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NOTE

Some Inequalities Involving the Constant e, and an Application to Carleman's Inequality

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Some inequalities involving the constant e are considered. A strengthened Carleman's inequality is proved.

Key Words: Maclaurin's expansion; Carleman's inequality.

1. INTRODUCTION

It is well known that

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e \quad \text{and} \quad \sum_{n=0}^{\infty}\frac{1}{n!} = e.$$

In estimating the convergence velocity, Sewell [1, p. 358] showed that

$$\frac{1}{(n+1)!} \le e - \sum_{n=0}^{\infty}\frac{1}{n!} < \frac{3}{(n+1)!}, \qquad n = 1, 2, \ldots. \tag{1.1}$$

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Kloosterman (see [4, Chapter 3.8.26]) proved the following inequality:

$$\left| e - \left(1 + \frac{1}{n}\right)^n \right| < \frac{e}{2n}, \qquad n = 1, 2, 3, \ldots . \tag{1.2}$$

In this paper, we consider some inequalities involving the constant e and give an improvement of (1.2). As an application, we prove a strengthened Carleman's inequality.

2. LEMMAS AND A THEOREM

LEMMA 2.1. *For every x in $0 < x \le \frac{1}{5}$, we have*

$$e - \frac{e}{2}x + \frac{11e}{24}x^2 - \frac{21e}{48}x^3 < (1+x)^{1/x} < e - \frac{e}{2}x + \frac{11e}{24}x^2, \tag{2.1}$$

Proof. We define a function g by $g(x) = (1/x)\ln(1+x)$ for $x \in (-1, 0) \cup (0, 1)$, and $g(0) = 1$.

It is convenient to calculate its derivatives explicitly term by term as follows:

$$g'(x) = -\sum_{n=2}^{\infty} (-1)^n \frac{(n-1)}{n} x^{n-2},$$

$$g''(x) = \sum_{n=3}^{\infty} (-1)^n \frac{(n-1)(n-2)}{n} x^{n-3},$$

$$g'''(x) = -\sum_{n=4}^{\infty} (-1)^n \frac{(n-1)(n-2)(n-3)}{n} x^{n-4},$$

$$g^{\mathrm{IV}}(x) = \sum_{n=5}^{\infty} (-1)^{n-1} \frac{(n-1)(n-2)(n-3)(n-4)}{n} x^{n-5}.$$

Whether from this or in the usual way, we find that

$$g'(0) = -\tfrac{1}{2}, \qquad g''(0) = \tfrac{2}{3}, \quad \text{and} \quad g''(0) = -\tfrac{3}{2}.$$

If $f(x) = \exp[g(x)]$ for $-1 < x < 1$, then $f(x) = (1 + x)^{1/x}$, and we obtain that

$$f'(x) = \exp[g(x)] \cdot g'(x), \qquad f'(0) = \exp[g(0)] \cdot g'(0) = -\frac{1}{2}e,$$

$$f''(x) = \exp[g(x)] \cdot \left[g'^2(x) + g''(x)\right],$$

$$f''(0) = \exp[g(0)] \cdot \left[g'^2(0) + g''(0)\right] = \frac{11e}{12},$$

$$f'''(x) = \exp[g(x)] \cdot \left[g'^3(x) + 3g'(x)g''(x) + g'''(x)\right],$$

$$f'''(0) = -\frac{21}{8}e,$$

$$f^{\mathrm{IV}}(x) = \exp[g(x)] \cdot \Big[g'^4(x) + 6g'^2(x)g''(x) + 4g'(x)g'''(x) + 3g''^2(x) + g^4(x)\Big].$$

We now write the Maclaurin expansion of $f(x)$ in the form

$$f(x) = e - \frac{e}{2}x + \frac{11e}{24}x^2 + \frac{1}{6}x^3 f'''(x\theta_1), \qquad x \in (-1, 1), 0 < \theta_1 < 1. \tag{2.2}$$

and

$$f(x) = e - \frac{e}{2}x + \frac{11e}{24}x^2 - \frac{21e}{48}x^3 + \frac{1}{24}x^4 f^{\mathrm{IV}}(\theta_2, x), \quad \text{for } -1 < x < 1 \text{ and } 0 < \theta_2 < 1. \tag{2.3}$$

For $x \in (0, \frac{1}{5}]$, we have

$$\left(\frac{n-1}{n}\right)x^{n-2} > \left(\frac{n}{n+1}\right)x^{n-1}, \qquad n \geq 2,$$

$$\frac{(n-1)(n-2)}{n}x^{n-3} > \frac{n(n-1)}{n+1}x^{n-2}, \qquad n \geq 3,$$

$$\frac{(n-1)(n-2)(n-3)}{n}x^{n-4} > \frac{n(n-1)(n-2)}{n+1}x^{n-3}, \qquad n \geq 4,$$

$$\frac{(n-1)(n-2)(n-3)(n-4)}{n}x^{n-5} > \frac{n(n-1)(n-2)(n-3)}{n}x^{n-4}, \qquad n \geq 5.$$

Thus, $g'(x) < 0$, $g''(x) > 0$, $g'''(x) < 0$, and $g^{\mathrm{IV}}(x) > 0$. Hence $f'''(x) \le 0$ and $f^{\mathrm{IV}}(x) > 0$. Then

$$x^3 f'''(x\theta_1) < 0 \quad \text{and} \quad x^4 f^{\mathrm{IV}}(x\theta_2) > 0 \qquad \text{for } 0 < x \le \tfrac{1}{5}.$$

In view of (2.2) and (2.3), we find that, for $0 < x \le \frac{1}{5}$,

$$f(x) < e - \frac{e}{2}x + \frac{11e}{24}x^2$$

and

$$f(x) > e - \frac{e}{2}x + \frac{11e}{24}x^2 - \frac{21e}{48}x^3.$$

Thus inequalities (2.1) follow.

LEMMA 2.2. *The following inequalities are true*:

$$\frac{1}{2n} - \frac{11}{24n^2} + \frac{21}{48n^3} < \frac{1}{2n + \frac{5}{3}}, \qquad n \ge 5, \tag{2.4}$$

$$\frac{1}{2n} - \frac{11}{24n^2} \ge \frac{1}{2(n+1)}, \qquad n \ge 11. \tag{2.5}$$

Proof. When $n \ge 5$,

$$\left(\frac{1}{2n} - \frac{11}{24n^2} + \frac{21}{48n^3}\right)\left(2n + \frac{5}{3}\right) = 1 - \frac{1}{n}\left(\frac{1}{12} - \frac{1}{9n} - \frac{105}{144n^2}\right) < 1.$$

This proves (2.4).

For $n \ge 11$,

$$\left(\frac{1}{2n} - \frac{11}{24n^2}\right)(2n+2) = 1 + \frac{1}{n}\left(\frac{1}{12} - \frac{11}{12n}\right) \ge 1.$$

This shows that (2.5) is true.

THEOREM 2.1. *For every positive integer n*,

$$\frac{1}{2(n+1)} < 1 - \frac{1}{e} \cdot \left(1 + \frac{1}{n}\right)^n < \frac{1}{2\left(n + \frac{5}{6}\right)}. \tag{2.6}$$

Later on, it will be convenient to use these inequalities in the form

$$e\left[1 - \frac{1}{2\left(n + \frac{5}{6}\right)}\right] < \left(1 + \frac{1}{n}\right)^n < e\left[1 - \frac{1}{2(n+1)}\right]. \qquad (2.7)$$

Proof. We substitute $x = 1/n$ in (2.1) so that, for $n \geq 5$,

$$e - \frac{e}{2n} + \frac{11e}{24n^2} - \frac{21e}{48n^3} < \left(1 + \frac{1}{n}\right)^n < e - \frac{e}{2n} + \frac{11e}{24n^2}.$$

These inequalities are equivalent to

$$\frac{1}{2n} - \frac{11}{24n^2} < 1 - \frac{1}{e}\left(1 + \frac{1}{n}\right)^n < \frac{1}{2n} - \frac{11}{24n^2} + \frac{21}{48n^3}. \qquad (2.8)$$

Using (2.4)–(2.5) and (2.8), it turns out that

$$1 - \frac{1}{e}\left(1 + \frac{1}{n}\right)^n < \frac{1}{2\left(n + \frac{5}{6}\right)}, \qquad \text{for } n \geq 5, \qquad (2.9)$$

and

$$\frac{1}{2(n+1)} < 1 - \frac{1}{e}\left(1 + \frac{1}{n}\right)^n, \qquad \text{for } n \geq 11. \qquad (2.10)$$

Inequality (2.9) is equivalent to

$$2.7182818^+ = e < \left(\frac{6n+5}{6n+2}\right)\left(1 + \frac{1}{n}\right)^n, \qquad n \geq 5. \qquad (2.11)$$

It is easy to check that (2.11) is valid for $n = 1, 2, 3, 4$. Hence (2.9) is true for every positive integer n.

Finally, inequality (2.10) is equivalent to

$$2.7182818^+ = e > \left(\frac{2}{2n+1}\right)\left(1 + \frac{1}{n}\right)^n, \qquad \text{for } n \geq 11. \qquad (2.12)$$

It is also easy to check that (2.12) is valid for $n = 1, 2, \ldots, 10$. Thus inequalities (2.6) are true, and so are inequalities (2.7). This completes the proof.

3. A STRENGTHENED CARLEMAN'S INEQUALITY

It is well known that the following Karlson inequality (see [4, p. 7]),

$$\left(\sum_{n=1}^{\infty} a_n\right)^4 < \pi^2\left(\sum_{n=1}^{\infty} a_n^2\right)\left(\sum_{n=1}^{\infty} n^2 a_n^2\right), \qquad 0 < \sum_{n=1}^{\infty} n^2 a_n^2 < \infty, \quad (3.1)$$

may be strengthened as

$$\left(\sum_{n=1}^{\infty} a_n\right)^4 < \pi^2\left(\sum_{n=1}^{\infty} a_n^2\right)\left(\sum_{n=1}^{\infty}\left(n - \frac{1}{2}\right)^2 a_n^2\right). \quad (3.2)$$

It is well known [2, Chapter 9.12] that the constant e is the best possible in the Carleman inequality

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e\left(\sum_{n=1}^{\infty} a_n\right) \quad (3.3)$$

where $a_n \geq 0$ and $0 < \sum_{n=1}^{\infty} a_n < \infty$.

Analogous to the strengthening of Karlson's inequality, (3.3) may be strengthened as a consequence of inequalities (2.7). Hence we prove the following.

THEOREM 3.1. *Let* $a_n \geq 0$, $n = 1, 2, \ldots,$ *and* $0 < \sum_{n=1}^{\infty} a_n < \infty$. *Then*

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty}\left[1 - \frac{1}{2(n+1)}\right] a_n. \quad (3.4)$$

Proof. Assume that $c_n > 0$ for $n = 1, 2, \ldots$. By the arithmetic–geometric average inequality, we have

$$(c_1 a_1 \cdot c_2 a_2 \cdot \cdots \cdot c_n a_n)^{1/n} \leq \frac{1}{n} \sum_{m=1}^{\infty} c_m a_m. \quad (3.5)$$

Then we find that

$$\begin{aligned}
\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} &= \sum_{n=1}^{\infty}\left(\frac{c_1 a_1 \cdot c_2 a_2 \cdots c_n a_n}{c_1 c_2 \cdots c_n}\right)^{1/n} \\
&= \sum_{n=1}^{\infty} (c_1 c_2 \cdots c_n)^{-1/n} \cdot (c_1 a_1 \cdot c_2 a_2 \cdots c_n a_n)^{1/n} \\
&\leq \sum_{n=1}^{\infty} (c_1 c_2 \cdots c_n)^{-1/n} \cdot \frac{1}{n} \sum_{m=1}^{\infty} c_m a_m \qquad (\text{by } (3.5)) \\
&= \sum_{m=1}^{\infty} c_m a_m \sum_{n=m}^{\infty} \frac{1}{n} (c_1 c_2 \cdots c_n)^{-1/n}. \qquad (3.6)
\end{aligned}$$

Set $c_m = (m+1)^m / m^{m-1}$, $m = 1, 2, \ldots$. Then

$$c_1 c_2 \cdots c_n = (n+1)^n,$$

and, hence,

$$\sum_{n=m}^{\infty} \frac{1}{n} (c_1 c_2 \cdots c_n)^{-1/n} = \sum_{n=m}^{\infty} \frac{1}{n(n+1)} = \frac{1}{m}.$$

Hence, the inequality (3.6) implies that

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} \le \sum_{m=1}^{\infty} \frac{1}{m} (c_m a_m) = \sum_{m=1}^{\infty} \left(1 + \frac{1}{m}\right)^m a_m.$$

By inequality (2.7), we obtain

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} \le e \sum_{m=1}^{\infty} \left[1 - \frac{1}{2(m+1)}\right] a_m. \tag{3.7}$$

Thus, inequality (3.4) is proved.

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