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A Symmetric Chain Decomposition of $L(4, n)$

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$L(m, n)$ is the set of integer m-tuples $(a, \ldots, a_m)$ with $0 \leqslant a_1 \leqslant \cdots \leqslant a_m \leqslant n$, ordered by $\underline{a} \leqslant \underline{b}$ when $a_i \leqslant b_i$ for all i. R. Stanley conjectured that $L(m, n)$ is a symmetric chain order for all (m, n). We verify this by construction for $m = 4$.

$L(m, n)$ is defined as the lattice formed by order ideals in the direct product of two chains with m and n elements, respectively. Equivalently, it is the collection of integer sequences $\underline{a} = (a_1, \ldots, a_m)$ satisfying $0 \leqslant a_1 \leqslant \cdots \leqslant a_m \leqslant n$, with ordering $\underline{a} \leqslant \underline{b}$ when $a_i \leqslant b_i$ for all i. The correspondence is simple. If the chain elements are $x_1 < \cdots < x_m$ and $y_1 < \cdots < y_n$, then the number of elements paired with x_i in the ideal corresponding to $\underline{a}$ is $n - a_i$. In other words, the antichain generating the ideal is $\{(x_1, y_{n-a_1}), \ldots, (x_m, y_{n-a_m})\}$.

Clearly, the rank of element $\underline{a}$ is $\sum a_i$, the rank of the entire lattice is mn, and the cardinality of the lattice is $\binom{m+n}{m}$. For any element $\underline{a}$, we define its conjugate $\underline{a}^* = (n - a_m, \ldots, n - a_1)$. Note that $\underline{a}^{**} = \underline{a}$. The ranks of an element and its conjugate sum to mn, so the sizes of the ranks are symmetric about the middle. Sylvester proved the sizes of the ranks are also unimodal. Stanley [4] proved a stronger property which implies unimodality, and he conjectured that $L(m, n)$ is a symmetric chain order. A symmetric chain order is one whose elements can be partitioned into chains which are saturated (skip no ranks) and symmetric about the middle rank. The conjecture is clearly true when $m = 1$ or $m = 2$. Lindström [2] provided an inductive construction to verify it for $m = 3$. Here we give a construction somewhat different from his which verifies the conjecture when $m = 4$. Since writing this paper, we have learned that Reiss [3] has independently verified the conjecture for $m = 3, 4$ by a different method of proof.

Let $S(m, n)$, the "shell" of $L(m, n)$, be those elements which begin with 0 or end with n. When these are removed from $L(m, n)$ the remainder is isomorphic to $L(m, n-2)$. The conjecture holds trivially when $n = 1$, and $L(m, 0)$ can be defined as having a single element. So, providing a symmetric chain decomposition of $S(m, n)$ proves the conjecture by induction. We use this approch here for $L(4, n)$. Unfortunately, when m is odd and n is even the rank sizes in $S(m, n)$ are not unimodal. So, for that case Lindström was forced to strip off two shells for his induction. For $m = 4$ this difficulty does not arise. It is possible that Lindström's construction generalizes for odd m and this construction generalizes for even m. When m and n both exceed 2, $L(m, n)$ is not an LYM-order, so Griggs' sufficient conditions for a symmetric chain order [1] cannot be applied.

THEOREM. *$L(4, n)$ is a symmetric chain order.*

It suffices to give a symmetric chain decomposition of $S(4, n)$. The chains will be of two types, C_{ij} and D_{ij} for suitable values of i and j. The chains are clearly saturated, so two steps will complete the proof.

(1) No element appears in more than one chain.

(2) The number of elements in the construction is the size of $S(m, n)$.

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Each chain is composed of six segments, with the top element of one segment and the bottom element of the next identical. Throughout a given segment only one position in the integer sequence changes. Table 1 explicitly defines the chains and gives the ranks where the changes between segments occur.

TABLE 1

Rank	C_{ij}	Segment	D_{ij}	Rank
$4n-6i-2j$	$(n-3i-j, n-2i-j, n-i, n)$		$(n-3i-j-2, n-2i-j-1, n-i, n)$	$4n-6i-2j-3$
	$\vdots$	6	$\vdots$	
$4n-6i-3j$	$(n-3i-j, n-2i-j, n-i-j, n)$		$(j+1, n-2i-j-1, n-i, n)$	$3n-3i$
	$\vdots$	5	$\vdots$	
$3n-3i-2j$	$(0, n-2i-j, n-i-j, n)$		$(j+1, i+j+1, n-i, n)$	$2n+2j+2$
	$\vdots$	4	$\vdots$	
$3n-3i-3j$	$(0, n-2i-j, n-i-j, n-j)$		$(j+1, i+j+1, 2i+j+1, n)$	$n+3i+3j+3$
	$\vdots$	3	$\vdots$	
$2n-2j$	$(0, i, n-i-j, n-j)$		$(0, i+j+1, 2i+j+1, n)$	$n+3i+2j+2$
	$\vdots$	2	$\vdots$	
$n+3i$	$(0, i, 2i+j, n-j)$		$(0, i+j+1, 2i+j+1, 3i+j+1)$	$6i+3j+3$
	$\vdots$	1	$\vdots$	
$6i+2j$	$(0, i, 2i+j, 3i+j)$		$(0, i+1, 2i+j+1, 3i+j+1)$	$6i+2j+3$

Segments must have length at least 0. That is, top and bottom elements may be identical, but the top element must not have rank below the bottom element. Examining the lengths of segments and ensuring that we have legal elements at the bottom of C_{ij} and the top of D_{ij} yields necessary conditions on i and j. We claim the desired decomposition is obtained by taking all chains for which these necessary conditions are satisfied.

$$S(4, n) = \{C_{ij}: 3i+2j \leq n, i \geq 0, j \geq 0\} \cup \{D_{ij}: 3i+2j \leq n-3, i \geq 0, j \geq 0\}$$

Figure 1 gives $S(4, 5)$ explicitly as an example.

5555						
4555						
3555	4455					
2555	4445		3455			
1555	3445	3355	2455			
0555	2445	3345	1455	2355		
0455	1445	3335	1355	2255	2345	
0355	0445	2335	1255	2245	1345	
0255	0444	1335	1155	2235	0345	1245
0155	0344	0335	1145	2225	0245	1235
0055	0244	0334	1135	1225	0145	0235
0045	0144	0333	1125	0225	0135	0234
0035	0044	0233	1115	0224	0125	0134
0025	0034	0133	0115	0223	0124	
0015	0024	0033	0114	0222	0123	
0005	0014	0023	0113	0122		
0004	0013	0022	0112			
0003	0012		0111			
0002	0011					
0001						
0000						
C_{00}	C_{01}	C_{02}	D_{00}	D_{01}	C_{10}	C_{11}

FIGURE 1. $S(4, 5)$

OUTLINE OF PROOF. To show the elements are all distinct, we express the D-chains in terms of the C-chains and then restrict our attention to the C-chains. Let C_{ij}^r be the element of C_{ij} of rank r, similarly for D_{ij}^r. We claim that chain $D_{i,j-1}$ is the conjugate of chain $C_{i,j}$ when the top and bottoms elements of the latter are removed. That is, $(D_{i,j-1}^r)^* = C_{i,j}^{4n-r}$. It suffices to perform the conjugation on the transition elements between segments of $D_{i,j-1}$. They become the transition elements of $C_{i,j}$. Note the top and bottom elements of $C_{i,j}$ are unaffected and are conjugates of each other. Whenever $D_{i,j-1}$ exists, C_{ij} exists. The affected C_{ij} are those where $j>0$ and $3i+2j<n$.

Distinctness now reduces to showing:

(1a) The elements of $\cup\{C_{ij}\}$ are all distinct.
(1b) The chains C_{i0} and $C_{i,(n-3i)/2}$ are self-conjugate.
(1c) There are no conjugate pairs among the elements of $\cup\{C_{ij}\}$, where $0<j<(n-3i)/2$, other than the tops and bottoms of chains.

(1b) is seen immediately by conjugating the transition elements in those chains. The other two statements require eliminating a large number of easy cases.

To show we have the correct number of elements, we proceed by induction. Simple counting verifies it for small n. In general, the size of $S(m, n)$ is $|L(m, n)| - |L(m, n-2)|$. So,

$$|S(4, n)| = \binom{n+4}{4} - \binom{n+2}{4} = \frac{(n+1)(n+2)(2n+3)}{6}.$$

This is the sum of a familiar sequence. Indeed,

$$|S(4, n)| - |S(4, n-1)| = (n+1)^2.$$

Now we examine the changes in the construction between $n-1$ and n. For all values of i and j such that C_{ij} or D_{ij} exists in the construction for $n-1$, a similarly indexed chain exists in the construction for n. Subtracting ranks, the number of elements in C_{ij} *is* $4(n-3i-j)+1$, and the number in D_{ij} is $4(n-3i-j)-5$. Each of these chains has 4 more elements than the similarly indexed chain in $S(4, n-1)$, if that chain exists. We will see there is a C_{ij} for every element of the middle rank which begins with 0 and a D_{ij} for every such element whose first position is not zero.

The chains which arise newly when n is reached are those C_{ij} for which $3i+2j=n$ and those D_{ij} for which $3i+2j=n-3$. For each value of i from 0 up to $\lfloor n/3 \rfloor$ or $\lfloor n/3 \rfloor - 1$, depending on parities, there will be one new C_{ij} or D_{ij}, but not both.

Verifying that the construction picks up the proper number of elements reduces to:

(2a) Computing (and multiplying by 4) the number of chains in the construction for $S(4, n-1)$—that is, the sum of the number of solutions to $3i+2j \leq n-1$ and $3i+2j \leq n-4$.
(2b) Computing the total number of elements in new chains.
(2c) Verifying the sum of new elements in (2a) and (2b) is $(n+1)^2$.

(2b) breaks into cases depending on the parity of n, and (2a) does the same with the parity of $\lfloor n/3 \rfloor$, so (2c) requires 6 cases, depending on the congruence class of n modulo 6.

DETAILS OF STEP 1. If (1a) does not hold, suppose $\underline{a} = C_{ij}^r = C_{kl}^r$. We have a number of cases to consider, depending on which segment contains $\underline{a}$ in each of the two chains. Let ${}^pC_{ij}$ denote segment p in C_{ij}. Equating the descriptions of the segments in Table 1 gives us a number of linear relationships between i, j, k, and l. If $\underline{a}$ comes from ${}^pC_{ij}$ and ${}^pC_{kl}$, equating the positions which do not change in that segment implies $i=k$ and $j=l$ in all six cases, by straightforward subtraction of equalities.

By symmetry we may assume $\underline{a}$ occurs in a lower numbered segment in C_{ij} than in C_{kl}. We allow the transition elements between segments to belong to either segment. So, if $\underline{a}$ is in ${}^pC_{ij}$ and ${}^qC_{kl}$, we may assume $\underline{a}$ is not the top element of ${}^pC_{ij}$ nor the bottom element of ${}^qC_{kl}$, else we have a case with smaller $q-p$. In particular, the rank of the top element in ${}^pC_{ij}$ must be strictly greater than the rank of the bottom element in ${}^qC_{kl}$.

Suppose $q=p+1$. In each case, comparing the ranks of transition elements as described above yields $f(i,j)>f(k,l)$ for some linear function f. Whenever $q=p+1$, two positions in the elements remain constant from the bottom of segment p to the top of segment q. This expresses two positions of $\underline{a}$ as identical linear functions of (i,j) and (k,l). In all five cases, we readily manipulate these to produce $f(i,j)=f(k\ l)$ for a contradiction.

If the first position of $\underline{a}$ is non-zero, $\underline{a}$ can occur only in segments 5 or 6. If it is zero, $\underline{a}$ occurs in segment 4 or below. Now we have eliminated all but 3 of the cases which might have $C^r_{ij}=C^r_{kl}$ with $(i,j)\neq(k,l)$. The remainder we handle individually.

If $\underline{a}$ is in ${}^2C_{ij}$ and ${}^4C_{kl}$, positions 2 and 3 require $i=n-2k-l$ and $n-i-j>n-k-l$. Adding these gives $n-j>2n-3k-2l\geq n$. Next suppose $\underline{a}$ is in ${}^1C_{ij}$ and ${}^3C_{kl}$. Equality of the last three positions requires $k<i$, $n-k-l=2i+j$, and $n-l\geq 3i+j$. Substituting for k and $n-l$ in the second of these gives $2i+j<2i+j$. Finally, suppose $\underline{a}$ is in ${}^1C_{ij}$ and ${}^4C_{kl}$. Comparing the top of ${}^1C_{ij}$ with the bottom of ${}^4C_{kl}$ yields $n+3i>3n-3k-3l$ or $i>k$. On the other hand, the middle two positions of $\underline{a}$ remain constant in both sections, so $i=n-2k-l$ and $2i+j=n-k-l$. Subtraction gives $i+j=k$ or $i\leq k$.

(1c) also breaks into cases depending on the segments. We assume $\underline{a}=C^r_{ij}=(C^{4n-r}_{kl})^*$, with $0<j<(n-3k)/2$ and $0<l<(n-3k)/2$, and will show this cannot happen. Here the arguments do not group together as cleanly. One element of such a conjugate pair occurs at least as high as the middle rank in one chain. Call this chain C_{ij}. The reader can easily work out the transition elements for the segments in C^*_{kl} to compare with C_{ij}. Examining the bottom ranks of segment 3, we see $2n-2j$ and $n+3k+2l$ are both less than $2n$, so $\underline{a}$ lies in segment 3, 4, 5 or 6 in both C_{ij} and C^*_{kl}. Assume $\underline{a}\in({}^pC_{ij}\cap{}^qC^*_{kl})$.

We first notice $p\leq 4$ is impossible. The first position of $\underline{a}$ is then 0, but the first position in segment 3 or higher of C^*_{kl} is always greater than 0 when $l>0$. We handle the remaining cases individually. Again we equate corresponding positions in $\underline{a}$. The requirements on j and l figure prominently. For example, $i+j\leq k$ and $i\geq k+l$ gives us a contradiction, as do $n-3i-j\leq l$ and $n-3k-l\leq j$. There are eight cases; we include three representative ones here. The others are handled by similar arguments.

$p=6$, $q=6$. $a_2\Rightarrow 2i+j=2k+l$. $a_3\Rightarrow i\leq k$. $a_1\Rightarrow 3i+j\geq 3k+l$. Subtracting a_2 implies $i\geq k$. So $(i,j)=(k,l)$, and this is the case where the top and bottom element of the chain are conjugate.

$p=5$, $q=6$. $a_3\Rightarrow i+j=k$. $a_2\Rightarrow 2i+j=2k+l$. Subtracting a_3 implies $i=k+l$, requiring $j=l=0$.

$p=5$, $q=3$. $a_2\Rightarrow n-2i-j=k+l$. $a_3\Rightarrow n-i-j=k+l$. Subtracting a_2 yields $i=k$. Substituting this in the two previous equations yields $n-3i-j=l$ and $n-3k-l=j$.

This completes the proof of (1).

DETAILS OF STEP 2. We begin with (2a). The top element of segment 4 in C_{ij} has rank $3n-3i-2j\geq 2n$, so every C_{ij} has a 0 in the first position of its middle rank element. The bottom rank of segment 3 in D_{ij} is $n+3i+2j+2\leq 2n-1$, so D_{ij} has a positive first position in its middle rank element. The non-decreasing sequences of length 4 which start with 0, end in k, and sum to $2n$ include from $(0,2n-2k,k,k)$ to $(0,\lfloor(2n-k)/2\rfloor,\lceil(2n-k)/2\rceil,k)$ when $n\geq k\geq\lceil 2n/3\rceil$. So, we want the number of C_{ij}'s to be $\sum_{\lceil 2n/3\rceil\leq k\leq n} k-\lceil(2n-k)/2\rceil+1$. Similarly, the middle-rank elements covered by D_{ij}'s include from $(k,k,n-2k,n)$ through $(k,\lfloor(n-k)/2\rfloor,\lceil(n-k)/2\rceil,n)$ for $1\leq k\leq\lfloor n/3\rfloor$, for a total of $\sum_{1\leq k\leq\lfloor n/3\rfloor}\lfloor(n-k)/2\rfloor-k+1$.

On the other hand, the number of solutions to $3i+2j \leq n$ is $\sum_{0\leq i\leq \lfloor n/3\rfloor} 1+\lfloor (n-3i)/2\rfloor$ and to $3i+2j\leq n-3$ is $\sum_{0\leq i\leq \lfloor n/3\rfloor -1} 1+\lfloor (n-3i-3)/2\rfloor$. These turn into the summations in the preceding paragraph when i is set to $n-k$ in the first case and to $k-1$ in the second. So, the middle rank elements are covered by the chains.

We wish to combine the summations. Separating the $i=0$ term from the first and adjusting the index in the second, the total number $f(n)$ of chains becomes

$$f(n)=1+\lfloor n/2\rfloor+2\sum_{1\leq i\leq \lfloor n/3\rfloor}(1+\lfloor (n-3i)/2\rfloor).$$

To compute the summation, we pair terms for consecutive values of i, with $i=\lfloor n/3\rfloor$ left over if it is odd. Further algebraic manipulation leads us to a closed form for $f(n)$. If $n\equiv r \bmod 6$, $0\leq r\leq 5$, then the total number of chains is

$$f(n)=\lfloor n/2\rfloor+(n+3)(n-r)/3-(n-r)(n-r+6)/6+\begin{cases}1; & r=0,1,2\\ 3; & r=3,4\\ 5; & r=5\end{cases}$$

Next we consider (2b). If n is even, a new chain C_{ij} occurs for even values of i with $0\leq i\leq \lfloor n/3\rfloor$, and a new D_{ij} for odd values of i with $1\leq i\leq \lfloor n/3\rfloor-1$. Similarly, when n is odd we have a new D_{ij} for even i with $0\leq i\leq \lfloor n/3\rfloor -1$ and a new C_{ij} for odd i with $1\leq i\leq \lfloor n/3\rfloor$.

To sum the number of elements in these chains, we can again pair consecutive terms. For the total number $g(n)$ of these elements, we have

$$g(n)=\begin{cases}|C_{0,n/2}|+\sum_{1\leq k\leq \lfloor n/6\rfloor}|D_{2k-1,(n-6k)/2}|+|C_{2k,(n-6k)/2}|; & n \text{ even}\\ \sum_{0\leq k\leq \lfloor (n-3)/6\rfloor}|D_{2k,(n-6k-3)/2}|+|C_{2k+1,(n-6k-3)/2}|; & n \text{ odd}\end{cases}$$

Since $|C_{ij}|=4(n-3i-j)+1$ and $|D_{ij}|=4(n-3i-j)-5$, the terms being summed become $4(n-6k)+8$ and $4(n-6k)-4$ for n even and odd, with $|C_{0,n/2}|=1+2n$. Further algebraic manipulation reduces the sum to

$$g(n)=\begin{cases}1+2n+2(n+2)(n-r)/3-(n-r)(n-r+6)/3; & r=0,2,4\\ 2(n-1)(n-r+6)/3-(n-r)(n-r+6)/3; & r=3,5\\ 2(n-1)^2/3-(n-7)(n-1)/3; & r=1\end{cases}$$

For (2c), we need only compute $4f(n-1)+g(n)$ to find the number by which the size of $S(m,n)$ exceeds the size of $S(m,n-1)$. We treat the congruence classes of n modulo 6 separately. Using the final expressions above for $f(n)$ and $g(n)$, in all six cases $4f(n-1)+g(n)$ reduces to $(n+1)^2$. We leave this as a pleasant exercise for the reader.

This completes the proof.

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