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New lower bounds on eigenvalue of the Hadamard product of an M-matrix and its inverse☆

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Some new lower bounds for the minimum eigenvalue of the Hadamard product of an M-matrix and its inverse are given. These bounds improve the results of [H.B. Li, T.Z. Huang, S.Q. Shen, H. Li, Lower bounds for the minimum eigenvalue of Hadamard product of an M-matrix and its inverse, Linear Algebra Appl. 420 (2007) 235–247].

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1. Introduction

A matrix $A = (a_{ij}) \in R^{n\times n}$ is called an M-matrix if $a_{ii} > 0, i \in N$; $a_{ij} \leqslant 0, i \neq j, i, j \in N$, A is nonsingular and $A^{-1} \geqslant 0$, where $N = \{1, 2, \ldots, n\}$ (see [1]).

If A is an M-matrix, then there exists a positive eigenvalue of A equal to $\tau(A) \equiv [\rho(A^{-1})]^{-1}$, where $\rho(A^{-1})$ is the Perron eigenvalue of the nonnegative matrix A^{-1}, $\tau(A) = \min\{|\lambda| : \lambda \in \sigma(A)\}$, $\sigma(A)$ denotes the spectrum of A (see [2]).

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For two matrices $A = (a_{ij}) \in R^{n\times n}$ and $B = (b_{ij}) \in R^{n\times n}$, the Hadamard product of A and B is the matrix $A \circ B = (a_{ij}b_{ij})$. If A and B are M-matrices, then it was proved in [2] that $A \circ B^{-1}$ is again an M-matrix.

Let $A = (a_{ij}) \in R^{n\times n}$ be an M-matrix. It was proved in [3] that $\tau(A \circ A^{-1}) \leqslant 1$.

Subsequently, Fiedler and Markham in [2] proved that $\tau(A \circ A^{-1}) \geqslant \frac{1}{n}$, and conjectured that $\tau(A \circ A^{-1}) \geqslant \frac{2}{n}$. Yong [4], Song [5] and Chen [6] have independently proved this conjecture.

Li in [7] improved the conjecture $\tau(A \circ A^{-1}) \geqslant \frac{2}{n}$ of Fiedler and Markham, and obtained the following result:

$$\tau(A \circ A^{-1}) \geqslant \min_i \left\{ \frac{a_{ii} - s_i R_i}{1 + \sum_{j\neq i} s_{ji}} \right\},$$

which depends only on the entries of matrix A, instead of the dimension of matrix A, where $s_{ji} = \frac{|a_{ji}|+\sum_{k\neq j,i}|a_{jk}|d_k}{a_{jj}}$, $j \neq i, j \in N$; and $s_i = \max_{j\neq i}\{s_{ij}\}$, $R_i = \sum_{k\neq i}|a_{ik}|$, $d_k = \frac{\sum_{j\neq k}|a_{kj}|}{|a_{kk}|}$, $i, k \in N$.

Recently, Huang in [8] proved the following inequality

$$\tau(A \circ B^{-1}) \geqslant \frac{1 - \rho(J_A)\rho(J_B)}{1 + (\rho(J_B))^2} \min_{1\leqslant i\leqslant n} \frac{a_{ii}}{b_{ii}},$$

where A and B are M-matrices and $\rho(J_A)$, $\rho(J_B)$ is the spectral radius of J_A and J_B. When $A = B$, the inequality provided another lower bound of $\tau(A \circ A^{-1})$, that is

$$\tau(A \circ A^{-1}) \geqslant \frac{1 - (\rho(J_A))^2}{1 + (\rho(J_A))^2}. \tag{1.1}$$

The bound (1.1) is a theoretical formula and it is difficult to calculate the lower bound of $\tau(A \circ A^{-1})$ by using this formula because of the difficulty of calculating the spectral radius of the Jacobi iterative matrix $\rho(J_A)$ when the order of A is large.

In this paper, we present some new lower bounds for $\tau(A \circ A^{-1})$. These bounds improve the results of Li in [7] and their calculations are easier than Huang's formula (1.1).

For any $i, k, l \in N$, denote

$$r_{li} = \frac{|a_{li}|}{|a_{ll}| - \sum_{k\neq l,i}|a_{lk}|}, \quad l \neq i; \qquad r_i = \max_{l\neq i}\{r_{li}\}, \quad i \in N.$$

$$c_{il} = \frac{|a_{il}|}{|a_{ll}| - \sum_{k\neq l,i}|a_{kl}|}, \quad l \neq i; \qquad c_i = \max_{l\neq i}\{c_{il}\}, \quad i \in N.$$

2. Some lemmas and notations

In this section, we give some lemmas that involve inequalities for the entries of A^{-1}. They will be useful in the following proofs.

Lemma 2.1 [4]. (a) *If $A = (a_{ij}) \in R^{n\times n}$ is a strictly row diagonally dominant matrix, that is, $|a_{ii}| > \sum_{j\neq i}|a_{ij}|$ for every $i \in N$, then $A^{-1} = (b_{ij})$ exists, and*

$$|b_{ji}| \leqslant \frac{\sum_{k\neq j}|a_{jk}|}{|a_{jj}|}|b_{ii}|, \qquad \text{for all } j \neq i.$$

(b) *If $A = (a_{ij}) \in R^{n\times n}$ is a strictly column diagonally dominant matrix, that is, $|a_{ii}| > \sum_{j\neq i}|a_{ji}|$ for every $i \in N$, then $A^{-1} = (b_{ij})$ exists, and*

$$|b_{ij}| \leqslant \frac{\sum_{k\neq j}|a_{kj}|}{|a_{jj}|}|b_{ii}|, \quad \text{for all } j \neq i.$$

Lemma 2.2. (a) *Let $A = (a_{ij}) \in R^{n\times n}$ be a strictly row diagonally dominant M-matrix. Then, for $A^{-1} = (b_{ij})$, we have*

$$b_{ji} \leqslant \frac{|a_{ji}| + \sum_{k \neq j,i} |a_{jk}| r_i}{a_{jj}} b_{ii}, \quad \text{for all } j \neq i.$$

(b) *Let $A = (a_{ij}) \in R^{n\times n}$ be a strictly column diagonally dominant M-matrix. Then, for $A^{-1} = (b_{ij})$, we have*

$$b_{ij} \leqslant \frac{|a_{ij}| + \sum_{k \neq j,i} |a_{kj}| c_i}{a_{jj}} b_{ii}, \quad \text{for all } j \neq i.$$

Proof. (a) For $i \in N$, let $r_i(\varepsilon) = \max_{l\neq i}\left\{\frac{|a_{li}|+\varepsilon}{a_{ll}-\sum_{k\neq l,j}|a_{lk}|}\right\}$. Since A is strictly diagonally dominant, then $\frac{|a_{li}|}{a_{ll}-\sum_{k\neq l,j}|a_{lk}|} < 1$. Hence, there exists $\varepsilon > 0$ such that $0 < r_i(\varepsilon) < 1$. Let $R_i(\varepsilon) = diag(r_i(\varepsilon), \ldots, r_i(\varepsilon), 1, r_i(\varepsilon), \ldots, r_i(\varepsilon))$.

For a given $i \in N$, one checks that the matrix $AR_i(\varepsilon)$ is again a strictly row diagonally dominant M-matrix. In fact, for $j \neq i$, we have

$$r_i(\varepsilon) > \frac{|a_{ji}|}{a_{jj} - \sum_{k\neq j,i}|a_{jk}|}, \qquad j \neq i, \quad j \in N.$$

So

$$|a_{ji}| + r_i(\varepsilon)\sum_{k\neq j,i}|a_{jk}| < r_i(\varepsilon)|a_{jj}|, \qquad j \neq i, \quad j \in N. \tag{2.1}$$

While, for $j = i$, we have

$$\sum_{k\neq i}|a_{ik}|r_i(\varepsilon) < \sum_{k\neq i}|a_{ik}| < a_{ii}. \tag{2.2}$$

From (2.1) and (2.2) we have proved that $AR_i(\varepsilon)$ is strictly row diagonally dominant, so it is also an M-matrix. By Lemma 2.1 (a), we derive the following inequality:

$$r_i^{-1}(\varepsilon)b_{ji} \leqslant \frac{|a_{ji}| + \sum_{k\neq j,i}|a_{jk}|r_i(\varepsilon)}{r_i(\varepsilon)a_{jj}} b_{ii}, \qquad j \neq i, \quad j \in N.$$

i.e.,

$$b_{ji} \leqslant \frac{|a_{ji}| + \sum_{k\neq j,i}|a_{jk}|r_i(\varepsilon)}{a_{jj}} b_{ii}, \qquad j \neq i, \quad j \in N.$$

Let $\varepsilon \to 0$ to obtain

$$b_{ji} \leqslant \frac{|a_{ji}| + \sum_{k\neq j,i}|a_{jk}|r_i}{a_{jj}} b_{ii}, \quad \text{for all } j \neq i, \; j \in N.$$

(b) For matrix $C_i(\varepsilon)A$, where $C_i(\varepsilon) = diag(c_i(\varepsilon), \ldots, c_i(\varepsilon), 1, c_i(\varepsilon), \ldots, c_i(\varepsilon))$, $i \in N$ and

$$c_i(\varepsilon) = \max_{l\neq i}\left\{\frac{|a_{il}| + \varepsilon}{a_{ll} - \sum_{k\neq l,i}|a_{kl}|}\right\}, \quad i \in N,$$

by Lemma 2.1 (b) and the same technique as in the above proof (a), Lemma 2.2 (b) is obtained. □

In the following, we need the notations

$$m_{ji} = \frac{|a_{ji}| + \sum_{k\neq j,i}|a_{jk}|r_i}{a_{jj}}, \quad j \neq i, \; j \in N; \qquad m_i = \max_{j\neq i}\{m_{ij}\}, \quad i \in N.$$

Lemma 2.3. *Let $A = (a_{ij}) \in R^{n\times n}$ be a strictly row diagonally dominant M-matrix. Then, for $A^{-1} = (b_{ij})$, we have*

$$\frac{1}{a_{ii}} \leqslant b_{ii} \leqslant \frac{1}{a_{ii} - \sum_{j\neq i}|a_{ij}|m_{ji}}, \quad \text{for all } i \in N.$$

Proof. (1) Let $B = A^{-1}$. Since A is an M-matrix, then $B \geqslant 0$. Since $AB = I$, we have

$$1 = \sum_{j=1}^{n} a_{ij}b_{ji} = a_{ii}b_{ii} - \sum_{j\neq i}|a_{ij}|b_{ji}, \quad \text{for all } i \in N.$$

Hence

$$a_{ii}b_{ii} \geqslant 1, \quad \text{for all } i \in N, \text{ that is, } \frac{1}{a_{ii}} \leqslant b_{ii}, \quad \text{for all } i \in N.$$

(2) By Lemma 2.2 (a), for all $i \in N$,

$$\begin{aligned} 1 &= a_{ii}b_{ii} - \sum_{j\neq i}|a_{ij}|b_{ji} \geqslant a_{ii}b_{ii} - \sum_{j\neq i}|a_{ij}|\frac{|a_{ji}|+\sum_{k\neq j,i}|a_{jk}|r_i}{a_{jj}}b_{ii} \\ &= \left(a_{ii} - \sum_{j\neq i}|a_{ij}|m_{ji}\right)b_{ii} \end{aligned}$$

i.e.,

$$b_{ii} \leqslant \frac{1}{a_{ii} - \sum_{j\neq i}|a_{ij}|m_{ji}}, \quad \text{for all } i \in N. \quad \square$$

Remark 2.1. Example 4.1 shows that Lemmas 2.2 and 2.3 are improvements of Theorems 2.1 and 2.3 of [7].

Lemma 2.4 [9]. *Let $A = (a_{ij}) \in C^{n\times n}$ and let $s_1, s_2, \cdots, s_n$ be positive real numbers. Then all the eigenvalues of A lie in the region*

$$\bigcup_{i=1}^{n}\left\{Z \in C : |Z - a_{ii}| \leqslant s_i \sum_{j\neq i}\frac{1}{s_j}|a_{ji}|\right\}.$$

Lemma 2.5 [2]. *If P is an irreducible M-matrix, and if $Pz \geqslant kz$ for a nonnegative nonzero vector z, then $k \leqslant \tau(P)$.*

Lemma 2.6 [10]. *If A^{-1} is a doubly stochastic matrix, then $Ae = e, A^Te = e$, where $e = (1, 1, \ldots, 1)^T$.*

3. Main results

In this section, we exhibit a new lower bound for $\tau(A \circ A^{-1})$, which improves the result of Li et al. in [7] and the conjecture of Fiedler and Markham.

Theorem 3.1. *Let $A = (a_{ij}) \in R^{n\times n}$ be an M-matrix, and suppose $A^{-1} = (b_{ij})$ is doubly stochastic. Then*

$$b_{ii} \geqslant \frac{1}{1 + \sum_{j\neq i}m_{ji}}, \quad i \in N.$$

Proof. Since A^{-1} is doubly stochastic and A is an M-matrix, by Lemma 2.6, we have

$$a_{ii} = \sum_{k\neq i}|a_{ik}| + 1 = \sum_{k\neq i}|a_{ki}| + 1, \quad i \in N \quad \text{and} \quad b_{ii} + \sum_{j\neq i}b_{ji} = 1, \quad i \in N.$$

The matrix A is strictly row diagonally dominant. Then, by Lemma 2.2 (a), for $i \in N$, we have

$$\begin{aligned} 1 &= b_{ii} + \sum_{j \neq i} b_{ji} \leqslant b_{ii} + \textstyle\sum_{j \neq i} \frac{|a_{ji}| + \sum_{k \neq j,i} |a_{jk}| r_i}{a_{jj}} b_{ii} \\ &= \left(1 + \sum_{j \neq i} \frac{|a_{ji}| + \sum_{k \neq j,i} |a_{jk}| r_i}{a_{jj}}\right) b_{ii} \\ &= \left(1 + \sum_{j \neq i} m_{ji}\right) b_{ii.} \end{aligned}$$

i.e., $b_{ii} \geqslant \frac{1}{1 + \sum_{j \neq i} m_{ji}}$, $i \in N$. □

Theorem 3.2. *Let $A = (a_{ij}) \in R^{n \times n}$ be an M-matrix, and let $A^{-1} = (b_{ij})$ be doubly stochastic. Then*

$$\tau(A \circ A^{-1}) \geqslant \min_i \left\{ \frac{a_{ii} - m_i R_i}{1 + \sum_{j \neq i} m_{ji}} \right\}.$$

Proof. (1) First, we assume that A is irreducible. Since A^{-1} is doubly stochastic, we obtain from Lemma 2.6 that

$$a_{ii} = \sum_{j \neq i} |a_{ij}| + 1 = \sum_{j \neq i} |a_{ji}| + 1 \quad \text{and} \quad a_{ii} > 1, \ i \in N.$$

Let

$$R_j^r = \sum_{k \neq j} |a_{jk}| r_i, \qquad j \neq i, \quad i \in N.$$

Then, for any $j \in N, j \neq i$

$$R_j^r = \sum_{k \neq j} |a_{jk}| r_i \leqslant |a_{ji}| + \sum_{k \neq j,i} |a_{jk}| r_i \leqslant R_j = \sum_{k \neq j} |a_{jk}| \leqslant a_{jj}.$$

Therefore, there exists a real number $\beta_{ji} (0 \leqslant \beta_{ji} \leqslant 1)$, such that

$$|a_{ji}| + \sum_{k \neq j,i} |a_{jk}| r_i = \beta_{ji} R_j + (1 - \beta_{ji}) R_j^r.$$

Hence

$$m_{ji} = \frac{\beta_{ji} R_j + (1 - \beta_{ji}) R_j^r}{a_{jj}}.$$

Let $\beta_j = \max_{i \neq j} \{\beta_{ji}\}$, $0 < \beta_j \leqslant 1$ (if $\beta_j = 0$, then A is reducible, which is a contradiction). Let

$$m_j = \max_{i \neq j} \{m_{ji}\} = \frac{\beta_j R_j + (1 - \beta_j) R_j^r}{a_{jj}}, \quad j \in N.$$

Since A is irreducible, then $R_j > 0$, $R_j^r > 0$ and $0 < m_j \leqslant 1$. Thus, by Lemma 2.4, there exists $i_0 \in N$ such that

$$|\lambda - a_{i_0 i_0} b_{i_0 i_0}| \leqslant m_{i_0} \sum_{j \neq i_0} \frac{1}{m_j} |a_{j i_0} b_{j i_0}|.$$

i.e.,

$$\begin{aligned} |\lambda| &\geqslant a_{i_0 i_0} b_{i_0 i_0} - m_{i_0} \sum_{j \neq i_0} \frac{1}{m_j} |a_{j i_0} b_{j i_0}| \\ &\geqslant a_{i_0 i_0} b_{i_0 i_0} - m_{i_0} \sum_{j \neq i_0} \frac{a_{jj}}{\beta_j R_j + (1 - \beta_j) R_j^r} |a_{j i_0}| \frac{|a_{j i_0}| + \sum_{k \neq j, i_0} a_{jk} r_{i_0}}{a_{jj}} b_{i_0 i_0} \end{aligned}$$

$$\begin{aligned}
&\geqslant a_{i_0i_0}b_{i_0i_0} - m_{i_0}\sum_{j\neq i_0}|a_{ij_0}|b_{i_0i_0}\\
&= (a_{i_0i_0} - m_{i_0}R_{i_0})b_{i_0i_0}\\
&\geqslant \frac{a_{i_0i_0} - m_{i_0}R_{i_0}}{1+\sum_{j\neq i_0} m_{ji_0}}\\
&\geqslant \min_i\left\{\frac{a_{ii} - m_iR_i}{1+\sum_{j\neq i} m_{ji}}\right\}.
\end{aligned}$$

(2) When A is reducible, without loss of generality, we can assume that A has the block upper triangular form

$$A = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1S}\\ & A_{22} & \dots & A_{2S}\\ & & \dots & \dots\\ & & & A_{SS}\end{bmatrix}$$

with irreducible diagonal blocks A_{ii}, $i = 1, 2, \dots, S$. Then A^{-1} is again block upper triangular with irreducible diagonal blocks A_{ii}^{-1}. Observing that $\tau(A \circ A^{-1}) = \min_k \tau(A_{kk} \circ A_{kk}^{-1})$ concludes the proof. □

Theorem 3.3. *Let $A = (a_{ij}) \in R^{n\times n}$ be an M-matrix with $a_{11} = a_{22} = \cdots = a_{nn}$, and suppose $A^{-1} = (b_{ij})$ is doubly stochastic. Then*

$$\min_i\left\{\frac{a_{ii} - m_iR_i}{1+\sum_{j\neq i}m_{ji}}\right\} \geqslant \min_i\left\{\frac{a_{ii} - s_iR_i}{1+\sum_{j\neq i}s_{ji}}\right\}.$$

Proof. Since A^{-1} is doubly stochastic, by Lemma 2.6, we have

$$a_{ii} = \sum_{k\neq i}|a_{ik}| + 1 = \sum_{k\neq i}|a_{ki}| + 1.$$

Then for every $i \in N$,

$$r_i = \max_{l\neq i}\left\{\frac{|a_{li}|}{|a_{ll}| - \sum_{k\neq l,i}|a_{lk}|}\right\} = \max_{l\neq i}\left\{\frac{|a_{li}|}{1+|a_{li}|}\right\} = \frac{\max_{l\neq i}|a_{li}|}{1+\max_{l\neq i}|a_{li}|}.$$

Since $f(x) = \frac{x}{1+x}$ is an increasing function on $(0, +\infty)$, we have

$$r_i = \frac{\max_{l\neq i}|a_{li}|}{1+\max_{l\neq i}|a_{li}|} \leqslant \frac{\sum_{k\neq i}|a_{ki}|}{1+\sum_{k\neq i}|a_{ki}|} = \frac{\sum_{k\neq i}|a_{ki}|}{a_{ii}} = \frac{\sum_{k\neq i}|a_{ik}|}{a_{ii}} = d_i, \quad i \in N.$$

Since A is an M-matrix with $a_{11} = a_{22} = \cdots = a_{nn}$ and $A^{-1} = (b_{ij})$ is doubly stochastic, we have

$$d_i = d_j, \quad j \neq i, \qquad a_{ii} = \sum_{k\neq i}|a_{ik}| + 1 = \sum_{k\neq j}|a_{jk}| + 1 = a_{jj}.$$

So $r_i \leqslant d_k$, $i, k \in N$. Then, we obtain

$$s_{ji} = \frac{|a_{ji}| + \sum_{k\neq j,i}|a_{jk}|d_k}{a_{jj}} \geqslant \frac{|a_{ji}| + \sum_{k\neq j,i}|a_{jk}|r_i}{a_{jj}} = m_{ji}, \quad j \neq i.$$

So

$$s_i = \max_{j\neq i}\{s_{ij}\} \geqslant \max_{j\neq i}\{m_{ij}\} = m_i, \quad i \in N.$$

Therefore

$$a_{ii} - s_iR_i \leqslant a_{ii} - m_iR_i \quad \text{and} \quad \frac{1}{1+\sum_{j\neq i}s_{ji}} \leqslant \frac{1}{1+\sum_{j\neq i}m_{ji}}.$$

Thus, for any $i \in N$, we have

$$\min_i \left\{ \frac{a_{ii} - m_i R_i}{1 + \sum_{j \neq i} m_{ji}} \right\} \geqslant \min_i \left\{ \frac{a_{ii} - s_i R_i}{1 + \sum_{j \neq i} s_{ji}} \right\}. \qquad \square$$

Remark 3.1. Theorem 3.3 shows that the result of Theorem 3.2 is better than the result $\tau(A \circ A^{-1}) \geqslant \min_i \left\{ \frac{a_{ii} - s_i R_i}{1 + \sum_{j \neq i} s_{ji}} \right\}$ of Theorem 3.1 of [7]. So, the result of Theorem of 3.1 is improved.

Theorem 3.4. *Let $A = (a_{ij}) \in R^{n \times n}$ be an M-matrix. Then*

$$\tau(A \circ A^{-1}) \geqslant \min_i \left\{ 1 - \frac{1}{a_{ii}} \sum_{j \neq i} |a_{ji}| m_{ji} \right\}.$$

Proof. If A is an irreducible M-matrix, then A^{-1} is positive and $A \circ A^{-1}$ is again irreducible. By a result of Sinkhorn [11], there exist diagonal matrices D_1 and D_2 with positive diagonal entries such that $D_1 A^{-1} D_2$ is doubly stochastic. The matrix $B = D_2^{-1} A D_1^{-1}$ is again an M-matrix and satisfies $\tau(A \circ A^{-1}) = \tau(B \circ B^{-1})$, for $B \circ B^{-1} = (D_2^{-1} A D_1^{-1}) \circ (D_1 A^{-1} D_2) = (D_1 D_2^{-1})(A \circ A^{-1})(D_1 D_2^{-1})^{-1}$.

So, for convenience and without loss of generality, we may assume that A is irreducible and $A^{-1} = (b_{ij})$ is doubly stochastic.

Since $A^{-1} = (b_{ij})$ is doubly stochastic, then, by Lemma 2.6, for every $i \in N$,

$$a_{ii} = \sum_{j \neq i} |a_{ij}| + 1 = \sum_{j \neq i} |a_{ji}| + 1.$$

Note that

$$\tau(A \circ A^{-1}) = \tau((A \circ A^{-1})^T) = \tau(A^T \circ (A^T)^{-1}).$$

Let

$$(A^T \circ (A^T)^{-1})e = (q_1, q_2, \ldots, q_n)^T,$$

where $e = (1, 1, \ldots, 1)^T$. Without loss of generality, let $q_1 = \min_i\{q_i\}$. Then, by Lemma 2.2, we have

$$\begin{aligned} q_1 &= \sum_{j=1}^{n} a_{j1} b_{j1} = a_{11} b_{11} - \sum_{j \neq 1} |a_{j1}| b_{j1} \\ &\geqslant a_{11} b_{11} - \sum_{j \neq 1} |a_{j1}| \frac{|a_{j1}| + \sum_{k \neq j,1} |a_{jk}| r_1}{a_{jj}} b_{11} \\ &= \left(a_{11} - \sum_{j \neq 1} |a_{j1}| m_{j1} \right) b_{11} \qquad \text{(by Lemma 2.3)} \\ &\geqslant \frac{a_{11} - \sum_{j \neq 1} |a_{j1}| m_{j1}}{a_{11}} \\ &= 1 - \frac{1}{a_{11}} \sum_{j \neq 1} |a_{j1}| m_{j1}. \end{aligned}$$

Therefore, by Lemma 2.5, we have

$$\tau(A \circ A^{-1}) = \tau(A^T \circ (A^T)^{-1}) \geqslant \min_i \left\{ 1 - \frac{1}{a_{ii}} \sum_{j \neq i} |a_{ji}| m_{ji} \right\}. \qquad \square$$

Remark 3.2. From the proof of Theorem 3.3, we know that $s_{ji} \geqslant m_{ji}, j \neq i$. So, we have

$$1 - \frac{1}{a_{ii}} \sum_{j \neq i} |a_{ji}| m_{ji} \geqslant 1 - \frac{1}{a_{ii}} \sum_{j \neq i} |a_{ji}| s_{ji}.$$

This shows that the result of Theorem 3.4 is better than the result $\tau(A \circ A^{-1}) \geqslant \min_i \left\{1 - \frac{1}{a_{ii}} \sum_{j \neq i} |a_{ji}| s_{ji}\right\}$ in Theorem 3.5 of [7].

4. Example

Consider the following M-matrix

$$A = \begin{bmatrix} 4 & -1 & -1 & -1 \\ -2 & 5 & -1 & -1 \\ 0 & -2 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{bmatrix}.$$

Since $Ae = e$ and $A^T e = e$, A^{-1} is doubly stochastic. By calculations we have

$$A^{-1} = \begin{bmatrix} 0.4 & 0.2 & 0.2 & 0.2 \\ 0.2333 & 0.3667 & 0.2 & 0.2 \\ 0.1667 & 0.2333 & 0.4 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.4 \end{bmatrix}.$$

(1) Upper bounds for entries of A^{-1}. First, by Lemma 2.1 (a), we obtain

$$A^{-1} \leqslant \begin{bmatrix} 1 & 0.75 & 0.75 & 0.75 \\ 0.8 & 1 & 0.8 & 0.8 \\ 0.75 & 0.75 & 1 & 0.75 \\ 0.75 & 0.75 & 0.75 & 1 \end{bmatrix} \circ \begin{bmatrix} b_{11} & b_{22} & b_{33} & b_{44} \\ b_{11} & b_{22} & b_{33} & b_{44} \\ b_{11} & b_{22} & b_{33} & b_{44} \\ b_{11} & b_{22} & b_{33} & b_{44} \end{bmatrix}.$$

If we apply Theorem 2.1 (a) of [7], we have

$$A^{-1} \leqslant \begin{bmatrix} 1 & 0.625 & 0.6375 & 0.6375 \\ 0.7 & 1 & 0.65 & 0.65 \\ 0.5875 & 0.6875 & 1 & 0.65 \\ 0.6375 & 0.625 & 0.5 & 1 \end{bmatrix} \circ \begin{bmatrix} b_{11} & b_{22} & b_{33} & b_{44} \\ b_{11} & b_{22} & b_{33} & b_{44} \\ b_{11} & b_{22} & b_{33} & b_{44} \\ b_{11} & b_{22} & b_{33} & b_{44} \end{bmatrix}.$$

If we apply Corollary 2.5 (2.7) of [7], we have

$$A^{-1} \leqslant \begin{bmatrix} 1 & 0.6667 & 0.5 & 0.5 \\ 0.6667 & 1 & 0.5 & 0.5 \\ 0.6667 & 0.6667 & 1 & 0.5 \\ 0.6667 & 0.6667 & 0.6667 & 1 \end{bmatrix} \circ \begin{bmatrix} b_{11} & b_{22} & b_{33} & b_{44} \\ b_{11} & b_{22} & b_{33} & b_{44} \\ b_{11} & b_{22} & b_{33} & b_{44} \\ b_{11} & b_{22} & b_{33} & b_{44} \end{bmatrix}.$$

Combining Theorem 2.1 (a) of [7] and Corollary 2.5 (2.7) of [7], we have

$$A^{-1} \leqslant \begin{bmatrix} 1 & 0.625 & 0.5 & 0.5 \\ 0.6667 & 1 & 0.5 & 0.5 \\ 0.5875 & 0.6667 & 1 & 0.5 \\ 0.6375 & 0.625 & 0.5 & 1 \end{bmatrix} \circ \begin{bmatrix} b_{11} & b_{22} & b_{33} & b_{44} \\ b_{11} & b_{22} & b_{33} & b_{44} \\ b_{11} & b_{22} & b_{33} & b_{44} \\ b_{11} & b_{22} & b_{33} & b_{44} \end{bmatrix}.$$

Now if we apply Lemma 2.2 (a), we have

$$A^{-1} \leqslant \begin{bmatrix} 1 & 0.583 & 0.5 & 0.5 \\ 0.6667 & 1 & 0.5 & 0.5 \\ 0.5 & 0.6667 & 1 & 0.5 \\ 0.583 & 0.583 & 0.5 & 1 \end{bmatrix} \circ \begin{bmatrix} b_{11} & b_{22} & b_{33} & b_{44} \\ b_{11} & b_{22} & b_{33} & b_{44} \\ b_{11} & b_{22} & b_{33} & b_{44} \\ b_{11} & b_{22} & b_{33} & b_{44} \end{bmatrix}. \tag{4.1}$$

Comparing the result of Lemma 2.2 (a) with the other results, we see that the result of Lemma 2.2 (a) is the best.

Theorem 2.3 of [7] and Lemma 3.2 of [7] give the following bounds for the diagonal entries of A^{-1}:

$$0.3419 \leqslant b_{11} \leqslant 0.5882; \quad 0.3404 \leqslant b_{22} \leqslant 0.5128,$$
$$0.3419 \leqslant b_{33} \leqslant 0.6061; \quad 0.3404 \leqslant b_{44} \leqslant 0.5882.$$

If we apply Theorem 3.1 and Lemma 2.3, we obtain better bounds:

$$0.3637 \leqslant b_{11} \leqslant 0.4430; \quad 0.3530 \leqslant b_{22} \leqslant 0.3870,$$
$$0.4 \leqslant b_{33} \leqslant 0.4; \quad 0.4 \leqslant b_{44} \leqslant 0.4.$$

(2) Lower bounds for $\tau(A \circ A^{-1})$.

If we apply the conjecture of Fiedler and Markham, we have

$$\tau(A \circ A^{-1}) \geqslant \frac{2}{n} = \frac{1}{2} = 0.5.$$

If we apply Theorem 3.1 of [7] we have

$$\tau(A \circ A^{-1}) \geqslant 0.6624.$$

If we apply Theorem 9 of [8] with $A = B$, we have

$$\tau(A \circ A^{-1}) \geqslant 0.2614.$$

The bound in our Theorem 3.2 is better:

$$\tau(A \circ A^{-1}) \geqslant 0.7999.$$

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