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An upper bound for the Ramsey numbers $r(K_3, G)$*

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Abstract

The Ramsey number $r(H, G)$ is defined as the minimum N such that for any coloring of the edges of the N-vertex complete graph K_N in red and blue, it must contain either a red H or a blue G. In this paper we show that for any graph G without isolated vertices, $r(K_3, G) \leqslant 2q+1$ where G has q edges. In other words, any graph on $2q+1$ vertices with independence number at most 2 contains every (isolate-free) graph on q edges. This establishes a 1980 conjecture of Harary. The result is best possible as a function of q.

1. Introduction

For graphs G and H, the Ramsey number $r(H, G)$ is defined as the minimum number N such that for any coloring of the edges of the N-vertex complete graph K_N in red and blue, it must contain either a red H or a blue G. Harary conjectured that $r(K_3, G) \leqslant 2q+1$, where q is the number of edges of G. This inequality is the best possible, since Chvátal [1] showed that $r(K_3, T_{n+1}) = 2n+1$ for any tree T_{n+1} on n edges. Also, it is well-known that $r(K_3, K_p) < 2\binom{p}{2}+1$.

Erdős et al. [2] showed that $r(K_3, G) \leqslant \lceil 8q/3 \rceil$. Sidorenko [3] improved this by showing that $r(K_3, G) \leqslant 5q/2 - 1$ (for $q \geqslant 4$). In this paper we establish Harary's conjecture.

Theorem 1.1. *For any graph G with q edges and without isolated vertices, $r(K_3, G) \leqslant 2q+1$.*

In other words, any graph on $2q+1$ vertices with independence number at most 2 contains every (isolate-free) graph on q edges.

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2. Preliminaries

Let G have q edges, p vertices and minimum degree δ. We prove the result by induction on q. In particular, let R be such that a red-blue coloring of K_R without a red K_3 always contains a blue copy of every graph on fewer edges than G and yet does not necessarily contain G. Then we find an upper bound on R, assuming it exists.

Like Sidorenko [3], we focus on the minimum degree. He established that $R \leqslant 2q$ when $\delta = 1$, so we will assume that $\delta \geqslant 2$.

Further, we use the same two-case approach as Sidorenko. Call a vertex a δ-vertex if it has degree δ. Then the first case is when G has adjacent δ-vertices.

Lemma 2.1. *If G has two adjacent δ-vertices then $R \leqslant 2q$.*

Proof. Let u_1 and u_2 be adjacent δ-vertices with neighborhoods W_1 and W_2 (themselves excluded). Let G' be the resultant graph when one contracts $u_1 u_2$ to form w. Consider a coloring of K_R that includes a blue G' (but no red K_3), and let X denote the remaining vertices.

Suppose there exist distinct vertices $x_1, x_2 \in X$ with x_i blue-adjacent to all of W_i $(i = 1, 2)$. Consider the three vertices x_1, x_2 and w. It is easy to see that if any two of these are joined by a blue edge, then we obtain a blue G. Therefore, these three vertices form a red K_3, a contradiction.

Thus, there exists an $i \in \{1, 2\}$ such that every vertex in X, except perhaps one, is red-adjacent to some vertex in W_i. We claim that a vertex has red-degree at most $p-1$; for otherwise its red-neighborhood would contain a blue K_p and hence G. Thus $|X| \leqslant (p-1)|W_i| + 1$. Hence

$$R \leqslant (p-1) + (p-1)(\delta - 1) + 1 = \delta p - (\delta - 1) \leqslant 2q - (\delta - 1),$$

as required. □

Now, consider a coloring of K_R without a red K_3 and without a blue G. Let t denote the size of the largest blue clique. It is trivial that the maximum red-degree is at most t, and that $t \leqslant p-1$. Another simple bound is given by the following lemma.

Lemma 2.2. $R \leqslant p + \delta t - 1$.

Proof. Let v be any δ-vertex. Then in K_R there is a blue $G - v$, with the remaining vertices constituting X, say. Let $w_1, \ldots, w_\delta$ be v's neighbors in this copy of $G - v$. If this copy does not directly extend to a blue G, then every vertex in X is red-adjacent to one of the w_i. Thus, the red neighborhoods of the w_i cover X, and hence $|X| \leqslant \delta t$. □

3. Independent δ-vertices

From now on we assume that the δ-vertices form an independent set of size s. We focus on the largest blue clique T in the coloring of K_{2q+1}, and argue that this can be extended to a blue copy of G. In this copy, the non-δ-vertices lie in T, while some δ-vertices lie in T and some outside. We use a greedy approach to show that there must be enough good vertices outside T.

We assume that the coloring of K_{2q+1} does not contain a red K_3. Let $Y = V(K_{2q+1}) - T$ have cardinality y, and let $f = p - t$ denote the number of vertices to be placed outside T. The proof is in three parts. We first establish conditions which ensure that T can be extended to a copy of G. We then derive some useful bounds, and verify that $y = 2q + 1 - t$ satisfies the conditions for $\delta \geqslant 3$. Finally, we handle the case when $\delta = 2$.

3.1. Conditions for extension

Suppose $y \geqslant t$. For I a δ-subset of T, let g_I denote the number of vertices in Y which are blue-adjacent to all of I. Every vertex in T is blue-adjacent to at least $y - t$ vertices in Y. We will assume that we have equality here. (For example, we may forbid our blue copy of G to use certain edges.) Thus,

$$y - \delta t \leqslant g_I \leqslant y - t.$$

Further, let $\bar{g}$ denote the average value of g_I.

Now, assume $t \geqslant p - s$. Consider a possible placement in T of the non-δ-vertices of G. Let $I_1, I_2, \ldots, I_s$ denote the resulting sets to which we need to attach δ-vertices $w_1, w_2, \ldots, w_s$. Assume $g_{I_1} \geqslant g_{I_2} \geqslant \cdots \geqslant g_{I_s}$. We can place $w_{f+1}, w_{f+2}, \ldots, w_s$ inside T without problems. Then we place w_f outside T; this requires $g_{I_f} \geqslant 1$. Next we place w_{f-1} outside T; if $g_{I_{f-1}} \geqslant 2$ then such a vertex is guaranteed to exist. So a greedy algorithm completes the placement of the δ-vertices provided:

$$g_{I_j} \geqslant f - j + 1 \quad \text{for } 1 \leqslant j \leqslant f. \tag{1}$$

For this it is sufficient that

$$\sum_{i=1}^{s} g_{I_i} \geqslant s(f-j) + (j-1)(y-t-f+j) + 1 \quad \text{for } j = 1, 2, \ldots, f. \tag{2}$$

(See Fig. 1.) The right-hand side of this expression is maximized at either $j = 1$ or $j = f$ where it has values $s(f-1)+1$ and $(f-1)(y-t)+1$, respectively. Further, by the above lower bound on g_I, if $y - \delta t \geqslant 0$ then we need only worry about $j \leqslant f - (y - \delta t)$.

By standard reasoning there exists a placement such that $\sum_{j=1}^{s} g_{I_j} \geqslant s\bar{g}$. Hence we have the following lemma.

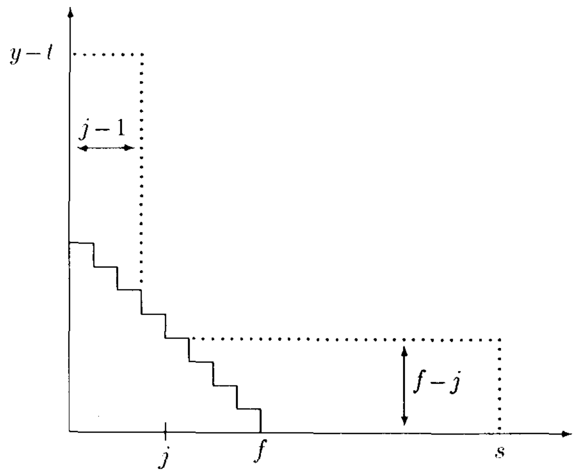

Fig. 1. We need the staircase to be under curve $\{g_{I_j}\}$.

Lemma 3.1. *Assume $y \geqslant t \geqslant p-s$. Then the following two conditions guarantee that the $\{g_{I_j}\}$ satisfy eq. (1), and thus that T can be extended to a copy of G:*
(C1) *$f \leqslant \bar{g}$, and*
(C2) *$f(y-t) \leqslant s\bar{g}$.*
If $\sigma = y - \delta t \geqslant 0$ then we may replace **C2** *by*
(C2′) *$\sigma s + (f-\sigma)(y-t-\sigma) \leqslant s\bar{g}$.*

3.2. Verification of conditions

Recall that $0 < s, t < p$ and $f = p - t$. Observe that $2q \geqslant (\delta+1)p - s$. By the independence of the δ-vertices, $q \geqslant \delta s$. Further, we may assume that $2q < p + \delta t$, else we are done by Lemma 2.2. Thus,

$$p + \delta t > 2q = y + t - 1 \geqslant \max(2\delta s, (\delta+1)p - s). \tag{3}$$

In particular, we have the following lemma.

Lemma 3.2. (a) $s \geqslant \delta f$,
(b) $y \geqslant p2\delta(\delta+1)/(2\delta+1) - t + 1$,
(c) $t > f(2\delta - 2)$.

Proof. Part (a) follows from $p + \delta t \geqslant (\delta+1)p - s$. The lower bound for y is minimized when $2\delta s = (\delta+1)p - s$; this yields (b). Now $p + \delta t \geqslant y + t \geqslant p2\delta(\delta+1)/(2\delta+1)$, so that $p/t \leqslant (2\delta^2 + \delta)/(2\delta^2 - 1)$. This implies that $p/t < (2\delta-1)/(2\delta-2)$ which rearranged gives (c). □

Hence $y \geqslant t \geqslant p - s$.

Lemma 3.3. *If $\bar{g} \geqslant (y-t)/\delta$, then Conditions* **C1** *and* **C2** *are satisfied.*

Proof. Condition **C1** holds since $\bar{g} \geqslant (y-t)/\delta \geqslant (2q-2t)/\delta \geqslant p-2t/\delta \geqslant f$. Condition **C2** holds since $s \geqslant \delta f$ (by Lemma 3.2).

So we need a bound on $\bar{g}$. Let d_i denote the blue-degree into T of the ith vertex of Y. Then $\sum_I g_I = \sum_i \binom{d_i}{\delta}$, while $\sum_i d_i = t(y-t)$. Hence,

$$\bar{g} \geqslant y\binom{t(y-t)/y}{\delta}\Big/\binom{t}{\delta} = (y-t)\prod_{j=1}^{\delta-1}\left(1-\frac{t^2}{y(t-j)}\right). \tag{4}$$

The above bound for $\bar{g}/(y-t)$ is minimized at y as small as possible; so take $y=(\delta-1)t$, a lower bound by Lemma 3.2. Then it is minimized for t as small as possible; so take $t=2(\delta-1)$, a lower bound by Lemma 3.2. Thus $\bar{g}/(y-t) \geqslant \prod_{j=1}^{\delta-1}(1-2/(2\delta-2-j)) = (\delta-3)/(4\delta-6)$.

For $\delta \geqslant 6$ we are thus home. If we are more careful, we can show that $\bar{g} \geqslant (y-t)/\delta$ for $\delta \geqslant 3$ (with one exceptional case). When $\delta = 2$ we must go back and verify the conditions of Lemma 3.1 directly. The details are given below.

3.3. Arithmetical details

From Lemma 3.2 and bound given by eq. (4) we obtain, as follows, $\bar{g} \geqslant (y-t)/\delta$ for $3 \leqslant \delta \leqslant 5$ except when $(\delta, f) = (3, 1)$. If $\delta = 5$, then $y \geqslant 60p/11 - t \geqslant 49(t+1)/11$, and $t \geqslant 9$. The expression $t^2/((t+1)(t-j))$ is minimized at t as small as possible. So plug in lower bounds for y and t and get $\bar{g}/(y-t) \geqslant 0.253$. Similarly for $\delta = 4$: $y \geqslant 40p/9 - t \geqslant 31(t+1)/9$ and $t \geqslant 7$, and plug in to get $\bar{g}/(y-t) \geqslant 0.251$. If $\delta = 3$ and $f \geqslant 2$, then $t \geqslant 9$. Since $y \geqslant 24(t+f)/7 - t \geqslant 17t/7 + 48/7$, it holds that $y(t-j) \geqslant y(t-2) \geqslant (17t^2 + 14t - 96)/7 \geqslant 17t^2/7$. Hence $\bar{g}/(y-t) \geqslant (10/17)^2 \geqslant 0.346$. When $(\delta, f) = (3, 1)$, we merely need $\bar{g} > 0$ (by eq. (2)). For this it is sufficient that $t(y-t) > 3y$. The expression $E = t(y-t) - 3y$ is minimized at y as small as possible, say $y = 17t/7 + 24/7$; and then at t as small as possible, viz. $t = 5$. E's value there is 43/7.

Thus, it remains to verify the conditions when $\delta = 2$. Note that $p \leqslant 10t/7$ (cf. proof of Lemma 3.2).

We consider first the case when $f = 1$. Here we need $\bar{g} > 0$ (by eq. (2)). For this it is sufficient that $t(y-t) > 2y$. The expression $E = t(y-t) - 2y$ is minimized at y as small as possible, say $y = 12(t+1)/5 - t + 1 = 7t/5 + 17/5$ (Lemma 3.2); and then at t as small as possible. If $t \geqslant 4$ then $E \geqslant 2$. The case when $t = 3$ is easily dispensed with. (Recall that $r(K_3, K_4) = 9$.) So from now on we assume that $f \geqslant 2$, and thus $t \geqslant 5$ (by Lemma 3.2).

We next verify Condition **C1**. By the bound of (4), it suffices to show that $f \leqslant (y-t)(yt - t^2 - y)/(yt - y)$. By rearranging it suffices to show that

$$y^2 t - yt^2 + t^3 - ytp \geqslant y^2 - yp.$$

Since $y \leqslant p + t$, the right-hand side of this expression is at most $t(p+t)$. On the other hand, the left-hand side L is minimized at the smallest value of y ($\partial L/\partial y = t(2y - p - t)$

and $y \geqslant (p+t)/2$ by Lemma 3.2). So take $y = 12p/5 - t$ (a lower bound by Lemma 3.2), where $L = t(84p^2 - 155pt + 75t^2)/25 \geqslant t(79p^2 - 150pt + 75t^2)/25 = t(4p^2 + 75f^2)/25$. So it is sufficient that $4p^2 + 300 \geqslant 25(p+t)$, which is true.

Finally, we verify Condition **C2**. Let $s = \alpha f$. By Lemma 3.2, $\alpha \geqslant 2$. We need to establish that $yt - y \leqslant \alpha(yt - t^2 - y)$, or equivalently that $F = t(\alpha(y-t) - y) - (\alpha - 1)y \geqslant 0$. The expression F is minimized at y as small as possible ($\partial F/\partial y = (t-1)(\alpha-1)$). We start with the case $y \leqslant 2t$. Then $\alpha \geqslant 3$ since $s \geqslant 3p - y - t$ by inequality (3). As $y \geqslant 3p - \alpha f - t$,

$$\alpha(y-t) - y \geqslant 3p\alpha - \alpha^2 f - \alpha t - \alpha t - 3p + \alpha f + t = (\alpha - 3)(p - s) + t \geqslant t.$$

Thus, it remains to verify that $t^2 \geqslant (\alpha - 1)y$. Since $y \leqslant 2t$, for this it is sufficient that $t \geqslant 2s/f$. By inequality (3), $y \geqslant 4s - t$ so that $t \geqslant 4s/3$. As $f \geqslant 2$ we are done.

Next we consider the case $y \geqslant 2t$. Then for $F \geqslant 0$ it is sufficient that $\alpha \geqslant 2(t-1)/(t-2)$. Hence, if $\alpha > 5/2$ and $t \geqslant 6$, we are done. The case $t = 5$ and $\alpha > 5/2$ is easily handled. (Since it follows that $p = 7$ and $f = 2$, whence $\alpha \geqslant 3 > 8/3$, as required.)

So consider $\alpha \leqslant 5/2$ and Condition **C2′**. By plugging in the bound of (3) and multiplying through by $y(t-1)/t$, it is sufficient to show that

$$st^3 - sty - pyt^2 + y^2t^2 - yt^3 + pyt - y^2t + yt^2 \geqslant 0.$$

The left-hand side L is minimized at y as small as possible ($\partial L/\partial y = t((2y - p - t)(t-1) - s)$). By Inequality (3), $y \geqslant 3p - s - t \geqslant 3p - 5(p-t)/2 - t$. Using this bound it follows that L is minimized at s as small as possible, so take $s = 2(p-t)$. Simplifying, the condition reduces to verifying that $5t^2 - 9t - pt - 3p \geqslant 0$. This is valid since $p \leqslant 10t/7$ and $t \geqslant 5$.

Notes added in proof

The result in this paper was obtained earlier and independently by A.F. Sidorenko (The Ramsey number of an N-edge graph versus triangle is at most $2N+1$, J. Combin. Theory Ser. B 58 (1993) 185–196) by different means.

References

[1] V. Chvátal, Tree-complete graph Ramsey numbers, J. Graph Theory 1 (1977) 93.

[2] P. Erdős, R.J. Faudree, C.C. Rousseau and R.H. Schelp, A Ramsey problem of Harary on graphs with prescribed size, Discrete Math. 67 (1987) 227–233.

[3] A.F. Sidorenko, An upper bound on the Ramsey number $r(K_3, G)$ depending only on the size of the graph G, J. Graph Theory 15 (1991) 15–17.