The spectral radius of tricyclic graphs with $n$ vertices and $k$ pendent vertices

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Abstract

In this paper, we determine graphs with the largest spectral radius among all the tricyclic graphs with $n$ vertices and $k$ pendent vertices.

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1. Introduction

All graphs considered in this paper are finite, undirected and simple. Let $G = (V, E)$ be a graph with $n$ vertices and let $A(G)$ be its adjacency matrix. Since $A(G)$ is symmetric, its eigenvalues are real. Without loss of generality, we can write them as $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$ and call them the eigenvalues of $G$. The characteristic polynomial of $G$ is just $\det(\lambda I - A(G))$, denoted by $\phi(G; \lambda)$. The largest eigenvalue $\lambda_1(G)$ is called the spectral radius of $G$, denoted by $\rho(G)$. If $G$ is connected, then $A(G)$ is irreducible and by the Perron–Frobenius theory of non-negative matrices, $\rho(G)$ has multiplicity one and there exists a unique positive unit eigenvector corresponding to $\rho(G)$. We shall refer to such an eigenvector as the Perron vector of $G$. 

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The investigation on the spectral radius of graphs is an important topic in the theory of graph spectra. For results on the spectral radius of graphs, one may refer to [1–14,16,17,19–23] and the references therein.

Chang et al. [4] determined graphs with the largest spectral radius among all the bicyclic graphs on \( n \) vertices with perfect matching. Yu and Tian [22] determined the graph with the largest spectral radius among all the bicyclic graphs on \( n \) vertices with a maximum matching of cardinality \( m \). Guo et al. [9,17] determined the graph with the largest spectral radius among all the unicyclic and bicyclic graphs with \( n \) vertices and \( k \) pendant vertices. Simić [19] determined the bicyclic graphs on prescribed number of vertices with spectral radius minimal.

Let \( G \) be a connected graph and \( T \) is a tree such that \( T \) is attached to a vertex \( v \) of \( G \). The vertex \( v \) is called the root of \( T \), or the root-vertex of \( G \). Throughout this paper, we assume that \( T \) does not include the root.

A tricyclic graph is a connected graph in which the number of edges equals the number of vertices plus two. Denote the set of tricyclic graphs on \( n \) vertices and \( k \) pendant vertices by \( \mathcal{F}_n^k \). In this paper, we study the spectral radius of tricyclic graphs on \( n \) vertices with \( k \) pendant vertices and determine the graph with the largest spectral radius in \( \mathcal{F}_n^k \).

2. Preliminaries

Denote by \( C_n \) and \( P_n \) the cycle and the path, respectively, each on \( n \) vertices. Let \( G - x \) or \( G - xy \) denote the graph that arises from \( G \) by deleting the vertex \( x \in V(G) \) or the edge \( xy \in E(G) \). Similarly, \( G + xy \) is a graph that arises from \( G \) by adding an edge \( xy \notin E(G) \), where \( x, y \in V(G) \). A pendant vertex of \( G \) is a vertex of degree 1. \( k \) paths \( P_{l_1}, P_{l_2}, \ldots, P_{l_k} \) are said to have almost equal lengths if \( l_1, l_2, \ldots, l_k \) satisfy \( |l_i - l_j| \leq 1 \) for \( 1 \leq i, j \leq k \). For \( v \in V(G) \), \( d(v) \) denotes the degree of vertex \( v \) and \( N(v) \) denotes the set of all neighbors of vertex \( v \) in \( G \). We know, by [15], that a tricyclic graph \( G \) contains at least 3 cycles and at most 7 cycles, furthermore, there do not exist 5 cycles in \( G \). Then let \( \mathcal{F}_n^k = \mathcal{F}_n^{k,3} \cup \mathcal{F}_n^{k,4} \cup \mathcal{F}_n^{k,6} \cup \mathcal{F}_n^{k,7} \), where \( \mathcal{F}_n^{k,i} \) denotes the set of tricyclic graphs in \( \mathcal{F}_n^k \) with exact \( i \) cycles for \( i = 3, 4, 6, 7 \).

In this section, we list some known results which will be used in this paper.

**Lemma 2.1** [16,20]. Let \( G \) be a connected graph and \( \rho(G) \) be the spectral radius of \( A(G) \). Let \( u, v \) be two vertices of \( G \) and \( d(v) \) be the degree of vertex \( v \). Suppose \( v_1, v_2, \ldots, v_s \in N(v) \setminus N(u) (1 \leq s \leq d(v)) \) and \( x = (x_1, x_2, \ldots, x_n) \) is the Perron vector of \( A(G) \), where \( x_i \) corresponds to the vertex \( v_i (1 \leq i \leq n) \). Let \( G^* \) be the graph obtained from \( G \) by deleting the edges \( vu \) and adding the edges \( uv_i (1 \leq i \leq s) \). If \( x_u \geq x_v \), then \( \rho(G) < \rho(G^*) \).

**Lemma 2.2** [10]. Let \( G, G', G'' \) be three connected graphs disjoint in pairs. Suppose that \( u, v \) are two vertices of \( G, u' \) is a vertex of \( G' \) and \( u'' \) is a vertex of \( G'' \). Let \( G_1 \) be the graph obtained from \( G, G', G'' \) by identifying, respectively, \( u \) with \( u' \) and \( v \) with \( u'' \). Let \( G_2 \) be the graph obtained from \( G, G', G'' \) by identifying vertices \( u, u' \). Let \( G_3 \) be the graph obtained from \( G, G', G'' \) by identifying vertices \( v, u, u'' \). Then either \( \rho(G_1) < \rho(G_2) \) or \( \rho(G_1) < \rho(G_3) \).

Let \( G \) be a connected graph, and \( uv \in E(G) \). The graph \( G_{u,v} \) is obtained from \( G \) by subdividing the edge \( uv \), i.e., adding a new vertex \( w \) and edges \( uw, uv \) in \( G - uv \). Hoffman and Smith define an internal path of \( G \) as a walk \( v_0v_1 \ldots v_s (s \geq 1) \) such that the vertices \( v_0, v_1, \ldots, v_s \) are distinct, \( d(v_0) > 2, d(v_s) > 2, \) and \( d(v_i) = 2 \), whenever \( 0 < i < s \). And \( s \) is called the length of the internal path. An internal path is closed if \( v_0 = v_s \).
Let $W_n$ be the tree on $n$ vertices obtained from a path $P_{n-4}$ (of length $n - 5$) by attaching two new pendent edges to each end vertex of $P_{n-4}$, respectively. In [13], Hoffman and Smith obtained the following result:

**Lemma 2.3** [13]. Let $uv$ be an edge of the connected graph $G$ on $n$ vertices.

(i) If $uv$ does not belong to an internal path of $G$, and $G \neq C_n$, then $\rho(G_{u,v}) > \rho(G)$;
(ii) If $uv$ belongs to an internal path of $G$, and $G \neq W_n$, then $\rho(G_{u,v}) < \rho(G)$.

**Lemma 2.4** [8,14]. Let $v$ be a vertex in a non-trivial connected graph $G$ and suppose that two paths of lengths $k, m, (k \geq m \geq 1)$ are attached to $G$ by their end vertices at $v$ to form $G^*_{k,m}$. Then $\rho(G^*_{k,m}) > \rho(G^*_{k+1,m-1})$.

**Lemma 2.5.** Let $G_1$ and $G_2$ be two graphs.

(i) [14] If $G_2$ is a proper spanning subgraph of $G_1$ and $G_1$ is a connected graph. Then $\phi(G_2; \lambda) > \phi(G_1; \lambda)$ for $\lambda \geq \rho(G_1)$;
(ii) [5,6] If $\phi(G_2; \lambda) > \phi(G_1; \lambda)$ for $\lambda \geq \rho(G_2)$, then $\rho(G_1) > \rho(G_2)$;
(iii) [13] If $G_2$ is a proper subgraph of $G_1$ and $G_1$ is a connected graph, then $\rho(G_2) < \rho(G_1)$.

**Lemma 2.6** [5,18]. Let $e = uv$ be an edge of $G$, and $C(e)$ be the set of all cycles containing $e$. The characteristic polynomial of $G$ satisfies

$$\phi(G; \lambda) = \phi(G - e; \lambda) - \phi(G - u - v; \lambda) - 2 \sum_{Z \in C(e)} \phi(G \setminus V(Z); \lambda).$$

**Lemma 2.7** [5]. The characteristic polynomial of $P_n$ satisfies the expression

$$\phi(P_n; \lambda) = \frac{1}{\sqrt{\lambda^2 - 4}}(x_1^{n+1} - x_2^{n+1}),$$

where $x_1 = \frac{1}{2}(\lambda + \sqrt{\lambda^2 - 4})$ and $x_2 = \frac{1}{2}(\lambda - \sqrt{\lambda^2 - 4})$ are the roots of the equation $x^2 - \lambda x + 1 = 0$.

**Lemma 2.8** [5,18]. Let $u$ be a vertex of $G$, and let $C(u)$ be the set of all cycles containing $u$. The characteristic polynomial of $G$ satisfies

$$\phi(G; \lambda) = \lambda\phi(G - u; \lambda) - \sum_{v \in N(u)} \phi(G - u - v; \lambda) - 2 \sum_{Z \in C(u)} \phi(G \setminus V(Z); \lambda).$$

**Lemma 2.9** [17]. Let $v$ be a vertex of $G$, let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of the graph $G$, and let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1}$ be the eigenvalues of $G - v$. Then the inequalities $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \lambda_n$ hold. If $G$ is connected, then $\lambda_1 > \mu_1$.

**Lemma 2.10** [17]. If the graphs $G$ and $H$ have exactly one eigenvalue greater than some constant $a$, and if $\phi(G, \rho(H)) > 0$, then $\rho(G) < \rho(H)$.

**Lemma 2.11.** If $k$ paths on $n$ vertices have almost equal lengths, then under the isomorphism relation the structure of this $k$ paths is unique.
Proof. If \( n \) is divisible by \( k \), then our result is obviously true. Otherwise, since \( k \) paths have almost equal lengths, we can assume that \( k_1 \) paths each of which contains \( m \) vertices, the rest paths each of which contains \( m - 1 \) vertices, where \( m = \lfloor \frac{n}{k} \rfloor + 1 \). Therefore,
\[
n = k_1 \cdot m + (k - k_1) \cdot (m - 1) = k_1 + k \cdot (m - 1).
\]

Note that \( n, k, m \) are fixed positive integers, therefore \( k_1 \) is determined correspondingly. This completes the proof. \( \square \)

Let \( B_3(1) \) denote a tricyclic graph in \( \mathcal{F}_n^k \) obtained from the graph \( G_0 \) in Fig. 1 by attaching \( k \) paths with almost equal lengths to vertex \( v \).

Let \( B_4(1) \) denote a tricyclic graph in \( \mathcal{F}_n^k \) obtained from the graph \( G_1 \) in Fig. 1 by attaching \( k \) paths with almost equal lengths to vertex \( z \).

Note that both \( B_3(1) \) and \( B_4(1) \) exist if and only if \( k \geq 1, n \geq k + 7 \).

Lemma 2.12. Provided that both \( B_3(1) \) and \( B_4(1) \) exist, the spectral radius of \( B_3(1) \) is greater than that of \( B_4(1) \).

Proof. Note that, in \( B_3(1) \), \( k \) paths contain exactly \( n - 7 \) vertices, while in \( B_4(1) \), \( k \) paths contain exactly \( n - 6 \) vertices, by Lemma 2.11 we may assume that \( k \) paths in \( B_3(1) \) are \( P_{l_1}, P_{l_2}, \ldots, P_{l_k} \), while those in \( B_4(1) \) are \( P_{l_{i+1}}, P_{l_2}, \ldots, P_{l_k} \). By Lemma 2.8,
\[
\phi(B_3(1); \lambda) = (\lambda^7 - 9\lambda^5 + 6\lambda^4 + 15\lambda^3 + 12\lambda^2 - 7\lambda - 6)\phi(P_{l_1}; \lambda)\phi(P_{l_2}; \lambda) \cdots \phi(P_{l_k}; \lambda)
\]
\[
- (\lambda^6 - 3\lambda^4 + 3\lambda^2 - 1) \sum_{i=1}^{k} \phi(P_{l_i}; \lambda)\phi(P_{l_{i+1}}; \lambda) \cdots \phi(P_{l_{i-1}}; \lambda) \cdots \phi(P_{l_k}; \lambda),
\]

(2.1)

\[
\phi(B_4(1); \lambda) = (\lambda^6 - 8\lambda^4 - 6\lambda^3 + 9\lambda^2 + 8\lambda)\phi(P_{l_{i+1}}; \lambda)\phi(P_{l_2}; \lambda) \cdots \phi(P_{l_k}; \lambda)
\]
\[
- (\lambda^5 - 3\lambda^3 + 2\lambda) \sum_{i=1}^{k} \phi(P_{l_{i+1}}; \lambda)\phi(P_{l_2}; \lambda) \cdots \phi(P_{l_{i-1}}; \lambda) \cdots \phi(P_{l_k}; \lambda).
\]

(2.2)

Together with Eqs. (2.1) and (2.2), we have
\[
\phi(B_4(1); \lambda) - \phi(B_3(1); \lambda) = f(\lambda) + g(\lambda),
\]

where
\[
f(\lambda) = (\lambda^6 - 8\lambda^4 - 6\lambda^3 + 9\lambda^2 + 8\lambda)\phi(P_{l_{i+1}}; \lambda)\phi(P_{l_2}; \lambda) \cdots \phi(P_{l_k}; \lambda)
\]
\[
- (\lambda^5 - 3\lambda^3 + 2\lambda) \cdot \phi(P_{l_1}; \lambda)\phi(P_{l_2}; \lambda) \cdots \phi(P_{l_k}; \lambda)
\]
\[
- (\lambda^7 - 9\lambda^5 - 6\lambda^4 + 15\lambda^3 + 12\lambda^2 - 7\lambda - 6)\phi(P_{l_1}; \lambda) \cdot \phi(P_{l_2}; \lambda) \cdots \phi(P_{l_k}; \lambda)
\]
Proof. Denote by \( \rho(B_i) \) for all \( i \).

Therefore, \( f(\lambda) = (\lambda^6 - 3\lambda^4 + 3\lambda^2 - 1)\phi(P_{l_1-1}; \lambda)\phi(P_{l_2}; \lambda) \cdots \phi(P_{l_k}; \lambda) \),
\[
g(\lambda) = (\lambda^6 - 3\lambda^4 + 3\lambda^2 - 1) \sum_{i=2}^{k} \phi(P_{l_i-1}; \lambda) \cdots \phi(P_{l_k}; \lambda) - (\lambda^5 - 3\lambda^3 + 2\lambda) \sum_{i=2}^{k} \phi(P_{l_i-1}; \lambda) \cdots \phi(P_{l_k}; \lambda).
\]

Furthermore, repeatedly using Lemma 2.8 we can simplify both \( f(\lambda) \) and \( g(\lambda) \) as following:
\[
f(\lambda) = (2\lambda^4 + 2\lambda^3 - \lambda^2 - 2\lambda - 1)\phi(P_{l_1-1}; \lambda)\phi(P_{l_2}; \lambda) \cdots \phi(P_{l_k}; \lambda) + (3\lambda^3 + 4\lambda^2 - 5\lambda - 6)\phi(P_{l_1-2}; \lambda)\phi(P_{l_2}; \lambda) \cdots \phi(P_{l_k}; \lambda),
\]
\[
g(\lambda) = \sum_{i=2}^{k} ((\lambda^2 - 1)\phi(P_{l_1}; \lambda) \cdots \phi(P_{l_i-1}; \lambda) \cdots \phi(P_{l_k}; \lambda) + (\lambda^5 - 3\lambda^3 + 2\lambda)\phi(P_{l_1-1}; \lambda)\phi(P_{l_2}; \lambda) \cdots \phi(P_{l_i-1}; \lambda) \cdots \phi(P_{l_k}; \lambda)).
\]

Note that, when \( \lambda \geq \rho(B_4(1)) = 2 \), by (iii) in Lemma 2.5 we have
\[
\phi(P_{l_1-2}; \lambda)\phi(P_{l_2}; \lambda) \cdots \phi(P_{l_k}; \lambda) > 0,
\]
and
\[
\phi(P_{l_1-1}; \lambda)\phi(P_{l_2}; \lambda) \cdots \phi(P_{l_i-1}; \lambda) \cdots \phi(P_{l_k}; \lambda) > 0
\]
for all \( i = 2, 3, \ldots, k \).

On the other hand, note that \( K_3 \) is a proper subgraph of \( B_4(1) \), by (iii) in Lemma 2.5, we have \( \rho(B_4(1)) > \rho(K_3) = 2 \). Hence, when \( \lambda \geq \rho(B_4(1)) = 2 \), it is easy to see
\[
2\lambda^4 + 2\lambda^3 - \lambda^2 - 2\lambda - 1 > 0, \quad 3\lambda^3 + 4\lambda^2 - 5\lambda - 6 > 0,
\]
\[
\lambda^2 - 1 > 0, \quad \lambda^5 - 3\lambda^3 + 2\lambda > 0.
\]

Therefore, \( f(\lambda) + g(\lambda) > 0 \), i.e., \( \phi(B_4(1); \lambda) > \phi(B_3(1); \lambda) \) for \( \lambda \geq \rho(B_4(1)) \). By Lemma 2.5, \( \rho(B_3(1)) > \rho(B_4(1)) \). This completes the proof. \( \square \)

Let \( B_7(1) \) be a tricyclic graph in \( \mathcal{T}^k_n \) created from \( K_4 \) by attaching \( k \) paths with almost equal lengths to a vertex, say \( v \), of \( K_4 \). Applying Lemma 2.8 to vertex \( v \) of \( B_7(1) \), we have
\[
\phi(B_7(1); \lambda) = (\lambda^4 - 6\lambda^2 - 8\lambda - 3)\phi(P_{l_1}; \lambda) \cdots \phi(P_{l_k}; \lambda)
\]
\[
- (\lambda^3 - 3\lambda - 2) \sum_{i=1}^{k} \phi(P_{l_1}; \lambda) \cdots \phi(P_{l_i-1}; \lambda) \cdots \phi(P_{l_k}; \lambda).
\]

Note that both \( B_3(1) \) and \( B_7(1) \) exist if and only if \( k \geq 1, n \geq k + 7 \).

**Lemma 2.13.** Provided that both \( B_3(1) \) and \( B_7(1) \) exist, the spectral radius of \( B_3(1) \) is greater than that of \( B_7(1) \).

**Proof.** Denote by \( l, (l \geq 2) \) the maximal number of vertices of a path attached to the vertex \( v \) of \( B_7(1) \). Suppose that the number of such paths is \( t \).
Let $B_3$ be the graph analogous to $B_3(1)$ in which all paths attached to vertex $v$ have $l - 1$ vertices. Let $B_7$ be the graph analogous to $B_7(1)$ in which all paths attached to vertex $v$ have $l$ vertices. Evidently, $B_3$ is an induced subgraph of $B_3(1)$ whereas $B_7(1)$ is an induced subgraph of $B_7$. Therefore, by Lemma 2.5,

$$\rho(B_3) \leq \rho(B_3(1))$$

with equality if and only if $n = (l - 1)k + 7$. Also,

$$\rho(B_7) \geq \rho(B_7(1))$$

with equality if and only if $n = lk + 4$.

Thus for the proof of Lemma 2.13 it is sufficient to show that $\rho(B_7) < \rho(B_3)$. We do this in the following.

Because of Lemma 2.9, the graphs $B_3$ and $B_7$ have exactly one eigenvalue greater than 3. (This is because all components of the subgraphs $B_3 - v$ and $B_7 - v$ are paths and $K_3$, and the spectral radii of paths are less than 2, while the spectral radius of $K_3$ is 2. Therefore $\lambda_2(B_3) < 3$ and $\lambda_2(B_7) < 3$. By direct calculation we check that in the case $n = 10, k = 3$, the greatest eigenvalues of $B_3$ and $B_7$ are greater than 3. Therefore the greatest eigenvalues of $B_3$ and $B_7$ are greater than 3 for all values of $n$ and $k$.) Consequently, Lemma 2.10 is applicable to $B_3$ and $B_7$ and it is sufficient to show that $\rho(B_7; \rho(B_3)) > 0$.

By applying Lemma 2.8 to the vertex $v$ of $B_3$ and $B_7$, respectively, we obtain

$$\phi(B_3; \lambda) = (\lambda^7 - 9\lambda^5 - 6\lambda^4 + 15\lambda^3 + 12\lambda^2 - 7\lambda - 6)(P_{l-1}; \lambda)^k$$

$$- k(\lambda^6 - 3\lambda^4 + 3\lambda^2 - 1)(P_{l-1}; \lambda)^{k-1}(P_{l-2}; \lambda),$$

$$\phi(B_7; \lambda) = (\lambda^4 - 6\lambda^2 - 8\lambda - 3)(P_{l}; \lambda)^k - k(\lambda^3 - 3\lambda - 2)(P_{l}; \lambda)^{k-1}(P_{l-1}; \lambda).$$

Denote the greatest eigenvalue of $B_3$ by $r$. For $n = 10$ and $k = 3$ the greatest eigenvalue of $B_3(1)$ is 3.3926. Therefore, for any $n$ and $k$, $r = \rho(B_3) \geq 3.3923$.

From the above expression for $\phi(B_3; \lambda)$ it is seen that $r$ satisfies the equation

$$(r^7 - 9r^5 - 6r^4 + 15r^3 + 12r^2 - 7r - 6)(P_{l-1}; r)$$

$$- k(r^6 - 3r^4 + 3r^2 - 1)(P_{l-2}; r) = 0$$

from which

$$k = \frac{(r^7 - 9r^5 - 6r^4 + 15r^3 + 12r^2 - 7r - 6)(P_{l-1}; r)}{(r^6 - 3r^4 + 3r^2 - 1)(P_{l-2}; r)}.$$

Now, the inequality $\phi(B_7; r) > 0$ holds if and only if

$$r(4^4 - 6r^2 - 8r - 3)(P_{l}; r) - k(r^3 - 3r - 2)(P_{l-1}; r) > 0$$

if and only if

$$(r^4 - 6r^2 - 8r - 3)(P_{l}; r) > k(r^3 - 3r - 2)(P_{l-1}; r)$$

if and only if

$$\frac{(r^7 - 9r^5 - 6r^4 + 15r^3 + 12r^2 - 7r - 6)(P_{l-1}; r)}{(r^6 - 3r^4 + 3r^2 - 1)(P_{l-2}; r)}(r^3 - 3r - 2)(P_{l-1}; r)$$

$$> \frac{(r^7 - 9r^5 - 6r^4 + 15r^3 + 12r^2 - 7r - 6)(P_{l-1}; r)}{(r^6 - 3r^4 + 3r^2 - 1)(P_{l-2}; r)}(r^3 - 3r - 2)(P_{l-1}; r).$$
We now demonstrate that for
\[(r^{10} - 9r^8 - 8r^7 + 18r^6 + 24r^5 - 10r^4 - 24r^3 - 3r^2 + 8r + 3)\phi(P_l; r)\phi(P_{l-2}; r) > (r^{10} - 12r^8 - 8r^7 + 42r^6 + 48r^5 - 40r^4 - 72r^3 - 3r^2 + 32r + 12)\phi(P_{l-1}; r)^2.\]

From Lemma 2.7 we get
\[\phi(P_n; r) = \frac{1}{\sqrt{r^2 - 4}}(r_1^{n+1} - r_2^{n+1}),\]
where \(r_1 = \frac{1}{2}(r + \sqrt{r^2 - 4})\) and \(r_2 = \frac{1}{2}(r - \sqrt{r^2 - 4})\) are the roots of the equation \(x^2 - rx + 1 = 0\). From the Vieta formulas, \(r_1 + r_2 = r; r_1r_2 = 1\) and therefore
\[r_1^2 + r_2^2 = (r_1 + r_2)^2 - 2r_1r_2 = r^2 - 2,\]
\[r_1^4 + r_2^4 = (r_1^2 + r_2^2)^2 - 2r_1^2r_2^2 = (r^2 - 2)^2 - 2.\]

In view of the above, \(\phi(B_l; r) > 0\) holds if and only if
\[\frac{1}{r^2 - 4}(r^{10} - 9r^8 - 8r^7 + 18r^6 + 24r^5 - 10r^4 - 24r^3 - 3r^2 + 8r + 3)(r_1^{l+1} - r_2^{l+1})(r_1^{l-1} - r_2^{l-1}) > \frac{1}{r^2 - 4}(r^{10} - 12r^8 - 8r^7 + 42r^6 + 48r^5 - 40r^4 - 72r^3 - 3r^2 + 32r + 12)(r_1^l - r_2^l)^2\]
if and only if
\[(r_1^{2l} + r_2^{2l})(3r^8 - 24r^6 - 24r^5 + 30r^4 + 48r^3 - 24r - 9) > r_1^{12} - 13r^{10} - 8r^9 + 60r^8 + 56r^7 - 130r^6 - 168r^5 + 97r^4 + 200r^3 + 15r^2 - 80r - 30.\]

We now demonstrate that for \(l \geq 2\) the series \(a_l = r_1^{2l} + r_2^{2l}\) strictly increases. Because \(r_1^{2l} + r_2^{2l} = \frac{r_1^{4l+1}}{r_1^l}\), we get that
\[\frac{a_{l+1}}{a_l} = \frac{r_1^{4l+4} + 1}{r_1^{4l+2} + r_1^2}\]
will be greater than unity (in which case \(a_l\) increases) if and only if
\[r_1^{4l+4} + 1 > r_1^{4l+2} + r_1^2\]
i.e., if
\[(r_1^{4l+2} - 1)(r_1^2 - 1) > 0\]
which is evidently obeyed since \(r_1 > 1\).

We have previously shown that \(\phi(B_l; r) > 0\) holds if and only if Inequality (2.3) holds. Now, if Inequality (2.3) is satisfied for \(l = 2\) it will be satisfied for all \(l \geq 2\).

For \(l = 2\) we get
\[(r_1^4 + r_2^4)(3r^8 - 24r^6 - 24r^5 + 30r^4 + 48r^3 - 24r - 9)\]
\[ r^{12} - 13r^{10} - 8r^9 + 60r^8 + 56r^7 - 130r^6 - 168r^5 + 97r^4 + 200r^3 + 15r^2 - 80r - 30 \]

if and only if
\[ 2r^{12} - 23r^{10} - 16r^9 + 72r^8 + 88r^7 - 38r^6 - 96r^5 - 46r^4 - 8r^3 + 21r^2 + 32r + 12 > 0. \]

Let
\[ f(r) = 2r^{12} - 23r^{10} - 16r^9 + 72r^8 + 88r^7 - 38r^6 - 96r^5 - 46r^4 - 8r^3 + 21r^2 + 32r + 12. \]

Note that \( r > 3 \), we have \( 2r^{12} > \frac{5}{3}r^{12} + r^{11} \). Therefore,
\[ f(r) > \frac{5}{3}r^{12} + r^{11} - 23r^{10} - 16r^9 + 72r^8 + 88r^7 - 38r^6 - 96r^5 - 46r^4 - 8r^3 + 21r^2 + 32r + 12 \]
\[ = \left( \frac{5}{3}r^{12} - 23r^{10} + 67r^8 \right) + (r^{11} - 16r^9 + 64r^7) + (5r^8 - 38r^6 - 46r^4) + (24r^7 - 96r^5 - 8r^3) + (21r^2 + 32r + 12). \]

It is easy to show when \( r > 3.1019 \), that \( \frac{5}{3}r^{12} - 23r^{10} + 67r^8 > 0 \), \( r^{11} - 16r^9 + 64r^7 > 0 \), \( 5r^8 - 38r^6 - 46r^4 > 0 \), \( 24r^7 - 96r^5 - 8r^3 > 0 \), \( 21r^2 + 32r + 12 > 0 \). Namely, when \( r > 3.1019 \), \( f(r) > 0 \). Note that \( r = \rho(B_3) \geq 3.3923 \), and so, we have demonstrated that \( \phi(B_7; \rho(B_3)) > 0 \), which, by Lemma 2.10, implies
\[ \rho(B_3) > \rho(B_7). \]

Therefore,
\[ \rho(B_3(1)) > \rho(B_7(1)). \]

Case 2.1 \( 1 \leq t \leq 2, k \geq 2 \).

In this case, it is straightforward to check that the maximal number of vertices of a path attached to the vertex \( v \) of \( B_7(1) \) is \( l \), while the minimal number of vertices of a path attached to the vertex \( v \) of \( B_3(1) \) is \( l - 2 \), where \( l \geq 3 \). Thus, let \( B'_3 \) be the graph analogous to \( B_3(1) \) in which all paths attached to vertex \( v \) have \( l - 2 \) vertices, whereas let \( B_7 \) be the graph analogous to \( B_7(1) \) in which all paths attached to vertex \( v \) have \( l \) vertices.

Evidently, \( B'_3 \) is an induced subgraph of \( B_3(1) \) whereas \( B_7(1) \) is an induced subgraph of \( B_7 \).

Therefore, by Lemma 2.5,
\[ \rho(B'_3(l)) \leq \rho(B_3(1)) \]
with equality if and only if \( n = (l - 2)k + 7 \). Also,
\[ \rho(B_7) \geq \rho(B_7(1)) \]
with equality if and only if \( n = lk + 4 \).

Thus for the proof of Lemma 2.13 it is sufficient to show that \( \rho(B_7) < \rho(B'_3) \). We do this in the following.

Because of Lemma 2.9, the graphs \( B'_3 \) and \( B_7 \) have exactly one eigenvalue greater than 3. (This is because all components of the subgraphs \( B'_3 - v \) and \( B_7 - v \) are paths and \( K_3 \), and the
spectral radii of paths are less than 2, while the spectral radius of $K_3$ is 2. Therefore $\lambda_2(B'_3) < 3$ and $\lambda_2(B_7) < 3$. By direct calculation we check that in the case $n = 9$, $k = 2$, the greatest eigenvalues of $B'_3$ and $B_7$ are greater than 3. Therefore the greatest eigenvalues of $B'_3$ and $B_7$ are greater than 3 for all values of $n$ and $k$.) Consequently, Lemma 2.10 is applicable to $B'_3$ and $B_7$ and it is sufficient to show that $\phi(B_7; \rho(B'_3)) > 0$.

By applying Lemma 2.8 to the vertex $v$ of $B'_3$ and $B_7$, respectively, we obtain
\[
\phi(B'_3; \lambda) = (\lambda^7 - 9\lambda^5 - 6\lambda^4 + 15\lambda^3 + 12\lambda^2 - 7\lambda - 6)\phi(P_{l-2}; \lambda)^k \\
- k(\lambda^6 - 3\lambda^4 + 3\lambda^2 - 1)\phi(P_{l-3}; \lambda),
\]
\[
\phi(B_7; \lambda) = (\lambda^4 - 6\lambda^2 - 8\lambda - 3)\phi(P_l; \lambda)^k - k(\lambda^3 - 3\lambda - 2)\phi(P_l; \lambda)^{k-1}\phi(P_{l-1}; \lambda).
\]
Denote the greatest eigenvalue of $B'_3$ by $r$. For $n = 9$ and $k = 2$ the greatest eigenvalue of $B'_3$ is 3.2635. Therefore, for any $n$ and $k$, $r = \rho(B'_3) \geq 3.2635$.

Similarly to Case 1, $\phi(B_7; r) > 0$ holds if and only if
\[
2r^{13} - 26r^{11} - 16r^{10} + 108r^9 + 112r^8 - 164r^7 - 240r^6 \\
+ 74r^5 + 208r^4 + 30r^3 - 64r^2 - 24r > 0.
\]
Let
\[
f(r) = 2r^{12} - 26r^{10} - 16r^9 + 108r^8 + 112r^7 - 164r^6 - 240r^5 \\
+ 74r^4 + 208r^3 + 30r^2 - 64r - 24 \\
= (2r^{12} - 26r^{10} - 16r^9 + 92r^8 + 88r^7) + (16r^8 - 164r^6 + 74r^4) \\
+ (24r^7 - 240r^5 + 144r^3) + (64r^3 - 64r) + (30r^2 - 24).
\]
It is easy to show when $r > 3.126$, that $16r^8 - 164r^6 + 74r^4 > 0$, $24r^7 - 240r^5 + 144r^3 > 0$, $64r^3 - 64r > 0$, $30r^2 - 24 > 0$.

Let
\[
g(r) = 2r^{12} - 26r^{10} - 16r^9 + 92r^8 + 88r^7.
\]
When $r > 3$, $g'(r) = 24r^{11} - 260r^9 - 144r^8 + 736r^7 + 616r^6 > 0$, this shows that $g(r)$ strictly increases when $r > 3$. Note that $g(3) > 0$, and so $g(r) > 0$ when $r > 3$. Namely, when $r > 3.1019$, $f(r) > 0$. Recall that $r = \rho(B'_3) \geq 3.2635$, therefore, we have demonstrated that $\phi(B_7; \rho(B'_3)) > 0$, which, by Lemma 2.10, implies
\[
\rho(B'_3) > \rho(B_7).
\]
Therefore,
\[
\rho(B_3(1)) > \rho(B_7(1)).
\]

Case 3. $t = 1, k = 1$. In this case, we have
\[
\phi(B'_3(1); \lambda) = (\lambda^7 - 9\lambda^5 - 6\lambda^4 + 15\lambda^3 + 12\lambda^2 - 7\lambda - 6)\phi(P_{l-3}; \lambda) \\
- (\lambda^6 - 3\lambda^4 + 3\lambda^2 - 1)\phi(P_{l-4}; \lambda),
\]
\[
\phi(B_7(1); \lambda) = (\lambda^4 - 6\lambda^2 - 8\lambda - 3)\phi(P_l; \lambda) - (\lambda^3 - 3\lambda - 2)\phi(P_{l-1}; \lambda).
\]
Because of Lemma 2.9, the graphs $B'_3(1)$ and $B_7(1)$ have exactly one eigenvalue greater than 3. (This is because all components of the subgraphs $B'_3(1) - v$ and $B_7(1) - v$ are paths and $K_3$,}
and the spectral radii of paths are less than 2, while the spectral radius of $K_3$ is 2. Therefore $\lambda_2(B_3(1)) < 3$ and $\lambda_2(B_7(1)) < 3$. By direct calculation we check that in the case $n = 8$, $k = 1$, the greatest eigenvalues of $B_3(1)$ and $B_7(1)$ are greater than 3. Therefore the greatest eigenvalues of $B_3(1)$ and $B_7(1)$ are greater than 3 for all values of $n$ and $k$. Consequently, Lemma 2.10 is applicable to $B_3(1)$ and $B_7(1)$ and it is sufficient to show that $\phi(B_7(1); \rho(B_3(1))) > 0$.

Denote the greatest eigenvalue of $B_3(1)$ by $r$. For $n = 8$ and $k = 1$ the greatest eigenvalue of $B_3(1)$ is 3.1326. Therefore, for any $n$ and $k$,

$$r = \rho(B_3(1)) \geq 3.1326.$$  

From the above expression for $\phi((B_3(1); \lambda)$ it is seen that $r$ satisfies the equation:

$$(r^7 - 9r^5 - 6r^4 + 15r^3 + 12r^2 - 7r - 6)\phi(P_{l-3}; r) - (r^6 - 3r^4 + 3r^2 - 1)\phi(P_{l-4}; r) = 0$$

from which

$$\frac{\phi(P_{l-4}; r)}{\phi(P_{l-3}; r)} = \frac{r^7 - 9r^5 - 6r^4 + 15r^3 + 12r^2 - 7r - 6}{r^6 - 3r^4 + 3r^2 - 1}.$$ 

Now, the inequality $\phi(B_7(1); r) > 0$ hold if and only if

$$(r^4 - 6r^2 - 8r - 3)\phi(P_{1}; r) - (r^3 - 3r - 2)\phi(P_{l-1}; r) > 0.$$ 

Note that

$$\phi(P_{l-1}; r) = (r^2 - 1)\phi(P_{l-3}; r) - r\phi(P_{l-4}; r),$$

so we have

$$\phi(B_7(1); r) = (r^7 - 9r^5 - 8r^4 + 13r^3 + 18r^2 + 3r - 2)\phi(P_{l-3}; r) - (r^6 - 8r^4 - 8r^3 + 6r^2 + 10r + 3)\phi(P_{l-4}; r).$$

That is to say, $\phi(B_7(1); r) > 0$ if and only if

$$(r^7 - 9r^5 - 8r^4 + 13r^3 + 18r^2 + 3r - 2)(r^6 - 3r^4 + 3r^2 - 1) > (r^7 - 9r^5 - 6r^4 + 15r^3 + 12r^2 - 7r - 6)(r^6 - 8r^4 - 8r^3 + 6r^2 + 10r + 3)$$

if and only if

$$5r^{11} + 6r^{10} - 50r^9 - 100r^8 + 66r^7 + 268r^6 + 76r^5 - 240r^4 - 175r^3 + 46r^2 + 78r + 20 > 0.$$ 

Let

$$f(r) = 5r^{11} + 6r^{10} - 50r^9 - 100r^8 + 66r^7 + 268r^6 + 76r^5 - 240r^4 - 175r^3 + 46r^2 + 78r + 20.$$ 

Similarly to the proof of Cases 1 and 2, we can also show that $f(r) > 0$, when $r = \rho(B_3(1))$, which, by Lemma 2.10, implies $\rho(B_3(1)) > \rho(B_7(1))$.

Combining Cases 1–3, we completes the proof. □
Let $B_6(1)$ (respectively, $B_6(2)$) denote a tricyclic graph in $\mathcal{F}_n^k$ obtained from the graph $G_0$ in Fig. 2 by attaching $k$ paths with almost equal lengths to vertex $y$ (respectively, $x$).

With the same method used in Lemma 2.13, we can prove the following lemma, we will not repeat the procedure here. Note that both $B_3(1)$ and $B_6(1)$ exist if and only if $k \geq 1, n \geq k + 7$.

**Lemma 2.14.** Provided that both $B_3(1)$ and $B_6(1)$ exist, the spectral radius of $B_3(1)$ is greater than that of $B_6(1)$.

### 3. Main results

**Theorem 3.1.** Let $G$ be a graph in $\mathcal{F}_n^{k,3}$, $k \geq 1$. Then $\rho(G) \leq \rho(B_3(1))$, and the equality holds if and only if $G \cong B_3(1)$.

**Proof.** The arrangement of three cycles, say $C_p, C_q, C_h$, in $G$ has seven possible cases; see Fig. 3. Choose $G \in \mathcal{F}_n^{k,3}$ such that the spectral radius of $G$ is as large as possible. Denote the vertex set of $G$ by $\{v_1, v_2, \ldots, v_n\}$ and the Perron vector of $G$ by $x = (x_1, x_2, \ldots, x_n)$, where $x_i$ corresponds to the vertex $v_i (1 \leq i \leq n)$. We first prove some facts.

**Fact 1.** The arrangement of three cycles in $G$ is (c) in Fig. 3.

**Proof of Fact 1.** Assume that the arrangement of the three cycles contained in $G$ is just (b) in Fig. 3. Then denote the path connecting two cycles, say $C_p, C_q$, by $v_1v_2 \ldots v_l$ with $l > 1$. Suppose that $v_1$ is on $C_p$, while $v_l$ is on $C_q$. Without loss of generality, we may assume that $x_1 \geq x_l$. Denote by $v_{l+1}$ a neighbor of $v_l$ which belongs to $C_q$. Let

$$G^* = G - \{v_l v_{l+1}\} + \{v_1 v_{l+1}\}.$$  

Then $G^* \in \mathcal{F}_n^{k,3}$. By Lemma 2.1, we have $\rho(G^*) > \rho(G)$, a contradiction. Hence $l = 1$. Using the same method, we can also show that $G$ cannot contains three cycles whose arrangement is as (d), (e), (f) or (g) in Fig. 3. Therefore the arrangement of three cycles in $G$ is either (a) or (c) in
Fig. 3. By Lemma 2.2, we know that $G$ cannot contains three cycles whose arrangement is that of (a). This completes the proof of Fact 1.

Furthermore, by Fact 1, we can prove by Lemma 2.2 that $G$ has exactly one tree, say $T$, attached and the root, say $v$, of the tree is the common vertex of the three cycles.

**Fact 2.** Each vertex $u$ of $T$ has degree $d(u) \leq 2$.

**Proof of Fact 2.** On the contrary, if there exists one vertex $v_i$ of $T$ such that $d(v_i) > 2$. Denote $N(v_i) = \{z_1, z_2, \ldots, z_l\}$ and $N(v) = \{w_1, w_2, \ldots, w_s\}$. Assume that $z_1, w_3$ belong to the path joining $v$ and $v_i$, and that $w_1$ (respectively, $w_2$) belongs to some cycle in $G$. If $x_v \geq x_i$, let

$$G^*_3 = G - \{v_iz_3, \ldots, v_iz_l\} + \{vz_3, \ldots, vz_l\}. $$

If $x_v < x_i$, let

$$G^*_4 = G - \{vw_1, vw_4, \ldots, vw_s\} + \{v_1w_1, v_1w_4, \ldots, v_1w_3\}. $$

Then $G^*_3, G^*_4 \in \mathcal{F}_{n, 3}$. By Lemma 2.1, we have $\rho(G^*_3) > \rho(G)$ and $\rho(G^*_4) > \rho(G)$, a contradiction. Hence $G$ is a graph with $k$ paths attached to $v$.

**Fact 3.** $k$ paths attached to $v$ have almost equal lengths.

**Proof of Fact 3.** Denote the $k$ paths attached to $v$ by $P_{l_1}, P_{l_2}, \ldots, P_{l_k}$, then we will show that $|l_i - l_j| \leq 1$ for $1 \leq i, j \leq k$. If there exist two paths, say $P_{l_1}$ and $P_{l_2}$, such that $l_1 - l_2 \geq 2$. Denote $P_{l_1} = vu_1 u_2 \ldots u_{l_1}$ and $P_{l_2} = vw_1 w_2 \ldots w_{l_2}$. Let

$$G^* = G - \{u_{l_1-1}u_{l_1}\} + \{w_{l_2}u_{l_1}\}. $$

Then $G^* \in \mathcal{F}_{n, 3}$. By Lemma 2.4, we have $\rho(G^*) > \rho(G)$, a contradiction.

**Fact 4.** All the cycles $C_p, C_q$ and $C_h$ in $G$ have length 3.

**Proof of Fact 4.** Assume that $p \geq 4$. Let $C_p = v_1v_2 \ldots v_{p-1}v$ and let $P_m = vu_1 u_2 \ldots u_m$ be a path attached to the graph $G$, where $m \geq 1$. Obviously, $G \neq C_n, G \neq W_n, vu_1 u_2 \ldots u_m$ is not an internal path.

Let

$$G^* = G - \{v_1v_2\} + \{vv_2, u_mv_1\}. $$

Then $G^* \in \mathcal{F}_{n, 3}$. By Lemma 2.3, we have $\rho(G^*) > \rho(G)$, a contradiction. Hence $p = 3$. Similarly, we can verify that $q = 3$ and $h = 3$.

Combining Facts 1–4, we have $G = B_3(1)$. This completes the proof.

**Theorem 3.2.** Let $G$ be a graph in $\mathcal{F}_{n, 4}^k$, $k \geq 1$. Then $\rho(G) \leq \rho(B_4(1))$, and the equality holds if and only if $G \cong B_4(1)$.

**Proof.** Let $P_{l+1}, P_{p+1}, P_{q+1}$ be three vertex-disjoint paths, where $l, p, q \geq 1$ and at most one of them is 1. Identifying the three initial vertices and terminal vertices of them, respectively, the resulting graph (e.g., (i) in Fig. 4), denoted by $P(l, p, q)$, is called a $\theta$-graph. Furthermore, let $C_h$ be a cycle. Connect $C_h$ and $P(l, p, q)$ by a path $P_s$ and denote the resulting graph by $G'$, where $s \geq 1$ and $G'$ has four types, see (ii)–(v) in Fig. 4. So, $\mathcal{F}_{n, 4}^k$ are those graphs each of which is obtained by attaching some trees to $G'$. 
Choose $G \in \mathcal{F}_n^{k,4}$ such that the spectral radius of $G$ is as large as possible. Denote the vertex set of $G$ by $\{v_1, v_2, \ldots, v_n\}$ and the Perron vector of $G$ by $x = (x_1, x_2, \ldots, x_n)$, where $x_i$ corresponds to the vertex $v_i (1 \leq i \leq n)$.

Similarly to the proof of Theorem 3.1 we can verify that $G$ is a $G'$-graph with $k$ paths of almost equal lengths attached to one vertex denoted by $v$, where $G'$ is either Type (ii) or Type (iii) and the cycle $C_h$ is of length 3. We will use $P_m = vu_1u_2 \cdots u_m$ to denote one of the $k$ paths attached to vertex $v$, where $m \geq 1$.

We can prove by Lemma 2.1 that $G'$ must be of Type (iii). For convenience, we assume that $a$ is just the common vertex of $P(l, p, q)$ and $C_h$. Then by Lemma 2.2 we obtain that $v = a$.

By the definition of graph $P(l, p, q)$, we have $l, p, q \geq 1$ and at most one of them is 1. We claim that one of $p, q, l$ is 1 and the other two are 2. Assume, on the contrary, that $l \geq 3$. Put $P_{l+1} = vv_2 \cdots v_{l+1}$. Obviously, $G \neq C_n, G \neq W_n, vv_2 \cdots v_{l+1}$ is an internal path, and $vu_1u_2 \cdots u_m$ is not an internal path. Let

$$G^* = G - \{vv_2, v_2v_3\} + \{vv_3, u_m v_2\}. $$

Then $G^* \in \mathcal{F}_n^{k,4}$. By Lemma 2.3, we have $\rho(G^*) > \rho(G)$, a contradiction. Hence $l \leq 2$. Similarly, we can verify that $p, q \leq 2$ and that one and only one of $l, p, q$ is 1. Thus $G = B_4(1)$. □

**Theorem 3.3.** Let $G$ be a graph in $\mathcal{F}_n^{k,6}$, $k \geq 1$. Then $\rho(G) \leq \rho(B_6(1))$, and the equality holds if and only if $G \cong B_6(1)$.

**Proof.** Since $G$ contains exactly six cycles, then it is straightforward to check that all of the six cycles either have exactly two vertices in common, or have exactly one vertex in common, or have no vertex in common; see Fig. 5.

Choose $G \in \mathcal{F}_n^{k,6}$ such that the spectral radius of $G$ is as large as possible. Denote the vertex set of $G$ by $\{v_1, v_2, \ldots, v_n\}$ and the Perron vector of $G$ by $x = (x_1, x_2, \ldots, x_n)$, where $x_i$ corresponds to the vertex $v_i (1 \leq i \leq n)$.

Similarly to the proof of Theorem 3.1 we can verify that there exactly exist $k$ paths of almost equal lengths attached to one vertex, say $v$, on a cycle of $G$.

We can prove by Lemma 2.1 that the six cycles contained in $G$ must be (a) in Fig. 5. Then label the common vertices of the six cycles as $v_1, v_2$, which are depicted in Fig. 5 (a).
Assume that \( P_{l+1}, P_{p+1}, P_{q+1}, P_{h+1} \) are the four paths connecting \( v_1 \) and \( v_2 \); see (a) in Fig. 5. Similarly to the proof of Theorem 4, we obtain that one of \( l, p, q, h \) is 1 and the other three are 2. When \( v = v_1 \) or \( v_2 \), then \( G = B_6(1) \). When \( v \notin \{v_1, v_2\} \), then \( G = B_6(2) \).

Now we show that \( \rho(B_6(1)) > \rho(B_6(2)) \). This completes the proof. \( \square \)

**Theorem 3.4.** Let \( G \) be a graph in \( \mathcal{F}^k_n \), \( k \geq 1 \). Then \( \rho(G) \leq \rho(B_7(1)) \), and the equality holds if and only if \( G \cong B_7(1) \).

**Proof.** The arrangement of seven cycles contained in \( G \) has only one case; see Fig. 6. Choose \( G \in \mathcal{F}^k_n \) such that the spectral radius of \( G \) is as large as possible. Denote the vertex set of \( G \) by \( \{v_1, v_2, \ldots, v_n\} \) and the Perron vector of \( G \) by \( x = (x_1, x_2, \ldots, x_n) \), where \( x_i \) corresponds to the vertex \( v_i(1 \leq i \leq n) \).

Similarly to the proof of Theorem 3.1 we can verify that there exactly exist \( k \) paths of almost equal lengths attached to one vertex, say \( v \), on a cycle of \( G \). For convenience, \( v_1, v_2, v_3, v_4 \) and \( P_1, \ldots, P_6 \) are just as shown in Fig. 6. We will use \( P_m = vw_1 \cdots w_m \) to denote one of the \( k \) paths attached to vertex \( v \), where \( m \geq 1 \). We can prove by Lemma 2.1 that \( v \) is in \( \{v_1, v_2, v_3, v_4\} \).

Now we want to show that \( l_1 = l_2 = \cdots = l_6 = 2 \). Assume, on the contrary, that \( l_1 \geq 3 \), then let \( P_{l_1} = v_1u_1u_2 \cdots u_s \) and \( u_s = v_2 \) and \( s \geq 2 \). Obviously, \( G \neq C_n, G \neq W_n, v_1u_1u_2 \cdots u_s \) is an internal path, and \( vw_1 \cdots w_m \) is not an internal path. Let

\[
G^* = G - \{v_1u_1, u_1u_2\} + \{v_1u_2, w_mB_7(1). \) This completes the proof. \( \square \)

From Lemmas 2.12–2.14 and Theorems 3.1–3.4, we get our main result in this paper.
Theorem 3.5. Let $G$ be a graph in $T^k_n$, $k \geq 1$. Then $\rho(G) \leq \rho(B_3(1))$, and the equality holds if and only if $G \cong B_3(1)$.

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