:\mathcal{I}})$, i.e. $|||a_n - a_m|||_{\mathcal{J}:\mathcal{I}}^q \leqslant \varepsilon^q$ for all $n, m \geqslant n(\varepsilon)$, $\varepsilon > 0$, thus

$$\begin{aligned}|||a_nx - a_mx|||_{\mathcal{J}} &\leqslant \frac{1}{\alpha_1}\|a_nx - a_mx\|_{\mathcal{J}} \leqslant \frac{1}{\alpha_1}\|a_n - a_m\|_{\mathcal{J}:\mathcal{I}}\|x\|_{\mathcal{I}}\\ &\leqslant \frac{\beta_2}{\alpha_1}|||a_n - a_n|||_{\mathcal{J}:\mathcal{I}}\|x\|_{\mathcal{I}} \leqslant \frac{\beta_2}{\alpha_1}\varepsilon\|x\|_{\mathcal{I}}\end{aligned} \tag{5}$$

for all $x \in \mathcal{I}$, $n, m \geqslant n(\varepsilon)$. Consequently, the sequence $\{a_nx\}_{n=1}^{\infty}$ is a Cauchy sequence in $(\mathcal{J}, d_{\mathcal{J}})$, $x \in \mathcal{I}$. Since the metric space $(\mathcal{J}, d_{\mathcal{J}})$ is complete, there exists an operator $z(x) \in \mathcal{J}$ such that $|||a_nx - z(x)|||_{\mathcal{J}}^p \to 0$ for $n \to \infty$. Since

$$\|a_nx - z(x)\|_{\mathcal{B}(H)} \leqslant \|a_nx - z(x)\|_{\mathcal{J}} \leqslant \beta_1|||a_nx - z(x)|||_{\mathcal{J}},$$

it follows that $\|a_nx - z(x)\|_{\mathcal{B}(H)} \to 0$.

Since

$$\|a_n - a_m\|_{\mathcal{B}(H)} \leqslant \|a_n - a_m\|_{\mathcal{J}:\mathcal{I}} \leqslant \beta_2 |||a_n - a_m|||_{\mathcal{J}:\mathcal{I}} \to 0$$

for $n, m \to \infty$, there exists $a \in \mathcal{B}(H)$ such that $\|a_n - a\|_{\mathcal{B}(H)} \to 0$ for $n \to \infty$. For an arbitrary $x \in \mathcal{I}$, we have $\|a_n x - ax\|_{\mathcal{B}(H)} \leqslant \|a_n - a\|_{\mathcal{B}(H)} \|x\|_{\mathcal{I}} \to 0$ for $n \to \infty$.

Thus, $ax = z(x)$ for all $x \in \mathcal{I}$. Since $z(x) \in \mathcal{J}$ for all $x \in \mathcal{I}$, it follows that $a \in \mathcal{J} : \mathcal{I}$, moreover, due to (5), $\|a_n x - ax\|_{\mathcal{J}} \leqslant \frac{\beta_1\beta_2}{\alpha_1} \varepsilon \|x\|_{\mathcal{I}}$ for $n \geqslant n(\varepsilon)$ and for all $x \in \mathcal{I}$. Consequently,

$$|||a_n - a|||_{\mathcal{J}:\mathcal{I}} \leqslant \frac{1}{\alpha_2} \|a_n - a\|_{\mathcal{J}:\mathcal{I}} = \frac{1}{\alpha_2} \sup\left\{ \|a_n x - ax\|_{\mathcal{J}} : x \in \mathcal{I}, \|x\|_{\mathcal{I}} \leqslant 1 \right\} \leqslant \frac{\beta_1\beta_2}{\alpha_1\alpha_2} \varepsilon$$

for $n \geqslant n(\varepsilon)$, i.e. $|||a_n - a|||_{\mathcal{J}:\mathcal{I}} \to 0$. Thus, the metric space $(\mathcal{J} : \mathcal{I}, d_{\mathcal{I}:\mathcal{J}})$ is complete, i.e. $(\mathcal{J} : \mathcal{I}, \| \cdot \|_{\mathcal{J}:\mathcal{I}})$ is a quasi-Banach space. □

Remark 4.12. Since the quasi-norms $\| \cdot \|_{\mathcal{J}}$ and $\| \cdot \|_{\mathcal{J}:\mathcal{I}}$ are symmetric, for all $a \in \mathcal{J} : \mathcal{I}$ the relations

$$\begin{aligned} \|a\|_{\mathcal{J}:\mathcal{I}} &= \|a^*\|_{\mathcal{J}:\mathcal{I}} = \sup\{\|a^* x\|_{\mathcal{J}} : x \in \mathcal{I}, \|x\|_{\mathcal{I}} \leqslant 1\} \\ &= \sup\{\|x^* a\|_{\mathcal{J}} : x \in \mathcal{I}, \|x\|_{\mathcal{I}} \leqslant 1\} = \sup\{\|xa\|_{\mathcal{J}} : x \in \mathcal{I}, \|x\|_{\mathcal{I}} \leqslant 1\} \end{aligned}$$

hold, i.e. for any $a \in \mathcal{J} : \mathcal{I}$ we have

$$\|a\|_{\mathcal{J}:\mathcal{I}} = \sup\{\|xa\|_{\mathcal{J}} : x \in \mathcal{I}, \|x\|_{\mathcal{I}} \leqslant 1\}. \tag{6}$$

When $\mathcal{I} \subseteq \mathcal{J}$ we have $\mathcal{J} : \mathcal{I} = \mathcal{B}(H)$ and for any $a \in \mathcal{J} : \mathcal{I}$ the mapping $T_a(x) = ax$ is a bounded linear operator from $\mathcal{I}$ into $\mathcal{J}$. As in the proof of Theorem 4.11 we may establish that $\|a\|_{\mathcal{J}:\mathcal{I}} = \sup\{\|ax\|_{\mathcal{J}} : x \in \mathcal{I}, \|x\|_{\mathcal{I}} \leqslant 1\}$ is a complete symmetric quasi-norm on $\mathcal{J} : \mathcal{I}$. In addition, in case $\mathcal{I} = \mathcal{J}$ we have

$$\begin{aligned} \|a\|_{\mathcal{I}:\mathcal{I}} &= \sup\{\|ax\|_{\mathcal{I}} : x \in \mathcal{I}, \|x\|_{\mathcal{I}} \leqslant 1\} \\ &\leqslant \sup\{\|a\|_{\mathcal{B}(H)} \|x\|_{\mathcal{I}} : x \in \mathcal{I}, \|x\|_{\mathcal{I}} \leqslant 1\} \leqslant \|a\|_{\mathcal{B}(H)}, \end{aligned}$$

i.e.

$$\|a\|_{\mathcal{I}:\mathcal{I}} \leqslant \|a\|_{\mathcal{B}(H)} \quad \text{for all } a \in \mathcal{I} : \mathcal{I}. \tag{7}$$

Thus, the norm $\| \cdot \|_{\mathcal{B}(H)}$ and the quasi-norm $\| \cdot \|_{\mathcal{I}:\mathcal{I}}$ are equivalent.

Now, let G and F be arbitrary symmetric quasi-Banach sequence spaces in l_∞. For every $\xi \in F : G$ set

$$\|\xi\|_{F:G} = \sup\{\|\xi\eta\|_F : \eta \in G, \|\eta\|_G \leqslant 1\}.$$

The following theorem is a "commutative" version of Theorem 4.11.

Theorem 4.13. *If $G \not\subseteq F$, then $(F : G, \| \cdot \|_{F:G})$ is a symmetric quasi-Banach sequence space in c_0 with the modulus of concavity, which does not exceed the modulus of concavity of the quasi-norm $\| \cdot \|_F$, in addition, $\|\xi\eta\|_F \leqslant \|\xi\|_{F:G} \|\eta\|_G$ for all $\xi \in F : G$, $\eta \in G$.*

Proof. Since $G \not\subseteq F$, it follows that $F \neq l_\infty$, $F : G \neq l_\infty$, and therefore, according to Corollary 4.8, $F : G$ is a solid rearrangement invariant space and $F : G \subset c_0$.

As in the proof of Theorem 4.11 it is established that $\| \cdot \|_{F:G}$ is a complete quasi-norm on $F : G$ with the modulus of concavity which does not exceed the modulus of concavity of the quasi-norm $\| \cdot \|_F$.

If $\xi, \eta \in F : G$ and $\xi^* \leqslant \eta^*$, then $\xi^* = a\eta^*$ for some $a \in l_\infty$ with $\|a\|_\infty \leqslant 1$. Hence,

$$\begin{aligned} \|\xi^*\|_{F:G} &= \|a\eta^*\|_{F:G} = \sup\{\|a\eta^*\zeta\|_F : \zeta \in G, \|\zeta\|_G \leqslant 1\} \\ &\leqslant \|a\|_\infty \sup\{\|\eta^*\zeta\|_F : \zeta \in G, \|\zeta\|_G \leqslant 1\} \leqslant \|\eta^*\|_{F:G}. \end{aligned}$$

Let us show that $\|\xi\|_{F:G} = \|\xi^*\|_{F:G}$ for all $\xi = \{\xi_n\}_{n=1}^\infty \in F : G$. Since $\xi \in c_0$ there exists a bijection $\pi : \mathbb{N} \to \mathbb{N}$ such that $U_\pi(\xi) := \{\xi_{\pi(n)}\}_{n=1}^\infty = \{\xi_n^*\}_{n=1}^\infty = \xi^*$. It is clear that the mapping $U_\pi : l_\infty \to l_\infty$ defined by the equality $U_\pi(\eta) = U_\pi(\{\eta_n\}_{n=1}^\infty) = \{\eta_{\pi(n)}\}_{n=1}^\infty$, $\eta = \{\eta_n\}_{n=1}^\infty \in l_\infty$, is a linear bijective mapping, such that $U_\pi(\eta\zeta) = U_\pi(\eta)U_\pi(\zeta)$, $\eta, \zeta \in l_\infty$. In addition, $U_\pi(G) = G$, $U_\pi(F) = F$, and $\|U_\pi(\eta)\|_G = \|\eta\|_G$, $\|U_\pi(\zeta)\|_F = \|\zeta\|_F$ for all $\eta \in G$, $\zeta \in F$.

Since $U_\pi(\xi) = \xi^*$, we have

$$\begin{aligned} \|\xi^*\|_{F:G} &= \sup\{\|U_\pi(\xi)\eta\|_F : \eta \in G, \|\eta\|_G \leqslant 1\} = \sup\{\|U_\pi(\xi U_\pi^{-1}(\eta))\|_F : \eta \in G, \|\eta\|_G \leqslant 1\} \\ &= \sup\{\|\xi U_\pi^{-1}(\eta)\|_F : \eta \in G, \|\eta\|_G \leqslant 1\} = \sup\{\|\xi\zeta\|_F : U_\pi(\zeta) \in G, \|U_\pi(\zeta)\|_G \leqslant 1\} \\ &= \sup\{\|\xi\zeta\|_F : \zeta \in G, \|\zeta\|_G \leqslant 1\} = \|\xi\|_{F:G}. \end{aligned}$$

Thus, from $\xi, \eta \in F : G$, $\xi^* \leqslant \eta^*$ it follows that

$$\|\xi\|_{F:G} = \|\xi^*\|_{F:G} \leqslant \|\eta^*\|_{F:G} = \|\eta\|_{F:G}.$$

The equality $\|\xi\|_{F:G} = 1$ is established similarly to the equality $\|p\|_{\mathcal{J}:\mathcal{I}} = 1$, where p is a one-dimensional projection from $\mathcal{B}(H)$ (see the proof of Theorem 4.11).

Consequently, $(F : G, \|\cdot\|_{F:G})$ is a symmetric quasi-Banach sequence space in c_0. The inequality $\|\xi\eta\|_F \leqslant \|\xi\|_{F:G}\|\eta\|_G$ immediately follows from the definition of $\|\cdot\|_{F:G}$. □

Let $\mathcal{I}, \mathcal{J}$ be symmetric quasi-Banach ideals of compact operators from $\mathcal{B}(H)$, $\mathcal{I} \not\subseteq \mathcal{J}$. By Proposition 4.9, $\mathcal{J} : \mathcal{I} = C_{E_{\mathcal{J}}:E_{\mathcal{I}}}$, i.e. $C_{E_{\mathcal{J}}:E_{\mathcal{I}}}$ is a two-sided ideal of compact operators from $\mathcal{B}(H)$. For every $a \in C_{E_{\mathcal{J}}:E_{\mathcal{I}}}$ we set

$$\|a\|_{C_{E_{\mathcal{J}}:E_{\mathcal{I}}}} := \|\{s_n(a)\}\|_{E_{\mathcal{I}}:E_{\mathcal{J}}}.$$

Proposition 4.14. *$\|\cdot\|_{C_{E_{\mathcal{J}}:E_{\mathcal{I}}}}$ is a symmetric quasi-norm on $C_{E_{\mathcal{J}}:E_{\mathcal{I}}}$.*

Proof. Obviously, $\|a\|_{C_{E_{\mathcal{J}}:E_{\mathcal{I}}}} \geqslant 0$ for all $a \in C_{E_{\mathcal{J}}:E_{\mathcal{I}}}$ and $\|a\|_{C_{E_{\mathcal{J}}:E_{\mathcal{I}}}} = 0 \Leftrightarrow a = 0$. If $a, b \in C_{E_{\mathcal{J}}:E_{\mathcal{I}}}$, $\lambda \in \mathbb{C}$, then

$$\|\lambda a\|_{C_{E_{\mathcal{J}}:E_{\mathcal{I}}}} = \|\{s_n(\lambda a)\}_{n=1}^{\infty}\|_{E_{\mathcal{J}}:E_{\mathcal{I}}} = |\lambda|\|a\|_{C_{E_{\mathcal{J}}:E_{\mathcal{I}}}}$$

and

$$\begin{aligned}\|a+b\|_{C_{E_{\mathcal{J}}:E_{\mathcal{I}}}} &= \|\{s_n(a+b)\}\|_{E_{\mathcal{J}}:E_{\mathcal{I}}} \overset{(3)}{\leq} \|\sigma_2(\{s_n(a)+s_n(b)\})\|_{E_{\mathcal{J}}:E_{\mathcal{I}}} \\ &\leqslant 2C\|\{s_n(a)\}+\{s_n(b)\}\|_{E_{\mathcal{J}}:E_{\mathcal{I}}} \\ &\leqslant 2C^2(\|\{s_n(a)\}\|_{E_{\mathcal{J}}:E_{\mathcal{I}}} + \|\{s_n(b)\}\|_{E_{\mathcal{J}}:E_{\mathcal{I}}}) \\ &= 2C^2(\|a\|_{C_{E_{\mathcal{J}}:E_{\mathcal{I}}}} + \|b\|_{C_{E_{\mathcal{J}}:E_{\mathcal{I}}}}).\end{aligned}$$

Hence, $\|\cdot\|_{C_{E_{\mathcal{J}}:E_{\mathcal{I}}}}$ is a quasi-norm on $C_{E_{\mathcal{J}}:E_{\mathcal{I}}}$ and the modulus of concavity of $\|\cdot\|_{C_{E_{\mathcal{J}}:E_{\mathcal{I}}}}$ does not exceed $2C^2$, where C is the modulus of concavity of the quasi-norm $\|\cdot\|_{E_{\mathcal{J}}}$.

Since $s_n(xay) \leqslant \|x\|_{\mathcal{B}(H)}\|y\|_{\mathcal{B}(H)}s_n(a)$ for all $a \in \mathcal{K}(H)$, $x, y \in \mathcal{B}(H)$, $n \in \mathbb{N}$ (see Proposition 2.2), it follows

$$\|xay\|_{C_{E_{\mathcal{J}}:E_{\mathcal{I}}}} = \|\{s_n(xay)\}\|_{E_{\mathcal{J}}:E_{\mathcal{I}}} \leqslant \|x\|_{\mathcal{B}(H)}\|y\|_{\mathcal{B}(H)}\|a\|_{C_{E_{\mathcal{J}}:E_{\mathcal{I}}}}.$$

It is clear that $\|p\|_{C_{E_{\mathcal{J}}:E_{\mathcal{I}}}} = 1$ for every one-dimensional projection p.

Thus, $\|\cdot\|_{C_{E_{\mathcal{J}}:E_{\mathcal{I}}}}$ is a symmetric quasi-norm on $C_{E_{\mathcal{J}}:E_{\mathcal{I}}}$. □

Remark 4.15. (i) If $\mathcal{I}, \mathcal{J}$ are symmetric Banach ideals of compact operators in $\mathcal{B}(H)$ and $\mathcal{I} \not\subseteq \mathcal{J}$, then $(\mathcal{J} : \mathcal{I}, \|\cdot\|_{\mathcal{J}:\mathcal{I}})$ is a symmetric Banach ideal of compact operators (Theorem 4.11), and therefore $(E_{\mathcal{J}:\mathcal{I}}, \|\cdot\|_{E_{\mathcal{J}:\mathcal{I}}})$ is a symmetric Banach sequence space in c_0 (Theorem 4.4).

(ii) If G, F are symmetric Banach sequence spaces in c_0 and $G \not\subseteq F$, then $(F : G, \|\cdot\|_{F:G})$ is a symmetric Banach sequence space in c_0 (Theorem 4.13), and therefore $(C_{F:G}, \|\cdot\|_{C_{F:G}})$ is a symmetric Banach ideal of compact operators from $\mathcal{B}(H)$ (Theorem 1.5).

Theorem 4.16. *Let $\mathcal{I}, \mathcal{J}$ be symmetric quasi-Banach ideals of compact operators from $\mathcal{B}(H)$ and $\mathcal{I} \not\subseteq \mathcal{J}$. Then*

(i) *$E_{\mathcal{J}:\mathcal{I}} = E_{\mathcal{J}} : E_{\mathcal{I}}$ and $\|\cdot\|_{E_{\mathcal{J}}:E_{\mathcal{I}}} \leqslant \|\cdot\|_{E_{\mathcal{J}:\mathcal{I}}} \leqslant 2C\|\cdot\|_{E_{\mathcal{J}}:E_{\mathcal{I}}}$, where C is the modulus of concavity of the quasi-norm $\|\cdot\|_{\mathcal{J}}$;*
(ii) *$\mathcal{J} : \mathcal{I} = C_{E_{\mathcal{J}}:E_{\mathcal{I}}}$ and $\|\cdot\|_{C_{E_{\mathcal{J}}:E_{\mathcal{I}}}} \leqslant \|\cdot\|_{\mathcal{J}:\mathcal{I}} \leqslant 2C\|\cdot\|_{C_{E_{\mathcal{J}}:E_{\mathcal{I}}}}$, where C is the modulus of concavity of the quasi-norm $\|\cdot\|_{E_{\mathcal{J}}}$.*

Proof. If $\xi = \xi^* \in E_{\mathcal{J}:\mathcal{I}}$, then $x_\xi \in \mathcal{J} : \mathcal{I}$ (see Theorem 4.4). Hence, for every $\eta = \eta^* \in E_{\mathcal{I}}$ we have $x_\eta \in \mathcal{I}$ and $x_{\xi\eta} = x_\xi x_\eta \in \mathcal{J}$, i.e. $\xi\eta \in E_{\mathcal{J}}$. Therefore, due to Proposition 4.7, $\xi \in E_{\mathcal{J}} : E_{\mathcal{I}}$, in addition,

$$\begin{aligned}\|\xi\|_{E_{\mathcal{J}:\mathcal{I}}} &= \|x_\xi\|_{\mathcal{J}:\mathcal{I}} = \sup\{\|x_\xi y\|_{\mathcal{J}} : y \in \mathcal{I}, \|y\|_{\mathcal{I}} \leqslant 1\} \\ &\geqslant \sup\{\|x_\xi x_\eta\|_{\mathcal{J}} : \eta \in E_{\mathcal{I}}, \|\eta\|_{E_{\mathcal{I}}} \leqslant 1\} \\ &= \sup\{\|x_{\xi\eta}\|_{\mathcal{J}} : \eta \in E_{\mathcal{I}}, \|\eta\|_{E_{\mathcal{I}}} \leqslant 1\} \\ &= \sup\{\|\xi\eta\|_{E_{\mathcal{J}}} : \eta \in E_{\mathcal{I}}, \|\eta\|_{E_{\mathcal{I}}} \leqslant 1\} = \|\xi\|_{E_{\mathcal{J}}:E_{\mathcal{I}}}.\end{aligned}$$

Conversely, if $\xi = \xi^* \in E_{\mathcal{J}} : E_{\mathcal{I}}$, then $x_\xi \in C_{E_{\mathcal{J}}:E_{\mathcal{I}}} = \mathcal{J} : \mathcal{I}$ (see Proposition 4.9), and so $\xi \in E_{\mathcal{J}:\mathcal{I}}$. Moreover,

$$\begin{aligned}\|\xi\|_{E_{\mathcal{J}:\mathcal{I}}} &= \|x_\xi\|_{\mathcal{J}:\mathcal{I}} = \sup\{\|x_\xi y\|_{\mathcal{J}} : y \in \mathcal{I}, \|y\|_{\mathcal{I}} \leqslant 1\} \\ &= \sup\{\|x_{\{s_{n(x_\xi y)}\}}\|_{\mathcal{J}} : y \in \mathcal{I}, \|y\|_{\mathcal{I}} \leqslant 1\} \\ &\overset{(4)}{\leq} \sup\{\|x_{\sigma_2(\{\xi s_{n(y)}\})}\|_{\mathcal{J}} : y \in \mathcal{I}, \|y\|_{\mathcal{I}} \leqslant 1\} \\ &\leqslant 2C\sup\{\|\xi\{s_n(y)\}\|_{E_{\mathcal{J}}} : y \in \mathcal{I}, \|y\|_{\mathcal{I}} \leqslant 1\} \\ &\leqslant 2C\sup\{\|\xi\eta\|_{E_{\mathcal{J}}} : \eta \in E_{\mathcal{I}}, \|\eta\|_{E_{\mathcal{I}}} \leqslant 1\} = 2C\|\xi\|_{E_{\mathcal{J}}:E_{\mathcal{I}}}.\end{aligned}$$

Thus, $E_{\mathcal{J}:\mathcal{I}} = E_{\mathcal{J}} : E_{\mathcal{I}}$ and $\|\xi\|_{E_{\mathcal{J}}:E_{\mathcal{I}}} \leqslant \|\xi\|_{E_{\mathcal{J}:\mathcal{I}}} \leqslant 2C\|\xi\|_{E_{\mathcal{J}}:E_{\mathcal{I}}}$ for all $\xi \in E_{\mathcal{J}:\mathcal{I}}$.
(ii) The equality $\mathcal{J} : \mathcal{I} = C_{E_{\mathcal{J}}:E_{\mathcal{I}}}$ is proven in Proposition 4.9. For an arbitrary $a \in \mathcal{J} : \mathcal{I}$ we have

$$\begin{aligned}\|a\|_{C_{E_{\mathcal{J}}:E_{\mathcal{I}}}} &= \|\{s_n(a)\}\|_{E_{\mathcal{I}}:E_{\mathcal{J}}}\\ &= \sup\{\|\{s_n(a)\}\eta\|_{E_{\mathcal{J}}} : \eta \in E_{\mathcal{I}}, \|\eta\|_{E_{\mathcal{I}}} \leqslant 1\}\\ &= \sup\{\|x_{\{s_n(a)\}}x_\eta\|_{\mathcal{J}} : x_\eta \in \mathcal{I}, \|x_\eta\|_{\mathcal{I}} \leqslant 1\}\\ &\leqslant \sup\{\|x_{\{s_n(a)\}}y\|_{\mathcal{J}} : y \in \mathcal{I}, \|y\|_{\mathcal{I}} \leqslant 1\}\\ &= \|x_{\{s_n(a)\}}\|_{\mathcal{J}:\mathcal{I}} = \|a\|_{\mathcal{J}:\mathcal{I}}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\|a\|_{\mathcal{J}:\mathcal{I}} &= \sup\{\|ay\|_{\mathcal{J}} : y \in \mathcal{I}, \|y\|_{\mathcal{I}} \leqslant 1\}\\ &= \sup\{\|\{s_n(ay)\}_{n=1}^\infty\|_{E_{\mathcal{J}}} : y \in \mathcal{I}, \|y\|_{\mathcal{I}} \leqslant 1\}\\ &\overset{(4)}{\leq} \sup\{\|\sigma_2(\{s_n(a)s_n(y)\}_{n=1}^\infty)\|_{E_{\mathcal{J}}} : y \in \mathcal{I}, \|y\|_{\mathcal{I}} \leqslant 1\}\\ &= 2C\sup\{\|\{s_n(a)s_n(y)\}\|_{E_{\mathcal{J}}} : y \in \mathcal{I}, \|y\|_{\mathcal{I}} \leqslant 1\}\\ &= 2C\|\{s_n(a)\}\|_{E_{\mathcal{J}}:E_{\mathcal{I}}} = 2C\|a\|_{C_{E_{\mathcal{J}}:E_{\mathcal{I}}}}. \quad \square\end{aligned}$$

Since $(\mathcal{J} : \mathcal{I}, \|\cdot\|_{\mathcal{J}:\mathcal{I}})$ is a quasi-Banach space (see Theorem 4.11) and quasi-norms $\|\cdot\|_{\mathcal{J}:\mathcal{I}}$ and $\|\cdot\|_{C_{E_{\mathcal{J}}:E_{\mathcal{I}}}}$ are equivalent (see Theorem 4.16(ii)), we have the following corollary.

Corollary 4.17. *For any symmetric quasi-Banach ideals $\mathcal{I}$, $\mathcal{J}$ of compact operators from $\mathcal{B}(H)$, $\mathcal{I} \not\subseteq \mathcal{J}$, the couple $(C_{E_{\mathcal{J}}:E_{\mathcal{I}}}, \|\cdot\|_{C_{E_{\mathcal{J}}:E_{\mathcal{I}}}})$ is a symmetric quasi-Banach ideal of compact operators from $\mathcal{B}(H)$.*

The following theorem gives the full description of the set $\mathrm{Der}(\mathcal{I}, \mathcal{J})$.

Theorem 4.18. (i) *Let $\mathcal{I}$ and $\mathcal{J}$ be symmetric quasi-Banach ideals of compact operators from $\mathcal{B}(H)$, $\mathcal{I} \not\subseteq \mathcal{J}$. Then any derivation δ from $\mathcal{I}$ into $\mathcal{J}$ has a form $\delta = \delta_a$ for some $a \in C_{E_{\mathcal{J}}:E_{\mathcal{I}}}$ and $\|a + \alpha\mathbb{1}\|_{\mathcal{B}(H)} \leqslant \|\delta_a\|_{\mathcal{B}(\mathcal{I},\mathcal{J})}$ for some $\alpha \in \mathbb{C}$. Conversely, if $a \in C_{E_{\mathcal{J}}:E_{\mathcal{I}}}$, then the restriction of δ_a on $\mathcal{I}$ is a derivation from $\mathcal{I}$ into $\mathcal{J}$. In addition, $\|\delta_a\|_{\mathcal{B}(\mathcal{I},\mathcal{J})} \leqslant 2C\|a\|_{\mathcal{J}:\mathcal{I}}$, where C is the modulus of concavity of the quasi-norm $\|\cdot\|_{\mathcal{J}}$;*

(ii) *Let G and F be symmetric Banach (respectively, F is a p-convex, G is a q-convex quasi-Banach with $0 < p, q < \infty$) sequence spaces in c_0 and $G \not\subseteq F$. Then any derivation $\delta : C_G \to C_F$ has a form $\delta = \delta_a$ for some $a \in C_{F:G}$ and $\|a + \alpha\mathbb{1}\|_{\mathcal{B}(H)} \leqslant \|\delta_a\|_{\mathcal{B}(C_G,C_F)}$ for some $\alpha \in \mathbb{C}$. Conversely, if $a \in C_{F:G}$, then the restriction of δ_a on C_G is a derivation from C_G into C_F. In addition, $\|\delta_a\|_{\mathcal{B}(C_G,C_F)} \leqslant 2C\|a\|_{C_F:C_G}$, where C is the modulus of concavity of the quasi-norm $\|\cdot\|_{C_F}$.*

Proof. (i) By Theorem 3.6, any derivation $\delta : \mathcal{I} \to \mathcal{J}$ has a form $\delta = \delta_a$ for some $a \in \mathcal{J} : \mathcal{I}$, in addition $\|a + \alpha\mathbb{1}\|_{\mathcal{B}(H)} \leqslant \|\delta_a\|_{\mathcal{B}(\mathcal{I},\mathcal{J})}$ for some $\alpha \in \mathbb{C}$. Since $\mathcal{J} : \mathcal{I} = C_{E_{\mathcal{J}}:E_{\mathcal{I}}}$ (see Theorem 4.16), we have $a \in C_{E_{\mathcal{J}}:E_{\mathcal{I}}}$.
Conversely, if $a \in C_{E_{\mathcal{J}}:E_{\mathcal{J}}}$, then $a \in \mathcal{J} : \mathcal{I}$, and, according to Theorem 3.6, $\delta_a(\mathcal{I}) \subset \mathcal{J}$.
Moreover,

$$\begin{aligned}\|\delta_a\|_{\mathcal{B}(\mathcal{I},\mathcal{J})} &= \sup\{\|\delta_a(x)\|_{\mathcal{J}} : x \in \mathcal{I}, \|x\|_{\mathcal{I}} \leqslant 1\}\\ &= \sup\{\|ax - xa\|_{\mathcal{J}} : x \in \mathcal{I}, \|x\|_{\mathcal{I}} \leqslant 1\}\\ &\leqslant \sup\{C(\|ax\|_{\mathcal{J}} + \|xa\|_{\mathcal{J}}) : x \in \mathcal{I}, \|x\|_{\mathcal{I}} \leqslant 1\}\\ &\overset{(6)}{=} 2C\sup\{\|ax\|_{\mathcal{J}} : x \in \mathcal{I}, \|x\|_{\mathcal{I}} \leqslant 1\} = 2C\|a\|_{\mathcal{J}:\mathcal{I}}.\end{aligned} \tag{8}$$

Item (ii) follows from (i) and Theorems 1.5 and 4.16. The inequality $\|\delta_a\|_{\mathcal{B}(C_F,C_G)} \leqslant 2C\|a\|_{C_G:C_F}$ is proven in the same manner. $\square$

We illustrate Theorem 4.18 with an example drawn from the theory of Lorentz and Marcinkiewicz sequence spaces. Let $\omega = \{\omega_n\}_{n=1}^\infty$ be a decreasing weight sequence of positive numbers. Letting $W(j) = \sum_{n=1}^j w_n$, $j \in \mathbb{N}$, we shall assume that $W(\infty) = \sum_{n=1}^\infty w_n = \infty$.
The Lorentz sequence space l^p_ω, $1 \leqslant p < \infty$, consists of all sequences $\xi = \{\xi_n\}_{n=1}^\infty \in c_0$ such that

$$\|\xi\|_{l^p_\omega} = \left(\sum_{n=1}^\infty (\xi_n^*)^p w_n\right)^{\frac{1}{p}} < \infty.$$

The Lorentz (Marcinkiewicz) sequence space m_W^p, $1 \leqslant p < \infty$, is the space of all sequences $\xi = \{\xi_n\}_{n=1}^{\infty} \in c_0$ satisfying

$$\|\xi\|_{m_W^p} = \sup_{k\geq 1} \left(\frac{\sum_{n=1}^{k} (\xi_n^*)^p}{W_k} \right)^{\frac{1}{p}} < \infty.$$

It is well known (see e.g. [21] and [22, Proposition 1]) that $(l_\omega^p, \|\cdot\|_{l_\omega^p})$ and $(m_W^p, \|\cdot\|_{m_W^p})$ are symmetric Banach sequence spaces in c_0.

Hence, $(C_{l_\omega^p}, \|\cdot\|_{C_{l_\omega^p}})$ and $(C_{m_W^p}, \|\cdot\|_{C_{m_W^p}})$ are symmetric Banach ideals of compact operators (Theorem 1.5). Since $l_1 : l_\omega = m_W^1$ (see e.g. [21]) it follows that $l_p : l_\omega^p = m_W^p$ for every $1 \leqslant p < \infty$ [22, Section 2]. By Theorem 4.16, $C_p : C_{l_\omega^p} = C_{m_W^p}$ and $\|a\|_{C_p:C_{l_\omega^p}} \leqslant 2\|a\|_{C_{m_W^p}}$ for all $a \in C_p : C_{l_\omega^p}$. From Theorem 4.18 (ii), we obtain the following example significantly extending similar results from [1].

Corollary 4.19. *A linear mapping $\delta : C_{l_\omega^p} \to C_p$, $1 \leqslant p < \infty$ is a derivation if and only if $\delta = \delta_a$ for some $a \in C_{m_W^p}$, in addition, $\|\delta\|_{\mathcal{B}(C_{l_\omega^p}, C_p)} \leqslant 2\|a\|_{C_p:C_{l_\omega^p}} \leqslant 4\|a\|_{C_{m_W^p}}$.*

In conclusion, note that, by Theorem 3.2, (8), any derivation δ from a symmetric quasi-Banach ideal $\mathcal{I}$ into a symmetric quasi-Banach ideal $\mathcal{J}$, such that $\mathcal{I} \subseteq \mathcal{J}$, has a form $\delta = \delta_a$ for some $a \in \mathcal{B}(H)$ and, in addition, $\|a\|_{\mathcal{B}(H)} \leqslant \|\delta_a\|_{\mathcal{B}(\mathcal{I},\mathcal{J})} \leqslant 2C\|a\|_{\mathcal{J}:\mathcal{I}}$, where C is the modulus of concavity of the quasi-norm $\|\cdot\|_{\mathcal{J}}$. Moreover, for the case when $\mathcal{I} = \mathcal{J}$ we have $\|a\|_{\mathcal{B}(H)} \leqslant \|\delta_a\|_{\mathcal{B}(\mathcal{I},\mathcal{I})} \leqslant 2C\|a\|_{\mathcal{B}(H)}$, where C is the modulus of concavity of the quasi-norm $\|\cdot\|_{\mathcal{I}}$ (see (7)). This complements results from [7].

References

[1] M.J. Hoffman, Essential commutants and multiplier ideals, Indiana Univ. Math. J. 30 (1981) 859–869.
[2] S. Sakai, C^*-Algebras and W^*-Algebras, Springer-Verlag, New York, Heidelberg, Berlin, 1971.
[3] E. Kissin, V.S. Shulman, Differential Schatten $*$-algebras. Approximation property and approximate identities, J. Oper. Theory 45 (2001) 303–334.
[4] J.W. Calkin, Two-sided ideals and congruences in the ring of bounded operators in Hilbert space, Ann. of Math. 42 (1941) 839–873.
[5] A.F. Ber, F.A. Sukochev, Commutator estimates in W^*-algebras, J. Funct. Anal. 262 (2012) 537–568.
[6] A.F. Ber, F.A. Sukochev, Commutator estimates in W^*-factors, Trans. Amer. Math. Soc. 364 (10) (2012) 5571–5587.
[7] A.F. Ber, F.A. Sukochev, Derivations in the Banach ideals of τ-compact operators, 18 April, 2012, p. 12. arXiv:1204.4052v1.
[8] B. Simon, Trace Ideals and their Applications, second ed., in: Mathematical Surveys and Monographs, vol. 120, American Mathematical Society, Providence, RI, 2005.
[9] N.J. Kalton, F.A. Sukochev, Symmetric norms and spaces of operators, J. Reine Angew. Math. 621 (2008) 81–121.
[10] D.J.H. Garling, On ideals of operators in Hilbert space, Proc. Lond. Math. Soc. (3) 17 (1967) 115–138.
[11] P.G. Dodds, T.K. Dodds, B. de Pagter, Non-commutative Köthe duality, Trans. Amer. Math. Soc. 339 (1993) 717–750.
[12] R.Y. Kadison, J.R. Ringrose, Fundamentals of the Theory of Operator Algebras I, II, Academic Press, Orlando, 1983.
[13] M. Reed, B. Simon, Methods of Modern Mathematical Physics. Vol. 1: Functional Analysis, Academic Press, 1980.
[14] M.A. Muratov, V.I. Chilin, Algebras of measurable and locally measurable operators, Proc. Inst. Math. of Nat. Acad. Sci. of Ukraine, Kiev 69 (2007) (Russian).
[15] I. Gohberg, M. Krein, Introduction to the Theory of Linear Nonselfadjoint Operators, in: Trans. Math. Monog., vol. 18, American Mathematical Society, Providence, RI, 1969.
[16] N.J. Kalton, Quasi-Banach spaces, in: Handbook of the Geometry of Banach Spaces. Vol. 2, Elsevier, 2003, pp. 1099–1130.
[17] H.H. Schaefer, Topological Vector Spaces, The Machmillan Company, Collier-Machmillan Limited, New York, London, 1966.
[18] B.Z. Vulich, Introduction to the Theory of Partially Ordered Spaces, Wolters-Noordhoff, Groningen, 1967.
[19] S.G. Krein, Y.I. Petunin, E.M. Semenov, Interpolation of Linear Operators, in: Trans. Math. Monogr., vol. 54, American Mathematical Society, Providence, RI, 1982.
[20] T. Fack, H. Kosaki, Generalised s-numbers of τ-measurable operators, Pacific J. Math. 123 (1986) 265–300.
[21] A. Kamińska, A.M. Parrish, Convexity and concavity constants in Lorentz and Marcinkiewicz spaces, J. Math. Anal. Appl. 343 (2008) 337–351.
[22] L. Maligranda, L.E. Persson, Generalized duality of some Banach function spaces, Indag. Math. 51 (1989) 323–338.