CPR graphs and regular polytopes

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Abstract

This paper studies C-group permutation representation graphs, or for short, CPR graphs. C-groups are the groups of abstract regular polytopes. CPR graphs are shown to be a useful tool for studying such polytopes. We establish general properties of CPR graphs. Moreover, we illustrate their use by constructing regular polyhedra with alternating groups $A_n$ as the automorphism groups.

1. Introduction

Regular polytopes have been studied since antiquity (cf. [1]). In the last 30 years, a combinatorial theory of abstract regular polytopes has been developed and has become an active area of research. The recent monograph [6] by McMullen and Schulte gives a comprehensive account of the subject; the notion of an abstract polytope was first introduced by Danzer and Schulte in [2] and was inspired by Grünbaum (see [3,9]).

In this paper we study certain (multi)graphs, called CPR graphs, that are associated with the automorphism group of an abstract regular polytope. The group $\Gamma(K)$ of such a polytope $K$ is a so-called string C-group, that is, a certain quotient of a Coxeter group with a string diagram (see [6]). A CPR graph $G$ is an edge-labeled multigraph that encodes a permutation representation of $\Gamma(K)$; here, the term CPR stands for “C-group permutation representation”.

In Section 2 we review some concepts concerning abstract regular polytopes and C-groups. Then, in Section 3, we introduce CPR graphs and illustrate that they are useful tools for working with string C-groups.
One of the difficulties we need to deal with while working with CPR graphs (and abstract regular polytopes in general) is the so-called intersection property, defined in Section 2. While in general rank this is quite hard, in the case of regular polyhedra we are able to obtain useful results concerning this property; some of them are started in Section 4.

Finally, in Section 5 we show how to work with CPR graphs. We describe a way to construct CPR graphs for regular polyhedra (and thus the polyhedra themselves) with automorphism groups isomorphic to the alternating groups $A_n$ ($n \geq 9$).

The power of the technique can also be seen through its application to the extension problems for regular polytopes (see [8]). We shall prove in [7] that every regular polytope $K$ can be extended to a regular polytope $P$ with facet type isomorphic to $K$ and any even number as the last entry in its Schl"afli symbol.

2. Abstract regular polytopes

An (abstract) polytope $K$ of rank $d$ is defined by the properties (I)–(IV).

(I) $(K, \leq)$ is a partially ordered set with a minimal element $F_{-1}$ and a maximal element $F_d$. This implies that $F_{-1} \leq F \leq F_d$ for every $F \in K$.

The elements of $K$ are called faces and any face $F$ is associated with $\{G \mid G \leq F\}$, the polytope section of the faces smaller than or equal to $F$. The polytope section $\{G \mid G \geq F\}$ of faces greater than or equal to $F$ is called the coface at $F$. The maximal totally ordered subsets of $K$ are called flags.

(II) Every flag of $K$ contains exactly $d + 2$ faces, including $F_{-1}$ and $F_d$.

This induces a rank function from $K$ to $\{-1, 0, \ldots, d\}$, where the rank $rk(F)$ of a face $F$ is the number of faces contained in the section $\{G \mid G \leq F\}$ distinct from $F_{-1}$ and $F$. The faces of ranks $0, 1, i, d - 1$ are called vertices, edges, $i$-faces and facets respectively. For polyhedra (rank 3 polytopes), the 2-faces are simply called faces. The coface at a vertex is also called the vertex figure.

(III) $K$ is connected in the following sense. For any two different flags $f$ and $g$ of $K$ there is a finite sequence of flags $f = f_1, f_2, \ldots, f_{n-1}, f_n$ such that $f \cap g$ is contained in $f_i$ ($i = 1, \ldots, n$) and $f_i$ differs from $f_{i+1}$ in exactly one face ($i = 1, \ldots, n - 1$).

(IV) For any two faces $F$ and $G$ of $K$ such that $G \leq F$ with $rk(F) - rk(G) = 2$, there are exactly two faces $F_1$ and $F_2$ such that $G < F_1 < F$ ($i = 1, 2$).

For any polytope $K$ if we replace the partial order $\leq$ by $\geq$ while keeping the faces unchanged, the new partially ordered set is again an abstract polytope $K^*$ called the dual of $K$.

A regular polytope is an abstract polytope such that the automorphism (order preserving permutation) group is transitive on the flags.

The faces (cofaces) of any regular polytope are also regular. Moreover, any two faces (cofaces) of the same rank are isomorphic.

The sections defined by two incident faces of ranks $i - 2$ and $i + 1$ of a regular polytope $K$ are isomorphic to regular $p_i$-gons. We will say that $\{p_1, \ldots, p_{d-1}\}$ is the Schl"afli type of $K$. Although in general $p_i$ is allowed to be infinite, we will not consider that case here.

The automorphism group $\Gamma(K)$ of a regular polytope $K$ of rank $d$ is generated by involutions $\rho_0, \ldots, \rho_{d-1}$, where $\rho_i$ is the unique automorphism that keeps all but the $i$-face of a fixed, base
flag $f = \{F_{-1}, \ldots, F_d\}$ fixed. These generators satisfy, at least, the relations

\[
\begin{align*}
\rho_i^2 &= \varepsilon \quad \text{for } 0 \leq i \leq d - 1, \\
(\rho_i \rho_j)^2 &= \varepsilon \quad \text{for } i, j \text{ such that } |i - j| \geq 2, \\
(\rho_i \rho_{i+1})^{p_{i+1}} &= \varepsilon \quad \text{for } 0 \leq i \leq d - 2.
\end{align*}
\]

(1)

For $I \subseteq \{0, \ldots, d - 1\}$ we define $\Gamma_I = \langle \rho_i \mid i \in I \rangle$ for $I \neq \emptyset$, and $\Gamma_\emptyset = \{\varepsilon\}$. Then the generators of $\Gamma(\mathcal{K})$ will also satisfy

\[
\Gamma_I \cap \Gamma_J = \Gamma_{I \cap J}
\]

for $I, J \subseteq \{0, \ldots, d - 1\}$. This is called the Intersection Property.

Groups generated by involutions satisfying (1) and (2) are called string C-groups. Every string C-group is the automorphism group of a regular polytope, and vice versa.

For further information about abstract regular polytopes and C-groups we refer the reader to [6].

3. CPR graphs

The term “CPR” graph stands for “(string) C-group Permutation Representation” graph and is explained next.

**Definition 3.1.** Let $\mathcal{K}$ be a regular $d$-polytope, and let $\pi$ be an embedding of $\Gamma(\mathcal{K})$ in the symmetric group $S_n$ for some $n$. The CPR graph $G$ of $\mathcal{K}$ given by $\pi$ is a $d$-edge-labeled multigraph (graph with multiple edges) with vertex set $V(G) = \{1, \ldots, n\}$ such that $ij$ is an edge of $G$ of label $k$ if $(\pi \rho_k)i = j$ (cf. [11]).

The loops (obtained for $i = j$) play no role here, so they will be ignored. If the embedding $\pi$ is clear from the context, we only refer to “a CPR graph of the polytope $\mathcal{K}$”. There is little possibility of confusion if we identify $\pi(\phi)$ with the element $\phi$ of $\Gamma(\mathcal{K})$.

The edges of each label $k$ form a matching on $G$ and represent pairs of vertices of $G$ interchanged by $\pi \rho_k$. This follows from the fact that the generators of $\Gamma(\mathcal{K})$ are involutions.

**Definition 3.2.** A $d$-edge-labeled multigraph $G$ with the properties that the set of edges of each label $i \in \{0, 1, \ldots, d - 1\}$ forms a matching $M_i$ on $G$, and that $M_i$ represents a different pairing of the vertices of $G$ from $M_j$ for $i \neq j$, will be called a proper $d$-edge-labeled graph.

**Remark 3.3.** Every CPR graph is a proper $d$-edge-labeled graph.

**Definition 3.4.** If $\Gamma(\mathcal{K})$ acts faithfully on $\mathcal{K}_j$, the set of $j$-faces of a regular polytope $\mathcal{K}$, then it can be embedded in $S_m$, where $m$ is the number of $j$-faces of $\mathcal{K}$. In this case, the graph will be called the $j$-face CPR graph of $\mathcal{K}$.

Note that if we consider $\Gamma(\mathcal{K})$ as a group of permutations on the flags of $\mathcal{K}$, then $G$ is just the Cayley graph of $\Gamma(\mathcal{K})$ (cf. [12]).
As an example, the figure above shows the vertex CPR graph (it is also the facet CPR graph) of the tetrahedron.

From now on, \( G_0, \ldots, G_{d-1} \) will denote a \( d \)-edge-labeled graph with edge labels 0, \ldots, \( d-1 \); and for any subset \( I = \{i_1, \ldots, i_m\} \) of \{0, \ldots, d-1\}, \( G_{i_1, \ldots, i_m} \) will denote the spanning subgraph (including all the vertices of \( G \)) of \( G_0, \ldots, G_{d-1} \) whose edge set consists of the edges with labels \( i \in I \).

Given a CPR graph \( G \) of a regular polytope \( K \) we can consider the faithful action of \( \Gamma(K) \) on the vertices \{1, \ldots, n\} of \( G \) induced by \( \pi \). This allows us to establish details of \( G \) given \( K \), as well as of \( K \) given \( G \).

**Proposition 3.5.** Let \( G = G_0, \ldots, G_{d-1} \) be a CPR graph of a regular polytope \( K \), and let \( |i-j| \geq 2 \). Then every connected component of \( G_{i,j} \) is either a single vertex, a single edge, a double edge, or an alternating square.

**Proof.** \( G_{i,j} \) is the union of the matchings determined by \( \rho_i, \rho_j \). Then the connected components of \( G_{i,j} \) are either alternating paths (including isolated vertices) or alternating even cycles (including double edges). The paths of length greater than 1 and the cycles of length greater than 4 come from the action of non-commuting generators or \( \Gamma(K) \) and thus are excluded here. \( \square \)

Now we will describe the way in which the subgroup \( \langle \rho_i, \rho_{i+1} \rangle \cong D_{p_{i+1}} \) acts on each connected component of \( G_{i,i+1} \). Each connected component is an alternating path (of even or odd length at most \( p_{i+1} - 1 \)) or an alternating cycle of even length (at most \( 2p_{i+1} \)).

Note that if the Schlafli type of \( K \) is \( \{p_1, \ldots, p_{d-1}\} \) and \( p_{i+1} \neq 2 \), then \( G_{i,i+1} \) contains at least one connected component which is a path of length greater than 1 or a cycle of length greater than 4.

Let \( C \) be a connected component of \( G_{i,i+1} \). If \( C \) is a path represented by the heavy lines in the figure below, then \( \langle \rho_i, \rho_{i+1} \rangle \) acts like the symmetry group (rotations and reflections) of the dotted regular \( p_{i+1} \)-gon formed as indicated in the figure.

If \( C \) is a cycle of length \( 2l \) greater than 4, then the elements of the form \( (\rho_i \rho_{i+1})^k \) act like a rotation in each of the two dotted regular \( l \)-gons formed by the two sets of alternating vertices of \( C \); these rotations have opposite directions. The remaining elements of \( \langle \rho_i, \rho_{i+1} \rangle \) interchange the two dotted \( l \)-gons by rotating the vertices of each \( l \)-gon along the \( 2l \)-gon, again in opposite directions.
This fact allows us to establish the Schl"afli type of a regular polytope given any of its CPR graphs.

**Definition 3.6.** Let \( G \) be a CPR graph. The action of an element \( \phi \) of \( \langle \rho_i, \rho_i + 1 \rangle \) on a connected component \( C \) of \( G_{i; i+1} \) is called the polygonal action of \( \phi \) on \( C \).

The following results relate the automorphism group of a regular polytope with the automorphism group of its CPR graphs as labeled graphs.

**Lemma 3.7.** Let \( G \) be a CPR graph of a regular polytope \( K \), let \( \Lambda(G) \) be its automorphism group (as an edge-labeled graph), and let \( O_v \) denote the orbit under \( \Lambda(G) \) of a vertex \( v \) of \( G \). Then the group

\[
N = \{ \phi \in \Gamma(K) : \phi(v) \in O_v \text{ for all } v \in V(G) \}
\]

is a normal subgroup of \( \Gamma(K) \).

**Proof.** First note that, if \( \lambda \in \Lambda(G) \), for all \( v \in V \) and for all \( i \), \( \lambda \) maps the edge \( \{v, \rho_i(v)\} \) with label \( i \) onto the edge \( \{\lambda(v), \lambda \rho_i(v)\} \) with label \( i \). This implies that \( \lambda \rho_i(v) = \rho_i \lambda(v) \).

Let \( \phi \in N \). Given a vertex \( v \), \( \phi \rho_i(v) = \lambda_v \rho_i(v) = \rho_i \lambda_v(v) \) for some \( \lambda_v \in \Lambda(G) \). Then \( \rho_i \phi \rho_i(v) = \lambda_v(v) \in O_v \). Since \( \rho_i N \rho_i = N \) for all \( i \), \( N \) is a normal subgroup of \( \Gamma(K) \).

**Proposition 3.8.** Let \( G, K \) and \( N \) be as in **Lemma 3.7**. Let \( G' \) be the \( d \)-edge-labeled graph with vertex set

\[
V(G') = \{O_v : v \in V(G)\},
\]

such that \( O_v O_w \) is an edge of \( G' \) labeled \( i \) if and only if \( v' w' \) is an edge of \( G \) labeled \( i \) for some \( v' \in O_v \) and \( w' \in O_w \). If \( G' \) is a CPR graph of a regular polytope \( K' \), then \( K' \) is the quotient of \( K \) determined by the subgroup \( N \) of \( \Gamma(K) \).

**Proof.** The group \( \Gamma(K) = \langle \rho_0, \ldots, \rho_{d-1} \rangle \) acts on \( G' \) in the following way:

\[
\phi(O_v) = O_{\phi(v)}, \quad v \in V(G).
\]

In particular, the involution on \( V(G') \) represented by the edges of \( G' \) with label \( j \) is induced by the generator \( \rho_j \) of \( \Gamma(K) \), so we have an epimorphism from \( \Gamma(K) \) to \( \Gamma(K') \). Now, by definition,

\[
N = \bigcap_{v \in V(G)} St_v
\]
where \( \text{St}_v \) is the stabilizer of \( O_v \) in \( \Gamma(K) \), but this is the kernel of the epimorphism. Hence,
\[
\Gamma(K') \cong \Gamma(K)/N,
\]
with the appropriate generators. Note that any generator \( \rho_j \) contained in \( N \) becomes trivial in the quotient, so the quotient might only give rise to a polytope of rank less than that of \( K \).

The last proposition does not guarantee the existence of a proper quotient polytope of a regular polytope given a CPR graph with nontrivial automorphism group. The subgroup \( N \) plays an important role and it might be trivial even if the automorphism group of the CPR graph is not.

Now we explain an important relation between the CPR graphs of a given regular polytope and certain subgroups of its automorphism group.

**Lemma 3.9.** Let \( \Gamma \) be a group, \( \Lambda \leq \Gamma \), and let \( \Gamma \) act on the left cosets of \( \Lambda \) in the following way:
\[
\alpha(\beta \Lambda) \mapsto (\alpha \beta) \Lambda.
\]
Then, \( \Lambda \) does not contain any nontrivial normal subgroup of \( \Gamma \) if and only if \( \Gamma \) acts faithfully on the left cosets of \( \Lambda \).

**Proof.** Note that \( \alpha \) fixes the left cosets \( \beta \Lambda \) if and only if \( \alpha \in \beta \Lambda \beta^{-1} \). Then \( \varepsilon \) is the only element fixing all the left cosets of \( \Lambda \) if and only if \( \{\varepsilon\} = \bigcap \beta \Lambda \beta^{-1} \). But the group on the right side is the largest normal subgroup of \( \Gamma \) contained in \( \Lambda \). \( \square \)

**Proposition 3.10.** Let \( \Lambda \) be a subgroup of the automorphism group \( \Gamma(K) \) of a regular polytope \( K \) such that \( \Lambda \) does not contain any non-trivial normal subgroup of \( \Gamma(K) \). Then, \( \Lambda \) determines a connected CPR graph for \( K \). Conversely, we can associate such a subgroup with any connected CPR graph \( G \) of \( K \).

**Proof.** To prove the first part of the theorem we make \( \Gamma(K) \) act on the left cosets of \( \Lambda \) as in Lemma 3.9. Then we construct a graph \( G_\Lambda \) whose vertices are the left cosets of \( \Lambda \) and whose edges are determined by the action of the generators of \( \Gamma(K) \) on the left cosets of \( \Lambda \). This graph is a CPR graph for \( K \) if and only if \( \Gamma(K) \) can be recovered from its action on the left cosets of \( \Lambda \); that is, if and only if \( \varepsilon \) is the only element fixing all the left cosets of \( \Lambda \). Now the proposition follows from Lemma 3.9.

To prove the second part of the theorem, let \( G \) be a CPR graph of \( K \) and let \( v_0 \) be a fixed vertex of \( G \). We now consider the stabilizer \( \Lambda \) of \( v_0 \) in \( \Gamma(K) \). Note that the stabilizer of any other vertex \( u \) of \( G \) is \( \phi \Lambda \phi^{-1} \), where \( \phi(v_0) = u \). Here we are using the fact that \( G \) is connected. Since \( \varepsilon \) is the only element of \( \Gamma(K) \) that fixes all the vertices of \( G \), we have
\[
\bigcap_{\phi \in \Gamma(K)} \phi \Lambda \phi^{-1} = \{\varepsilon\}.
\]
Hence, \( \Lambda \) does not contain any normal subgroup of \( \Gamma(K) \). \( \square \)

The last proposition can be used to find all connected CPR graphs for a given regular polytope \( K \) from the subgroup lattice of \( \Gamma(K) \). Note that conjugate subgroups \( \Lambda \) give rise to the same CPR graph (up to isomorphism).

**Definition 3.11.** Let \( \phi \in \langle \rho_i, \rho_{i+1} \rangle \). If \( \phi \) is of the form \( (\rho_i \rho_{i+1})^k \), then \( \phi \) is called \( (i, i+1) \)-even; otherwise, \( \phi \) is called \( (i, i+1) \)-odd.
The following results will be used in Section 4 and are immediate consequences of the definition of polygonal action.

**Lemma 3.12.** (a) Each \((i, i + 1)\)-odd element is an involution fixing at most two vertices of any connected component of \(G_{i,i+1}\), for any CPR graph \(G = G_{0,\ldots,d-1}\).

(b) No \((i, i + 1)\)-even element has a fixed point in a connected component of \(G_{i,i+1}\) where it does not act as the identity.

**Lemma 3.13.** Let \(C\) be a connected component of \(G_{i,i+1}\) with at least three vertices, let \(\phi \in \langle \rho_i, \rho_{i+1} \rangle\), and let \(\phi|_C = (\rho_k)|_C\) for \(k = i\) or \(k = i + 1\). Then \(\phi\) is \((i, i + 1)\)-odd.

**Proof.** It can be seen easily that the polygonal action of an \((i, i + 1)\)-even element \(\phi\) on \(C\) can coincide with that of \(\rho_k\) in at most two vertices. \(\square\)

4. CPR graphs of polyhedra

In this section we show some particular results for CPR graphs of regular polyhedra that will be used in the Section 5. In other words, from now on \(d = 3\).

**Lemma 4.1.** Let \(G = G_{0,1,2}\) be a CPR graph, and let \(C\) and \(D\) be connected components of \(G_{0,1}\) and \(G_{1,2}\) respectively, that have at least one edge of label 1 in their intersection. Let \(\phi \in \langle \rho_0, \rho_1 \rangle \cap \langle \rho_1, \rho_2 \rangle\) be such that \(\phi|_C = (\rho_1)|_C\) and \(\phi|_D \neq (\rho_1)|_D\). Then every vertex of \(C\) is incident to an edge of label 2. Moreover, if \(C\) and \(D\) have at least three vertices, then \(\phi\) is \((0, 1)\)-odd and \((1, 2)\)-even.

**Proof.** If \(C \cap D\) has no vertex incident to an edge of label 1 and an edge of label 2, then \(D\) only consists of the edge of label 1 with its two vertices, and \(\phi|_D = (\rho_1)|_D\). Hence we can assume that \(C \cap D\) has at least one vertex incident to an edge of label 1 and an edge of label 2.

Let \(v_0\) be a vertex of \(C \cap D\) incident to an edge of label 1 and an edge of label 2. Then, \(\phi\) cannot be \((1, 2)\)-odd because the only reflection interchanging \(v_0\) and \(\phi(v_0) = \rho_1(v_0)\) in the polygonal action of \(\langle \rho_1, \rho_2 \rangle\) on \(D\) is \(\rho_1\). Hence \(\phi\) must be \((1, 2)\)-even. However, the only \((1, 2)\)-even element that interchanges two vertices that are also interchanged by \(\rho_1\) is \((\rho_1\rho_2)^{(n/2)}\), with \(n\) the Schl"afli symbol induced by \(D\), \(n\) even. In this case, \(\phi|_D = (\rho_1\rho_2)^{(n/2)}\), this possibility occurring only if \(D\) is a path of odd length \(n - 1\) and the edge of label 1 incident to \(v_0\), \(u_0v_0\) (say), is the central edge of \(D\). Thus \(\phi|_D = (\rho_1\rho_2)^{(n/2)}\). Then this implies that \(u_0\) is also incident to an edge of label 2. As \(\phi\) is \((1, 2)\)-even, it cannot act like \(\rho_1\) in any connected component of \(G_{1,2}\) with at least three vertices (see Lemma 3.13); moreover, any edge of label 1 in \(C\) is adjacent either to two edges of label 2, or to none.

If \(xw\) is an edge of label 0 and \(x\) is incident to an edge of label 2, then, by *Proposition 3.5*, \(w\) is also incident to an edge of label 2.

Finally, the last two paragraphs and the existence of \(v_0\) imply that every vertex of \(C\) is incident to an edge of label 2 and that \(\phi\) is \((1, 2)\)-even; and *Lemma 3.13* proves that if \(C\) has at least three vertices, then \(\phi\) is \((0, 1)\)-odd. \(\square\)

**Lemma 4.2.** Let \(G = G_{0,1,2}\) be a CPR graph, let \(C\) and \(D\) be connected components of \(G_{0,1}\) and \(G_{1,2}\) respectively, and let \(C \cap D \neq \emptyset\). Let \(\phi \in \langle \rho_0, \rho_1 \rangle \cap \langle \rho_1, \rho_2 \rangle\) be such that \(\phi|_C = (\rho_1)|_C\) and \(\phi|_D \neq (\rho_1)|_D\). Then every vertex of \(C\) is incident to an edge of label 2, and, if \(D\) has at least three vertices, then \(\phi\) is \((0, 1)\)-odd and \((1, 2)\)-even.
Theorem 4.3. Let $G = G_{0,1,2}$ be a connected, proper 3-edge-labeled graph satisfying the condition that every connected component of $G_{0,2}$ is either a single vertex, a single edge, a double edge, or an alternating square. If $G_{0,1}$ (or $G_{1,2}$) has two connected components with at least two vertices such that their numbers of vertices are relatively prime, then $G$ is a CPR graph.

Proof. Let $\phi \in \langle \rho_0, \rho_1 \rangle \cap \langle \rho_1, \rho_2 \rangle$. Suppose to the contrary that $\phi \notin \langle \rho_1 \rangle$. If $\phi$ is (0, 1)-odd, then $\phi \rho_1$ is (0, 1)-even,

$$\phi \rho_1 \in \langle \rho_0, \rho_1 \rangle \cap \langle \rho_1, \rho_2 \rangle,$$

and $\phi \rho_1 \notin \langle \rho_1 \rangle$. Thus we can suppose $\phi$ to be (0, 1)-even.

By assumption, $G_{0,1}$ has two connected components with $n$ and $m$ vertices such that $(n, m) = 1$. Hence, one number, $n$ (say), is odd; then $n \geq 3$. The cycle representation of $\phi$ consists only of cycles of length $l$ in each connected component $C$ of $G_{0,1}$, and no vertex fixed under $\phi$. Note that $l$ is a divisor of $n$, and $l \geq 3$, so $\phi$ is also (1, 2)-even. This implies that the cyclic representation of $\phi$ consists of cycles of length $l$ in each component of $G_{1,2}$ intersecting $C$, so $\phi$ moves all the vertices of such components. It follows from the connectedness of $G$ that then $\phi$ is a product of disjoint cycles of length $l$ without fixed points, so $l$ is a divisor of both $m$ and $n$, but this is impossible. □

Theorem 4.4. Let $G = G_{0,1,2}$ be a connected, proper 3-edge-labeled graph satisfying the condition that every connected component of $G_{0,2}$ is either a single vertex, a single edge, a double edge, or an alternating square. If $G$ has a vertex invariant under $\langle \rho_1, \rho_2 \rangle$ (or dually, $\langle \rho_0, \rho_1 \rangle$), then $G$ is a CPR graph.

Proof. Let $v_0$ be a vertex invariant under $\langle \rho_1, \rho_2 \rangle$, let $C_0$ be the connected component on $G_{0,1}$ containing $v_0$, and let $\phi \in \langle \rho_0, \rho_1 \rangle \cap \langle \rho_1, \rho_2 \rangle$. We can assume that $\phi$ is (0, 1)-odd; otherwise we could take $\rho_1 \phi$. Note that $\phi(v_0) = v_0$, by assumption on $v_0$. Then necessarily $\phi|_{C_0} = (\rho_1)|_{C_0}$.

Suppose to the contrary that $\phi \neq \rho_1$. Then, there exist connected components $C$ and $D$ of $G_{0,1}$ and $G_{1,2}$ respectively with at least three vertices, such that $\phi|_C = (\rho_1)|_C$ and $\phi|_D \neq (\rho_1)|_D$;
Lemma 4.2 shows that $\phi$ acts like $\rho_1$ on every connected component $D$ of $G_{1,2}$ with $C_0 \cap D \neq \emptyset$. But since $\phi$ is $(1, 2)$-even and $\phi|_D = (\rho_1)|_D$, each such component $D$ can have at most two vertices. Then $G$ has no edge of label 2, but this is not possible. This establishes that $\phi$ acts like $\rho_1$ in every connected component of $G_{0,1}$ and $G_{1,2}$.

Hence, the only $(0, 1)$-odd element of $\langle \rho_0, \rho_1 \rangle \cap \langle \rho_1, \rho_2 \rangle$ is $\rho_1$. This completes the proof. $\Box$

**Theorem 4.5.** Let $G = G_{0,1,2}$ be a connected, proper 3-edge-labeled graph satisfying the condition that every connected component of $G_{0,2}$ is either a single vertex, a single edge, a double edge, or an alternating square. If $G$ has an edge $u_0v_0$ of label 1 such that no edge of label 2 (or dually, 0) is incident to either $v_0$ or $u_0$ and such that $u_0v_0$ is not the central edge of a connected component of $G_{0,1}$ that is a path of odd length, then $G_{0,1,2}$ is a CPR graph.

**Proof.** Let $C_0$ be the connected component of $u_0v_0$ in $G_{0,1}$, and let $\phi \in \langle \rho_0, \rho_1 \rangle \cap \langle \rho_1, \rho_2 \rangle$. We can assume that $\phi$ is $(0, 1)$-odd; otherwise we could take $\rho_1 \phi$. We know then that $\phi|_{C_0} = (\rho_1)|_{C_0}$, because the polygonal action of $\rho_1$ on $C_0$ is the only one that interchanges $v_0$ and $u_0$ (we can rule out the possibility of $C_0$ being an odd-path connected component of $G_{0,1}$, where $\phi$ could fix $u_0$ and $v_0$).

Again, Lemma 4.2 shows that $\phi$ acts like $\rho_1$ on every connected component of $G_{1,2}$ intersecting $C_0$. We suppose that $\phi \neq \rho_1$ and continue the proof in the same way as in Theorem 4.4. $\Box$

5. Polyhedra with automorphism groups $A_n$

CPR graphs are a useful tool for constructing regular polyhedra. For example, it is possible to determine the 64 regular polyhedra with automorphism group $S_7$ (cf. [5]) by first constructing all the possible connected 3-edge-labeled graphs that satisfy the conditions on $G_{0,2}$ described in Proposition 3.5 and then verifying the intersection property and the isomorphism with $S_7$ for the corresponding group.

In this section we construct regular polyhedra with automorphism groups isomorphic to $A_n$ with $n \geq 9$.

The following facts are useful for determining whether a CPR graph $G$ belongs to a regular polyhedron with group $S_n$ or $A_n$, where $n$ is the number of vertices of $G$.

**Lemma 5.1.** If a subgroup $\Gamma$ of $S_n$ contains the transposition $(n - 1 \ n)$ as well as a subgroup acting transitively on $\{1, \ldots, n - 1\}$ while keeping $n$ fixed, then $\Gamma = S_n$.

**Lemma 5.2.** If a subgroup $\Gamma$ of $S_n$ contains the 3-cycle $(n - 2 \ n - 1 \ n)$ as well as a subgroup acting transitively on $\{1, \ldots, n - 2\}$ while keeping $n - 1, n$ fixed, then $A_n \leq \Gamma$.

It is easy to see that neither $A_3$ nor $A_4$ can be the group of a regular polyhedron, while $A_5$ is the group of the hemi-dodecahedron, the hemi-icosahedron and the hemi-great dodecahedron. Moreover, none of $A_6$, $A_7$ or $A_8$ can be the group of a regular polyhedron; this can be proved using CPR graphs with six, seven and eight vertices (see [4] and [10]).

For $n \geq 9$ we can find regular polyhedra with group $A_n$. Consider the following graphs on $n = 4k + 1$ vertices.
We know that these graphs are CPR graphs because of Theorem 4.3, and we can check that $A_n$ is a subgroup of the automorphism groups of these regular polyhedra in the following way.  

Note that $\phi := (\rho_0 \rho_1)^4$ is a 3-cycle that permutes only the three vertices in the connected component $C$ of $G_{0,1}$ on the left, while the group generated by $\psi_1 := (\rho_0 \rho_1)^3$ and $\psi_2 := (\rho_1 \rho_2)^3$ is transitive on the remaining vertices of the graph as well as one of those of $C$. By Lemma 5.2, $A_n$ is contained in the group of the polyhedron of this graph. To see whether the group is $A_n$ or $S_n$, it is enough to see whether the number of edges labeled 1 is even or odd, respectively.  

This gives us a regular polyhedron with group $A_n$ for $n = 8k + 1$ and $S_n$ for $n = 8k + 5$. We need to modify these graphs to also obtain regular polyhedra with groups $A_n$ for the remaining integers $n \geq 9$. There are several ways to do this.  

For example, we add an edge of label 1 as a diagonal of one square of the graph with the exception of the first and the last squares (as shown in the picture below). In this situation, $\phi := (\rho_0 \rho_1)^8$ is a 3-cycle that permutes the vertices of $C$, while the subgroup generated by $\psi_1 := (\rho_0 \rho_1)^3$ and $\psi_2 := (\rho_1 \rho_2)^3$ is transitive on the set consisting of the remaining vertices as well as one vertex moved by $\phi$. This helps to construct graphs with groups $A_n$ from those with automorphism groups $S_n$. It is important to note that we cannot add two diagonals of label 1 to a pair of adjacent squares; in this case, no power of $\rho_0 \rho_1$ would be a 3-cycle.  

We can add more vertices to the graph and connect them to any square, except the first and the last, by an edge of label 1. In this situation, $\phi := (\rho_0 \rho_1)^{40}$ is a 3-cycle that permutes the vertices of $C$, while $\psi_1, \psi_2$ can be taken as before. Actually, we can add two vertices to the same square, but we cannot add vertices to adjacent squares in opposite sides of the graph, or add a vertex to a square next to another square with diagonal of label 1, again because no power of $\rho_0 \rho_1$ will be a 3-cycle.  

With these two modifications we can generate families of regular polyhedra with groups $A_n$ for $n \geq 21$. The CPR graphs for polyhedra with groups $A_n$, $9 \leq n \leq 20$, have to be constructed separately. We next give a list of CPR graphs for polyhedra with groups $A_9, A_{10}, \ldots, A_{25}$. The
edges of the squares are labeled 0 and 2 alternately and the other edges are labeled 1. For $A_{12}$ and $A_{18}$ the labels are indicated. From these graphs it is easy to see how to construct more families of graphs for regular polyhedra with group $A_n$ for larger $n$. 

$A_9$ $A_{10}$ $A_{11}$ $A_{12}$ $A_{13}$ $A_{14}$ $A_{15}$ $A_{16}$ $A_{17}$ $A_{18}$
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References