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Extremal energies of integral circulant graphs via multiplicativity

T.A. Le, J.W. Sander *

Institut für Mathematik und Angewandte Informatik, Universität Hildesheim, D-31141 Hildesheim, Germany

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ABSTRACT

The energy of a graph is the sum of the moduli of the eigenvalues of its adjacency matrix. Integral circulant graphs can be characterised by their order n and a set $\mathcal{D}$ of positive divisors of n in such a way that they have vertex set $\mathbb{Z}/n\mathbb{Z}$ and edge set $\{(a, b) : a, b \in \mathbb{Z}/n\mathbb{Z},\ \gcd(a-b, n) \in \mathcal{D}\}$. Among integral circulant graphs of fixed prime power order p^s, those having minimal energy $\mathcal{E}_{\min}(p^s)$ or maximal energy $\mathcal{E}_{\max}(p^s)$, respectively, are known. We study the energy of integral circulant graphs of arbitrary order n with so-called multiplicative divisor sets. This leads to good bounds for $\mathcal{E}_{\min}(n)$ and $\mathcal{E}_{\max}(n)$ as well as conjectures concerning the true value of $\mathcal{E}_{\min}(n)$.

1. Introduction

The class of *integral circulant graphs*, i.e. graphs having a circulant adjacency matrix with integral eigenvalues, is an object comprising algebraic, arithmetic and combinatorial features. In 2007 Klotz and T. Sander [13] generalised the concept of unitary Cayley graphs to what they called *gcd graphs*: For a given integer $n > 1$ and a set $\mathcal{D}$ of positive divisors of n they defined the corresponding graph to have vertex set $\mathbb{Z}/n\mathbb{Z}$ and edge set $\{(a, b) : a, b \in \mathbb{Z}/n\mathbb{Z},\ \gcd(a-b, n) \in \mathcal{D}\}$, where $\mathbb{Z}/n\mathbb{Z}$ denotes the additive group of residue classes $\bmod n$. The work of So [23] implies that the class of gcd graphs can be

* Corresponding author.
E-mail address: sander@imai.uni-hildesheim.de (J.W. Sander).

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identified with the class of integral circulant graphs. Therefore, these graphs were labelled $\mathrm{ICG}(n, \mathcal{D})$ for suitable $n \in \mathbb{N}$ and $\mathcal{D} \subseteq D(n) := \{d > 0 : \ d|n\}$. Since $\mathrm{ICG}(n, \mathcal{D})$ has loops in case $n \in \mathcal{D}$, it is usually assumed that $\mathcal{D} \subseteq D^*(n) := D(n)\backslash\{n\}$. Quite a lot of interesting results on this class of graphs have been obtained in recent years (see [18] for references). In particular, the examination of the spectra of integral circulant graphs attracted a lot of attention (cf. [1,2,12,20,22]), where the *spectrum* $Spec(G)$ of a graph G is the set of eigenvalues of its adjacency matrix.

The eigenvalues of circulant matrices can be evaluated explicitly as exponential sums, i.e. sums of powers of primitive roots of unity (cf. [7, Theorem 3.2.2]). Klotz and T. Sander [13, Theorem 16], deduced from this that the eigenvalues $\lambda_k(n, \mathcal{D})$ $(1 \leqslant k \leqslant n)$ of the integral circulant graph $\mathrm{ICG}(n, \mathcal{D})$ are given by

$$\lambda_k(n, \mathcal{D}) = \sum_{d \in \mathcal{D}} c\left(k, \frac{n}{d}\right) \quad (1 \leqslant k \leqslant n), \tag{1}$$

where

$$c(k, n) := \sum_{\substack{j \bmod n \\ (j,n)=1}} \exp\left(\frac{2\pi i\, kj}{n}\right)$$

is the well-known Ramanujan sum (cf. [3, Chapter 8.3–8.4]). Recently, the authors [14] observed that

$$\lambda_k(n, \mathcal{D}) = (\mathbb{1} *_{\mathcal{D}} c(k, \cdot))(n) \quad (1 \leqslant k \leqslant n), \tag{2}$$

where $\mathbb{1}$ is the constant function and $*_{\mathcal{D}}$ denotes the so-called $\mathcal{D}$-*convolution* of arithmetic functions introduced by Narkiewicz [17], which is a generalisation of the classical Dirichlet convolution. This representation of the eigenvalues will be of great importance for our purpose.

In general, given non-empty sets $A(n) \subseteq D(n)$ for all positive integers n, the *(arithmetical) convolution A* or the *A-convolution* of two arithmetic functions $f, g \in \mathbb{C}^{\mathbb{N}}$ is defined as

$$(f *_A g)(n) = \sum_{d \in A(n)} f(d) g\left(\frac{n}{d}\right).$$

All convolutions considered in the literature are required to be *regular*, a property which basically guarantees that $f *_A g$ is multiplicative for multiplicative functions f and g, and the inverse of $\mathbb{1}$ with respect to $*_A$, i.e. an analogue of the Möbius function μ exists. Narkiewicz [17] proved that regularity of an A-convolution is, besides some minor technical requirements, essentially equivalent with the following two conditions:

(α) A is *multiplicative*, i.e. $A(mn) = A(m)A(n) := \{ab : \ a \in A(m), b \in A(n)\}$ for all coprime $m, n \in \mathbb{N}$.
(β) A is *semi-regular*, i.e. for every prime power p^s with $s \geqslant 1$ there exists a divisor $t = t_A(p^s)$ of s, called the *type* of p^s, such that $A(p^s) = \{1, p^t, p^{2t}, \ldots, p^{jt}\}$ with $j = \frac{s}{t}$.

Both of these properties will occur in a natural fashion along our way, but for our purposes concerning the eigenvalues of $\mathrm{ICG}(n, \mathcal{D})$ the multiplicativity of the divisor sets $\mathcal{D}$ will be the guiding feature.

In 1978 Gutman [10] established the mathematical concept of the *energy*

$$E(G) := \sum_{\lambda \in Spec(G)} |\lambda|$$

of a graph G, but it is rooted in chemistry way back in the 1930s (see [15] for connections between *Hückel molecular orbital theory* and graph spectral analysis, and [5] for a mathematical survey). We abbreviate the energy of an integral circulant graph by setting

$$\mathcal{E}(n, \mathcal{D}) := E(\mathrm{ICG}(n, \mathcal{D})) = \sum_{\lambda \in Spec(\mathrm{ICG}(n,\mathcal{D}))} |\lambda| = \sum_{k=1}^{n} |\lambda_k(n, \mathcal{D})|. \tag{3}$$

It is of particular interest to determine for any fixed positive integer n the extremal energies

$$\mathcal{E}_{\min}(n) := \min\{\mathcal{E}(n, \mathcal{D}) : \ \mathcal{D} \subseteq D^*(n)\}$$

and

$$\mathcal{E}_{\max}(n) := \max\{\mathcal{E}(n, \mathcal{D}) : \ \mathcal{D} \subseteq D^*(n)\}$$

as well as the divisor sets producing these energies. For prime powers $n = p^s$ this problem was completely settled by the second author and T. Sander in [18, Theorem 3.1] and [19, Theorem 1.1], respectively. The basis of these results was an explicit formula evaluating $\mathcal{E}(p^s, \mathcal{D})$ (cf. [18, Theorem 2.1]). Due to the lack of a comparable energy formula for arbitrary n, it seems much more difficult to deal with $\mathcal{E}_{\min}(n)$ and $\mathcal{E}_{\max}(n)$ in general. So far a crucial obstacle was that the energy revealed no signs of multiplicativity with respect to n. In this note we shall overcome that deficiency by using the theory of *multiplicative divisor sets* [14], which is closely linked with the eigenvalue representation in (2). This will at least provide us with good bounds for $\mathcal{E}_{\min}(n)$ and $\mathcal{E}_{\max}(n)$ as well as divisor sets producing the corresponding energies. In case of the minimal energy, we even conjecture to have found $\mathcal{E}_{\min}(n)$ and its associated divisor sets for all n.

The final section of this article addresses perspectives on further research, providing open problems and conjectures. As a motivation for the reader, we prove a simple formula for the energy of some integral circulant graphs whose divisor sets are not multiplicative (cf. Theorem 6.1).

2. Terminology and statement of results

The product of non-empty sets $A_1, \ldots, A_t$ of integers is defined as

$$\prod_{i=1}^{t} A_i := \{a_1 \cdot \ldots \cdot a_t : \ a_i \in A_i \quad (1 \leqslant i \leqslant t)\}.$$

For infinitely many such sets $A_1, A_2, \ldots$, we require that $A_i = \{1\}$ for all but finitely many i and define

$$\prod_{i=1}^{\infty} A_i := \prod_{\substack{i=1 \\ A_i \neq \{1\}}}^{\infty} A_i.$$

Let us call a set $\mathcal{A}$ of positive integers a *multiplicative set* if it is the product of non-empty finite sets $A_i \subset \{1, p_i, p_i^2, p_i^3, \ldots\}$, $1 \leqslant i \leqslant t$ say, with pairwise distinct primes $p_1, \ldots, p_t$. In other words, a given set $\mathcal{A}$ is multiplicative if and only if $\mathcal{A} = \prod_{p \in \mathbb{P}} \mathcal{A}_p$, where $\mathcal{A}_p := \{p^{e_p(a)} : \ a \in \mathcal{A}\}$ for each prime p, and $e_p(a)$ denotes the order of the prime p in a. Observe that $\mathcal{A}_p \neq \{1\}$ only for those finitely many primes dividing at least one of the $a \in \mathcal{A}$.

By use of (2), the authors proved in [14, Theorem 4.2] that given a multiplicative divisor set $\mathcal{D} \subseteq D(n)$ we have for the energy of $\mathrm{ICG}(n, \mathcal{D})$ as defined in (3)

$$\mathcal{E}(n, \mathcal{D}) = \prod_{p \in \mathbb{P},\, p | n} \mathcal{E}(p^{e_p(n)}, \mathcal{D}_p). \tag{4}$$

We shall investigate minimal and maximal energies of integral circulant graphs with respect to multiplicative divisor sets. To this end, we define

$$\tilde{\mathcal{E}}_{\min}(n) := \min\{\mathcal{E}(n, \mathcal{D}) : \ \mathcal{D} \subseteq D^*(n) \text{ multiplicative}\}$$

and

$$\tilde{\mathcal{E}}_{\max}(n) := \max\{\mathcal{E}(n, \mathcal{D}) : \ \mathcal{D} \subseteq D^*(n) \text{ multiplicative}\}.$$

Clearly,

$$\mathcal{E}_{\min}(n) \leqslant \tilde{\mathcal{E}}_{\min}(n) \leqslant \tilde{\mathcal{E}}_{\max}(n) \leqslant \mathcal{E}_{\max}(n) \tag{5}$$

for all positive integers n. Moreover, for prime powers p^s any divisor set $\mathcal{D} \subseteq D(p^s)$ is trivially multiplicative, hence $\tilde{\mathcal{E}}_{\min}(p^s) = \mathcal{E}_{\min}(p^s)$ and $\tilde{\mathcal{E}}_{\max}(p^s) = \mathcal{E}_{\max}(p^s)$.

Theorem 2.1. *Let $n \geqslant 2$ be an integer with prime factorisation $n = p_1^{s_1} \cdot \ldots \cdot p_t^{s_t}$. Then*

$$\tilde{\mathcal{E}}_{max}(n) = \prod_{i=1}^{t} \theta(p_i^{s_i}),$$

where for any prime power p^s

$$\theta(p^s) := \begin{cases} \frac{1}{(p+1)^2}\Big((s+1)(p^2-1)p^s + 2(p^{s+1}-1)\Big) & \text{if } 2 \nmid s, \\ \frac{1}{(p+1)^2}\Big(s(p^2-1)p^s + 2(2p^{s+1} - p^{s-1} + p^2 - p - 1)\Big) & \text{if } 2|s. \end{cases}$$

Moreover, for multiplicative sets $\mathcal{D} \subseteq D^*(n)$, we have $\mathcal{E}(n, \mathcal{D}) = \tilde{\mathcal{E}}_{max}(n)$ if and only if $\mathcal{D} = \prod_{i=1}^{t} \mathcal{D}^{(i)}$ with

$$\mathcal{D}^{(i)} = \begin{cases} \{1, p_i^2, p_i^4, \ldots, p_i^{s_i-3}, p_i^{s_i-1}\} & \text{if } 2 \nmid s_i, \ p_i \geqslant 3, \\ \{1, 2^2, 2^4, \ldots, 2^{s_i-3}, 2^{s_i-1}\} \text{ or } \{1, 2, 2^3, \ldots, 2^{s_i-4}, 2^{s_i-2}, 2^{s_i-1}\} & \text{if } 2 \nmid s_i, \ p_i = 2, \\ \{1, p_i^2, p_i^4, \ldots, p_i^{s_i-4}, p_i^{s_i-2}, p_i^{s_i-1}\} \text{ or } \{1, p_i, p_i^3, \ldots, p_i^{s_i-3}, p_i^{s_i-1}\} & \text{if } 2|s_i. \end{cases}$$

It should be observed that the maximising factor divisor sets $\mathcal{D}^{(i)}$ are (almost) semi-regular (cf. (β) in Section 1).

In order to describe the multiplicative divisor sets minimising the energy, let us agree to call $\mathcal{D} \subseteq D(p^s)$ for a prime power p^s *uni-regular*, if $\mathcal{D} = \{p^u, p^{u+1}, \ldots, p^{v-1}, p^v\}$ for some integers $0 \leqslant u \leqslant v \leqslant s$, which is a special case of semi-regularity as defined in (β) if $u = 0$ and $v = s$.

Theorem 2.2. *Let $n \geqslant 2$ be an integer with prime factorisation $n = p_1^{s_1} \cdot \ldots \cdot p_t^{s_t}$ and $p_1 < p_2 < \cdots < p_t$. Then*

$$\tilde{\mathcal{E}}_{min}(n) = 2n\left(1 - \frac{1}{p_1}\right).$$

Moreover, for multiplicative sets $\mathcal{D} \subseteq D^*(n)$, we have $\mathcal{E}(n, \mathcal{D}) = \tilde{\mathcal{E}}_{min}(n)$ if and only if $\mathcal{D} = \prod_{i=1}^{t} \mathcal{D}^{(i)}$ with $\mathcal{D}^{(1)} = \{p_1^u\}$ for some $u \in \{0, 1, \ldots, s_1 - 1\}$ and arbitrary uni-regular sets $\mathcal{D}^{(i)}$ with $p_i^{s_i} \in \mathcal{D}^{(i)}$ for $2 \leqslant i \leqslant t$.

Remarks 2.1.

(i) It was observed by So [23] that $\mathrm{ICG}(n, \mathcal{D})$ is connected if and only if the elements of $\mathcal{D}$ are coprime. Assuming connectivity in Theorem 2.2, minimising sets $\mathcal{D} = \prod_{i=1}^{t} \mathcal{D}^{(i)}$ then necessarily have $\mathcal{D}^{(1)} = \{1\}$ and $\mathcal{D}^{(i)} = \{1, p_i, p_i^2, \ldots, p_i^{s_i}\}$ for $2 \leqslant i \leqslant t$, i.e. all $\mathcal{D}^{(i)}$ with $2 \leqslant i \leqslant t$ have to be semi-regular sets of type 1 (cf. (β) in Section 1).

(ii) The proof of Theorem 2.2 reveals that for $\mathcal{D} \subseteq D(n)$, i.e. possibly $n \in \mathcal{D}$, we would have $\tilde{\mathcal{E}}_{\min}(n) = n$ with minimising sets $\mathcal{D} = \prod_{i=1}^{t} \mathcal{D}^{(i)}$, where each $\mathcal{D}^{(i)}$ is an arbitrary uni-regular set containing $p_i^{s_i}$.

(iii) One could prove Theorem 2.2 by referring to the energy formula for $\mathrm{ICG}(p^s, \mathcal{D})$ containing loops (cf. [14, Proposition 5.1]). Instead we shall take a closer look at the second largest $|\lambda|$ with $\lambda \in \mathit{Spec}(\mathrm{ICG}(n, \mathcal{D}))$. This provides more insight into the underlying structure.

As a consequence of (5), we immediately obtain from Theorems 2.1 and 2.2 the desired bounds for the extremal energies of integral circulant graphs with arbitrary divisor sets.

Corollary 2.1. *Let $n \geqslant 2$ be an integer with prime factorisation $n = p_1^{s_1} \cdot \ldots \cdot p_t^{s_t}$ and $p_1 < p_2 < \cdots < p_t$. Then*

(i) $\mathcal{E}_{max}(n) \geqslant \tilde{\mathcal{E}}_{max}(n) = \prod_{i=1}^{t} \theta(p_i^{s_i})$;

(ii) $\mathcal{E}_{min}(n) \leqslant \tilde{\mathcal{E}}_{min}(n) = 2n\left(1 - \frac{1}{p_1}\right)$.

Examples show that, while equality between $\mathcal{E}_{\max}(n)$ and $\tilde{\mathcal{E}}_{\max}(n)$ does occur occasionally, we usually have $\mathcal{E}_{\max}(n) > \tilde{\mathcal{E}}_{\max}(n)$ (cf. Example 3.1). This phenomenon is due to the fact that maximising divisor sets normally are not multiplicative. Yet $\tilde{\mathcal{E}}_{\max}(n)$ falls short of $\mathcal{E}_{\max}(n)$ by less than a comparatively small factor. To state this result, we denote by φ Euler's totient function and by $\tau(n)$ the number of positive divisors of n, while $\omega(n)$ is the number of distinct prime factors of n. As before, $e_p(n)$ denotes the order of the prime p in n.

Theorem 2.3. *Let n be a positive integer. Then*

$$\text{(i)}\quad \mathcal{E}_{max}(n) \leqslant n \sum_{d|n} \frac{\varphi(d)\tau(d)}{d} = n \prod_{p \in \mathbb{P},\, p|n} \left(\tfrac{1}{2}\left(1 - \tfrac{1}{p}\right)(e_p(n)+1)(e_p(n)+2) + \tfrac{1}{p}\right);$$

$$\text{(ii)}\quad \mathcal{E}_{max}(n) < \left(\tfrac{3}{4}\right)^{\omega(n)} n\, \tau(n)^2;$$

$$\text{(iii)}\quad \mathcal{E}_{max}(n) \leqslant \tilde{\mathcal{E}}_{max}(n)\, \tau(n).$$

The proof of Theorem 2.3 (ii) will show that

$$\left(\tfrac{1}{4}\right)^{\omega(n)} \tau(n)^2 < \sum_{d|n} \frac{\varphi(d)\tau(d)}{d} < \left(\tfrac{3}{4}\right)^{\omega(n)} \tau(n)^2, \tag{6}$$

and (ii) is deduced from (i) by use of the upper bound in (6). We like to point out that the constants $\frac{1}{4}$ and $\frac{3}{4}$ in the lower and upper bound of (6), respectively, cannot be improved for all n, although we expect $\sum_{d|n} \frac{\varphi(d)\tau(d)}{d} \approx \left(\frac{1}{2}\right)^{\omega(n)} \tau(n)^2$ for most n. Yet, each of the bounds is sharp for integers with certain arithmetic properties (cf. Remark 3.1).

As far as the magnitude of $\mathcal{E}_{\max}(n)$ is concerned, it is well known that $\tau(n) = O(n^{\varepsilon})$ for any real $\varepsilon > 0$. In fact the true maximal order of $\tau(n)$ is approximately $n^{\frac{\log 2}{\log\log n}}$. On average, $\tau(n)$ is of order $\log n$, but for almost all n it is considerably smaller, because the normal order of $\tau(n)$ is roughly $(\log n)^{\log 2} \approx (\log n)^{0.693}$. For all these results as well as for average and normal order of $\omega(n)$ the reader is referred to [11, §§ 18.1–18.2, § 22.13 and § 22.11].

As opposed to sets maximising the energy of integral circulant graphs, numerical calculations suggest that divisor sets minimising the energy are always multiplicative. According to that, equality in Corollary 2.1(ii) should hold for all n. We shall specify this observation in two conjectures at the end of the paper.

3. Maximal energies for multiplicative divisor sets

In [14] the authors restricted the study of integral circulant graphs to multiplicative divisor sets. By use of this concept, it is quite easy to determine $\tilde{\mathcal{E}}_{\max}(n)$.

Proof of Theorem 2.1. Let $n = p_1^{s_1} \cdot \ldots \cdot p_t^{s_t}$ be fixed, and let $\mathcal{D} \subseteq D^*(n)$ be a multiplicative set satisfying $\mathcal{E}(n, \mathcal{D}) = \tilde{\mathcal{E}}_{\max}(n)$. By (4) (cf. [14, Theorem 4.2]), it follows that

$$\tilde{\mathcal{E}}_{\max}(n) = \mathcal{E}(n, \mathcal{D}) = \prod_{p \in \mathbb{P},\, p|n} \mathcal{E}(p^{e_p(n)}, \mathcal{D}_p) = \prod_{i=1}^{t} \mathcal{E}(p_i^{s_i}, \mathcal{D}_{p_i}).$$

This implies that $\mathcal{E}(p^{s_i}, \mathcal{D}_{p_i}) = \mathcal{E}_{\max}(p_i^{s_i})$ for $1 \leqslant i \leqslant t$. From [19, Theorem 1.1] we conclude that $\mathcal{E}_{\max}(p_i^{s_i}) = \theta(p_i^{s_i})$ for all i, and the only corresponding divisor sets $\mathcal{D}^{(i)}$ are just the ones listed in our assertion. $\square$

While equality between $\mathcal{E}_{\max}(n)$ and $\tilde{\mathcal{E}}_{\max}(n)$ does occur occasionally, we usually have $\mathcal{E}_{\max}(n) > \tilde{\mathcal{E}}_{\max}(n)$. This is illustrated by:

Example 3.1. We have $\theta(p) = 2(p-1)$ for each prime p. By use of Theorem 2.1, we easily obtain $\tilde{\mathcal{E}}_{\max}(6) = \theta(2) \cdot \theta(3) = 8$, $\tilde{\mathcal{E}}_{\max}(105) = \theta(3) \cdot \theta(5) \cdot \theta(7) = 384$ and $\tilde{\mathcal{E}}_{\max}(21) = \theta(3) \cdot \theta(7) = 48$. Numerical evaluation of (3) yields $\mathcal{E}_{\max}(6) = 10$, $\mathcal{E}_{\max}(105) = 520$, but $\mathcal{E}_{\max}(21) = 48 = \tilde{\mathcal{E}}_{\max}(21)$.

In order to be able to compare $\mathcal{E}_{\max}(n)$ and $\tilde{\mathcal{E}}_{\max}(n)$ and finally prove Theorem 2.3 completely, we start by establishing an upper bound for $\mathcal{E}_{\max}(n)$.

Proof of Theorem 2.3 (i) and (ii). (i) By (3), (2) and the well-known Hölder identity for Ramanujan sums (cf. [3, Chapter 8.3–8.4]), we have for any n and any divisor set $\mathcal{D} \subseteq D(n)$

$$\mathcal{E}(n, \mathcal{D}) = \sum_{k=1}^{n} \left| \sum_{d \in \mathcal{D}} \mu\left(\frac{n}{(n, kd)}\right) \frac{\varphi(\frac{n}{d})}{\varphi\left(\frac{n}{(n,kd)}\right)} \right|,$$

where μ denotes the Möbius function. Let $n/\mathcal{D} := \{\frac{n}{d} : d \in \mathcal{D}\} \subseteq D(n)$ be the set of complementary divisors of all $d \in \mathcal{D}$ with respect to n. Then

$$\begin{aligned}\mathcal{E}(n, \mathcal{D}) &= \sum_{k=1}^{n} \left| \sum_{d \in n/\mathcal{D}} \mu\left(\frac{d}{(d, k)}\right) \frac{\varphi(d)}{\varphi\left(\frac{d}{(d,k)}\right)} \right| \\ &\leqslant \sum_{k=1}^{n} \sum_{d \in n/\mathcal{D}} \frac{\varphi(d)}{\varphi\left(\frac{d}{(d,k)}\right)} = \sum_{d \in n/\mathcal{D}} \varphi(d) \sum_{g|d} \frac{1}{\varphi(\frac{d}{g})} \sum_{\substack{k=1 \\ (k,d)=g}}^{n} 1\,.\end{aligned}$$

Since

$$\sum_{\substack{k=1 \\ (k,d)=g}}^{n} 1 = \frac{n}{d} \sum_{\substack{r=1 \\ (r,d)=g}}^{d} 1 = \frac{n}{d} \sum_{\substack{r=1 \\ (r,\frac{d}{g})=1}}^{\frac{d}{g}} 1 = \frac{n}{d} \cdot \varphi\left(\frac{d}{g}\right),$$

we conclude that

$$\mathcal{E}(n, \mathcal{D}) \leqslant n \sum_{d \in n/\mathcal{D}} \frac{\varphi(d)\tau(d)}{d} \leqslant n \sum_{d|n} \frac{\varphi(d)\tau(d)}{d} .$$

Since this holds for all n, the inequality in (i) follows. By the fact that φ, τ and the identity function id are all multiplicative, this property is carried over to $\frac{\varphi \cdot \tau}{id}$ and its summatory function

$$f(n) := \sum_{d|n} \frac{\varphi(d)\tau(d)}{d} \qquad (n \in \mathbb{N}). \tag{7}$$

Given a prime power p^s, we obviously have

$$\begin{aligned} f(p^s) &= \sum_{j=0}^{s} \frac{\varphi(p^j) \cdot \tau(p^j)}{p^j} = 1 + \sum_{j=1}^{s} \frac{(p^j - p^{j-1})(j+1)}{p^j} \\ &= \frac{1}{2}\left(1 - \frac{1}{p}\right)(s+1)(s+2) + \frac{1}{p}. \end{aligned} \tag{8}$$

This completes the proof of (i).

(ii) With regard to (i) it suffices to prove the upper bound in (6), but in order to justify the remark following Theorem 2.3 we shall verify the lower bound as well. By (8), we have for all primes p and all positive integers s that

$$\frac{1}{4} \leqslant \frac{1}{2}\left(1 - \frac{1}{p}\right) < \frac{f(p^s)}{(s+1)^2} \leqslant \frac{3}{4}\left(1 - \frac{2}{3p}\right) < \frac{3}{4}, \tag{9}$$

thus $\frac{1}{4}\tau(p^s)^2 < f(p^s) < \frac{3}{4}\tau(p^s)^2$. By the multiplicativity of $\tau(n)$ and the additivity of $\omega(n)$, which implies the multiplicativity of $c^{\omega(n)}$ for any positive constant c, this proves (6). $\square$

Remark 3.1. It is easy to see that $\frac{f(p^s)}{(s+1)^2}$ comes close to the lower bound $\frac{1}{4}$ in (9) for $p = 2$ and large s and close to the upper bound $\frac{3}{4}$ for large p and $s = 1$. Hence the lower bound in (6) is approximated for integers n that are high powers of 2, while the upper bound is approached for squarefree integers n having large prime factors.

Proposition 3.1. *Let p be a prime, and let s be a positive integer. Then*

$$g(p^s) := \frac{p^s f(p^s)}{\tilde{\mathcal{E}}_{max}(p^s)} \leqslant \begin{cases} \frac{p+1}{2p}(s+2) & \text{for } 2 \nmid s, \\ \frac{p+1}{2p} \cdot \frac{(s+1)(s+2)}{s} & \text{for } 2|s \end{cases}$$

for the function f defined in (7). More precisely, we have in particular $g(2) = 2$, $g(p^2) \leqslant 3$ and $g(2^4) = \frac{64}{17}$.

Proof. By [19, Theorem 1.1] we know that $\tilde{\mathcal{E}}_{\max}(p^s) = \mathcal{E}_{\max}(p^s) = \theta(p^s)$ as defined in Theorem 2.1.

Case 1: $2 \nmid s$.
By use of (8) we obtain

$$g(p^s) \leqslant \frac{\left(\frac{1}{2}\left(1-\frac{1}{p}\right)(s+1)(s+2)+\frac{1}{p}\right)(p+1)^2}{(s+1)(p^2-1)+2p-\frac{2}{p^s}}$$
$$\leqslant \frac{\left(1-\frac{1}{p}\right)(s+1)(s+2)(p+1)^2}{2(s+1)(p^2-1)} = \frac{p+1}{2p}(s+2).$$

Observe that the second inequality is an identity for $p = 2$ and $s = 1$.
Case 2: $2|s$.
Applying again (8), we conclude

$$g(p^s) \leqslant \frac{\left(\frac{1}{2}\left(1-\frac{1}{p}\right)(s+1)(s+2)+\frac{1}{p}\right)(p+1)^2}{s(p^2-1)+4p-\frac{2}{p}+\frac{2}{p^{s-2}}-\frac{2}{p^{s-1}}-\frac{2}{p^s}}$$
$$\leqslant \frac{\left(1-\frac{1}{p}\right)(s+1)(s+2)(p+1)^2}{2s(p^2-1)} = \frac{p+1}{2p} \cdot \frac{(s+1)(s+2)}{s}.$$

The special values for $g(p^s)$ are the results of straightforward computations. □

Corollary 3.1. *Let n be a positive integer with prime factorisation $n = p_1^{s_1} \cdot \ldots \cdot p_t^{s_t}$. Then we have*

$$\frac{\mathcal{E}_{max}(n)}{\tilde{\mathcal{E}}_{max}(n)} \leqslant \tau(n) \prod_{i=1}^{t} \frac{p_i+1}{2p_i} \prod_{\substack{i=1 \\ 2\nmid s_i}}^{t} \left(1+\frac{1}{s_i+1}\right) \prod_{\substack{i=1 \\ 2|s_i}}^{t} \left(1+\frac{2}{s_i}\right).$$

Proof. We have by Theorem 2.3 (i), Theorem 2.1 and Proposition 3.1 that

$$\begin{aligned} \frac{\mathcal{E}_{\max}(n)}{\tilde{\mathcal{E}}_{\max}(n)} &\leqslant \frac{nf(n)}{\tilde{\mathcal{E}}_{\max}(n)} = \prod_{i=1}^{t} g(p_i^{s_i}) \\ &\leqslant \prod_{i=1}^{t} \frac{p_i+1}{2p_i} \prod_{\substack{i=1 \\ 2\nmid s_i}}^{t} (s_i+2) \prod_{\substack{i=1 \\ 2|s_i}}^{t} \frac{(s_i+1)(s_i+2)}{s_i}. \end{aligned} \tag{10}$$

Since $\tau(n) = \prod_{i=1}^{t}(s_i+1)$, the corollary follows. □

Proof of Theorem 2.3 (iii). We cannot use Corollary 3.1 directly, but by (10) we know that

$$\frac{\mathcal{E}_{\max}(n)}{\tilde{\mathcal{E}}_{\max}(n)} \leqslant \prod_{p\in\mathbb{P},\, p|n} g(p^{e_p(n)}).$$

Since $\tau(n) = \prod_{p\in\mathbb{P},\, p|n}(e_p(n)+1)$, it suffices to show that $g(p^s) \leqslant s+1$ for any prime power p^s. This will be verified by use of the different bounds obtained in Proposition 3.1.
Case 1: $2 \nmid s$.
We have $g(p^s) \leqslant \frac{p+1}{2p}(s+2) \leqslant s+1$ for all p and s except for $p = 2, s = 1$, but $g(2) = 2$ closes the gap.
Case 2: $s = 2$.
This case is settled by the fact that $g(p^2) \leqslant 3$.

Case 3: $2|s$ and $s \geqslant 4$.
Here we have $g(p^s) \leqslant \frac{p+1}{2p} \cdot \frac{(s+1)(s+2)}{s} \leqslant s+1$ for all p and s except for $p=2$, $s=4$, but we know that $g(2^4) = \frac{64}{17} \leqslant 5$. $\square$

4. Spectral properties of integral circulant graphs

Before we prove some more intriguing results on eigenvalues of integral circulant graphs, which will be applied in the subsequent section, let us state without proof some more or less obvious facts.

- An integral circulant graph $\mathrm{ICG}(n, \mathcal{D})$ is apparently *regular*, more precisely $\Phi(n, \mathcal{D})$-regular, where

$$\Phi(n, \mathcal{D}) := (\mathbb{1} *_{\mathcal{D}} \varphi)(n) = \sum_{d \in \mathcal{D}} \varphi\left(\frac{n}{d}\right) \tag{11}$$

  is the nth eigenvalue $\lambda_n(n, \mathcal{D})$ of $\mathrm{ICG}(n, \mathcal{D})$ as defined in (1). This can easily be deduced from the Perron–Frobenius theorem (cf. [9, Chapter 8.8]) and the fact that $c(n, n) = \varphi(n)$.
- By the observation of So [23] that $\mathrm{ICG}(n, \mathcal{D})$ with $\mathcal{D} = \{d_1, \ldots, d_r\}$, say, is connected if and only if $\gcd(d_1, \ldots, d_r) = 1$, connectivity can readily be checked. If $\mathrm{ICG}(n, \mathcal{D})$ is connected, then $\Phi(n, \mathcal{D})$ is the largest eigenvalue of $\mathrm{ICG}(n, \mathcal{D})$, the so-called *spectral radius* of the graph, and it occurs with multiplicity 1 (cf. [4, Proposition 3.1]).
- Given a prime power $p^s \neq 2$ and any divisor set $\mathcal{D} \subseteq D(p^s)$, it is easily seen that $-\Phi(p^s, \mathcal{D})$ is not an eigenvalue of $\mathrm{ICG}(p^s, \mathcal{D})$. Consequently we conclude by [9, Theorem 8.8.2] that $\mathrm{ICG}(p^s, \mathcal{D})$ is not bipartite and hence the spectrum of $\mathrm{ICG}(p^s, \mathcal{D})$ is not symmetric about 0.

Besides multiplicativity, our proofs concerning $\mathcal{E}_{\min}(n)$ will be based on knowledge about the second largest modulus of the eigenvalues of $\mathrm{ICG}(n, \mathcal{D})$ (cf. Remark 2.1 (iii)), i.e. about

$$\Lambda(n, \mathcal{D}) := \max\{|\lambda| : \ \lambda \in Spec(\mathrm{ICG}(n, \mathcal{D})), \ |\lambda| < \Phi(n, \mathcal{D})\}, \tag{12}$$

which we only define if $\mathrm{ICG}(n, \mathcal{D})$ has eigenvalues differing in modulus. It will be crucial for us to gather some facts about $\Lambda(n, \mathcal{D})$. The first step is

Lemma 4.1. *Let n be a positive integer and $\mathcal{D} \subseteq D(n)$ with $n \in \mathcal{D}$.*

(i) Then $\mathcal{E}(n, D) \geqslant n$.
(ii) $\mathrm{ICG}(n, \mathcal{D})$ has a negative eigenvalue if and only if $\mathcal{E}(n, \mathcal{D}) > n$.

Proof. Thanks to linear algebra we know that $\sum_{k=1}^{n} \lambda_k(n, \mathcal{D})$ equals the trace of the adjacency matrix of $\mathrm{ICG}(n, \mathcal{D})$. Since $n \in \mathcal{D}$ by hypothesis, $\mathrm{ICG}(n, \mathcal{D})$ has a loop at every vertex, i.e. all diagonal entries of its adjacency matrix are 1. Hence $\sum_{k=1}^{n} \lambda_k(n, \mathcal{D}) = n$, and consequently

$$\mathcal{E}(n, \mathcal{D}) = \sum_{k=1}^{n} |\lambda_k(n, \mathcal{D})| \geqslant \Big| \sum_{k=1}^{n} \lambda_k(n, \mathcal{D}) \Big| = n, \tag{13}$$

which proves (i). Equality in (13) only holds if all $\lambda_k(n, \mathcal{D})$ have the same sign. We know that $\lambda_n(n, \mathcal{D}) = \Phi(n, \mathcal{D}) > 0$, and this implies (ii). $\square$

Lemma 4.2. *Let p^s be a prime power and $\mathcal{D} = \{p^{a_1}, p^{a_2}, \ldots, p^{a_{r-1}}, p^{a_r}\}$ with integers $0 \leqslant a_1 < a_2 < \cdots < a_{r-1} < a_r \leqslant s$. Then*

$$\lambda_k(p^s, \mathcal{D}) = \sum_{\substack{i=1 \\ a_i \geqslant s-j}}^{r} \varphi(p^{s-a_i}) - \sum_{\substack{i=1 \\ a_i = s-j-1}}^{r} p^{s-a_i-1} \tag{14}$$

for $k \in \{1, 2, \ldots, p^s\}$, where $j := e_p(k)$.

Proof. We have for $k \in \{1, 2, \ldots, p^s\}$ that

$$\lambda_k(p^s, \mathcal{D}) = \sum_{d \in \mathcal{D}} c\left(k, \frac{p^s}{d}\right) = \sum_{i=1}^{r} c(k, p^{s-a_i}). \tag{15}$$

We use two well-known properties of Ramanujan sums (cf. [3] or [21]), namely $c(k, n) = c(\gcd(k, n), n)$ for all k and n, and

$$c(p^u, p^v) = \begin{cases} \varphi(p^v) & \text{if } u \geqslant v, \\ -p^{v-1} & \text{if } u = v-1, \\ 0 & \text{if } u \leqslant v-2 \end{cases}$$

for primes p and non-negative integers u and v. On setting $m := \frac{k}{p^j}$, i.e. $k = p^j m$ with $0 \leqslant j \leqslant s$ and $m \geqslant 1,\ p \nmid m$, it follows that

$$c(k, p^{s-a_i}) = c(p^j m, p^{s-a_i}) = c(p^{\min\{j, s-a_i\}}, p^{s-a_i}) = \begin{cases} \varphi(p^{s-a_i}) & \text{if } j \geqslant s-a_i, \\ -p^{s-a_i-1} & \text{if } j = s-a_i-1, \\ 0 & \text{if } j \leqslant s-a_i-2. \end{cases}$$

Inserting this into (15), we obtain (14). $\square$

Proposition 4.1. *Let p^s be a prime power and $\mathcal{D} \subseteq D(p^s)$ with $p^s \in \mathcal{D}$, and set $r := |\mathcal{D}|$.*

(i) For $r = 1$, i.e. $\mathcal{D} = \{p^s\}$, we have $\mathrm{Spec}(ICG(p^s, \mathcal{D})) = \{1\}$.
(ii) Let $r \geqslant 2$. Then $\mathcal{D}$ is uni-regular if and only if $\Lambda(p^s, \mathcal{D}) = 0$. In this case the maximal eigenvalue $\Phi(p^s, \mathcal{D}) = p^{s-a_1} = p^{r-1}$ has multiplicity p^{a_1} where p^{a_1} is the smallest element in $\mathcal{D}$.
(iii) If $r \geqslant 2$ and $\mathcal{D}$ is not uni-regular, then $\mathcal{E}(p^s, \mathcal{D}) > p^s$.

Proof. Let $\mathcal{D} = \{p^{a_1}, p^{a_2}, \ldots, p^{a_{r-1}}, p^{a_r}\}$ with $0 \leqslant a_1 < a_2 < \cdots < a_{r-1} < a_r = s$. Hence in case $r = 1$, that is $\mathcal{D} = \{p^s\}$, we trivially have $\lambda_k(p^s, \mathcal{D}) = c(k, 1) = 1$ for all k, which proves (i).

As from now we assume $r \geqslant 2$. On setting $j := e_p(k)$, we apply Lemma 4.2 and distinguish two cases.

Case 1: $s - a_\ell \leqslant j \leqslant s - a_{\ell-1} - 2$ for some $1 \leqslant \ell \leqslant r$, where $a_0 := -2$.
Using the notation $\mathcal{D}(x) := \{d \in \mathcal{D} :\ d \geqslant x\}$, we obtain from Lemma 4.2 that

$$\lambda_k(p^s, \mathcal{D}) = \sum_{\substack{i=1 \\ a_i \geqslant a_\ell}}^{r} \varphi(p^{s-a_i}) = \Phi(p^s, \mathcal{D}(p^{a_\ell})) \tag{16}$$

with $\Phi(n, \mathcal{D})$ as defined in (11).

Case 2: $j = s - a_\ell - 1$ for some $1 \leqslant \ell \leqslant r - 1$.
Now Lemma 4.2 yields

$$\lambda_k(p^s, \mathcal{D}) = \sum_{\substack{i=1 \\ a_i \geqslant a_\ell + 1}}^{r} \varphi(p^{s-a_i}) - p^{s-a_\ell-1} = \Phi(p^s, \mathcal{D}(p^{a_{\ell+1}})) - p^{s-a_\ell-1}. \tag{17}$$

In order to prove (ii), we first assume that $\mathcal{D} = \{p^{s-r+1}, p^{s-r+2}, \ldots, p^{s-1}, p^s\}$ is uni-regular. We observe that for $a_\ell = a_{\ell-1} + 1$ $(2 \leqslant \ell \leqslant s)$ the corresponding interval considered in Case 1 is empty. Hence Case 1 occurs only if $\ell = 1$, i.e. for $s - a_1 \leqslant j \leqslant s$, and then

$$\lambda_k(p^s, \mathcal{D}) = \Phi(p^s, \mathcal{D}(p^{a_1})) = \Phi(p^s, \mathcal{D}) = \lambda_{p^s}(p^s, \mathcal{D}), \tag{18}$$

which is the largest eigenvalue. This reflects the phenomenon that the largest eigenvalue has multiplicity greater than 1 if $a_1 > 0$, that is, the elements of $\mathcal{D}$ are not coprime or, equivalently, $\mathrm{ICG}(n, \mathcal{D})$ is disconnected (see Remark 2.1 (i)). More precisely, we have for each $j = s - u$, $0 \leqslant u \leqslant a_1$, exactly $\varphi(p^u)$ integers $k = p^j m$ with $p \nmid m$ and $1 \leqslant k \leqslant p^s$. Hence the multiplicity of the largest eigenvalue $\Phi(p^s, \mathcal{D})$ is precisely $\sum_{u=0}^{a_1} \varphi(p^u) = p^{a_1}$. By (18) we know that $\Phi(p^s, \mathcal{D}) = \Phi(p^s, \mathcal{D}(p^{a_1}))$, and since $a_1 = s - r + 1$ in $\mathcal{D} = \{p^{s-r+1}, p^{s-r+2}, \ldots, p^{s-1}, p^s\}$, the asserted formulas for $\Phi(p^s, \mathcal{D})$ in (ii) follow.

By the argument above, eigenvalues other than the largest one can only appear in Case 2. For $\mathcal{D} = \{p^{s-r+1}, p^{s-r+2}, \ldots, p^{s-1}, p^s\}$ and $j = r - \ell - 1$ $(1 \leqslant \ell \leqslant r)$, we obtain by (17)

$$\lambda_k(p^s, \mathcal{D}) = \Phi(p^s, \mathcal{D}(p^{s-r+\ell+1})) - p^{r-\ell-1} = \sum_{i=0}^{r-\ell-1} \varphi(p^i) - p^{r-\ell-1} = 0.$$

This proves $\Lambda(p^s, \mathcal{D}) = 0$.

To complete the proof of (ii), it remains to show that $\Lambda(p^s, \mathcal{D}) \neq 0$ for any non-regular set $\mathcal{D}$. Such a divisor set can be written as $\mathcal{D} = \{p^{a_1}, \ldots, p^{a_\ell}, p^{a_{\ell+1}}, \ldots, p^{a_r}\}$ with $0 \leqslant a_1 < a_2 < \cdots < a_r = s$, $a_{\ell+1} - a_\ell \geqslant 2$ and $a_{i+1} - a_i = 1$ for some $1 \leqslant \ell \leqslant r - 1$ and all $i = \ell + 1, \ldots, r - 1$. Then for all $k = p^{s-a_\ell-1} m$, $p \nmid m$, i.e. $j = s - a_\ell - 1$, we have by (17) in Case 2

$$\lambda_k(p^s, \mathcal{D}) = \Phi(p^s, \mathcal{D}(p^{a_{\ell+1}})) - p^{s-a_\ell-1} = \sum_{i=\ell+1}^{r} \varphi(p^{s-a_i}) - p^{s-a_\ell-1} < 0, \tag{19}$$

hence $\Lambda(p^s, \mathcal{D}) \neq 0$.

It remains to verify (iii). Since $\mathcal{D}$ is not uni-regular, we have negative eigenvalues by (19), and Lemma 4.1 (ii) proves $\mathcal{E}(p^s, \mathcal{D}) > p^s$. $\square$

5. Minimal energies for multiplicative divisor sets

It was shown in [18, Theorem 3.1] that for a prime power p^s

$$\mathcal{E}_{\min}(p^s) = 2p^s\Big(1 - \frac{1}{p}\Big). \tag{20}$$

Observe that the minimum is extended over divisor sets $\mathcal{D} \subseteq D^*(p^s)$, i.e. over loopless graphs. Moreover, the p^s-minimal divisor sets were identified precisely as the singleton sets $\mathcal{D} = \{p^j\}$ with $0 \leq j \leqslant s-1$. For our purpose we shall require a corresponding result for graphs $\mathrm{ICG}(p^s, \mathcal{D})$ containing loops, that is with $p^s \in \mathcal{D}$.

Proposition 5.1. *Let p^s be a prime power. Then*

$$\hat{\mathcal{E}}_{min}(p^s) := \min\{\mathcal{E}(p^s, \mathcal{D}) : \, p^s \in \mathcal{D} \subseteq D(p^s)\} = p^s, \tag{21}$$

where the minimising divisor sets are exactly the uni-regular ones.

The reader might notice that (20) implies $\hat{\mathcal{E}}_{\min}(p^s) \leqslant \mathcal{E}_{\min}(p^s)$, where equality only holds in case $p=2$.

Proof of Proposition 5.1. For $r = 1$, the assertion follows immediately from Proposition 4.1(i). Hence assume that $r \geqslant 2$. By Proposition 4.1(iii) it suffices to show that we have $\mathcal{E}(p^s, \mathcal{D}) = p^s$ for each uni-regular divisor set $\mathcal{D} = \{p^{a_1}, p^{a_1+1}, \ldots, p^{s-1}, p^s\}$. We know from Proposition 4.1(ii) that $\mathrm{ICG}(n, \mathcal{D})$ has only two different eigenvalues, namely $\Phi(p^s, \mathcal{D}) = p^{s-a_1}$ with multiplicity p^{a_1} and 0 (consequently with multiplicity $p^s - p^{a_1}$). Therefore, $\mathcal{E}(p^s, \mathcal{D}) = p^{a_1} \cdot p^{s-a_1} = p^s$, as required. □

Proof of Theorem 2.2. Let $\mathcal{D} \subseteq D^*(n)$ be a multiplicative set such that $\mathcal{E}(n, \mathcal{D}) = \tilde{\mathcal{E}}_{\min}(n)$. Then $\mathcal{D} = \prod_{i=1}^t \mathcal{D}^{(i)}$ for certain divisor sets $\mathcal{D}^{(i)} \subseteq D(p_i^{s_i})$ $(1 \leqslant i \leqslant t)$, and $\tilde{\mathcal{E}}_{\min}(n) = \prod_{i=1}^t \mathcal{E}(p_i^{s_i}, \mathcal{D}^{(i)})$ by [14, Corollary 4.1(ii)]. By the minimality of $\mathcal{E}(n, \mathcal{D})$ it follows from [18, Theorem 3.1] and our Proposition 5.1 that for $1 \leqslant i \leqslant t$

$$\mathcal{E}(p_i^{s_i}, \mathcal{D}^{(i)}) = \begin{cases} 2p_i^{s_i}(1 - \frac{1}{p_i}) & \text{if } p_i^{s_i} \notin \mathcal{D}^{(i)}, \\ p_i^{s_i} & \text{if } p_i^{s_i} \in \mathcal{D}^{(i)}, \end{cases}$$

where $\mathcal{D}^{(i)}$ is either a singleton set $\{p_i^{u_i}\}$ for some $0 \leqslant u_i \leqslant s_i - 1$ or a uni-regular set containing $p_i^{s_i}$, respectively. This yields

$$\tilde{\mathcal{E}}_{\min}(n) = n \prod_{\substack{i=1 \\ p_i^{s_i} \notin \mathcal{D}^{(i)}}}^{t} 2\left(1 - \frac{1}{p_i}\right), \tag{22}$$

and our assumption $n \notin \mathcal{D}$ implies that $p_i^{s_i} \notin \mathcal{D}^{(i)}$ for at least one i. Under this restriction, and since p_1 is the smallest of the primes involved, it is easily seen that the righthand side of (22) becomes minimal if $p_1^{s_1} \notin \mathcal{D}^{(1)}$ and $p_i^{s_i} \in \mathcal{D}^{(i)}$ for $2 \leqslant i \leqslant t$ with the corresponding divisor sets $\mathcal{D}^{(i)}$ just mentioned. □

6. Concluding remarks, problems and conjectures

It would be most desirable to have an explicit formula for $\mathcal{E}(n, \mathcal{D})$ in case of arbitrary positive integers n and arbitrary divisor sets $\mathcal{D}$, comparable with the one available for prime powers $n = p^s$ in [18]. Such a formula is missing thanks to the lack of a multiplicative behaviour of $\mathcal{E}(n, \mathcal{D})$ with respect to n. In order to make good this deficit, we developed the concept of multiplicative divisor sets. This enabled us to obtain a closed formula for $\mathcal{E}(n, \mathcal{D})$ at least for multiplicative sets $\mathcal{D}$. The next result shows that we can even go a little beyond multiplicative divisor sets.

We denote the complement of some $\mathcal{D} \subseteq D(n)$ by $\overline{\mathcal{D}} := D(n)\backslash\mathcal{D}$. Apparently, either $\mathcal{D}$ or $\overline{\mathcal{D}}$ contains n, hence either $\mathrm{ICG}(n, \mathcal{D})$ or $\mathrm{ICG}(n, \overline{\mathcal{D}})$ has loops. This, however, collides with the standard notion of *graph complements*. We overcome this minor obstacle by considering for divisor sets $\mathcal{D} \subseteq D(n)$ the set $\mathcal{D}^* := \mathcal{D}\backslash\{n\} \subseteq D^*(n)$. Then, assuming $n \in \mathcal{D} \subseteq D(n)$ and thus $n \notin \overline{\mathcal{D}}$, the graph complement $\overline{\mathrm{ICG}(n, \mathcal{D}^*)}$ of $\mathrm{ICG}(n, \mathcal{D}^*)$ satisfies $\overline{\mathrm{ICG}(n, \mathcal{D}^*)} = \mathrm{ICG}(n, \overline{\mathcal{D}})$. Observe that for a multiplicative divisor set $\mathcal{D} \subsetneq D(n)$ with $n \in \mathcal{D}$, the set $\overline{\mathcal{D}}$ is not multiplicative if n is not a prime power.

Theorem 6.1. *Let n be a positive integer, and let $\mathcal{D} \subsetneq D(n)$ be multiplicative with $n \in \mathcal{D}$. Then*

$$\mathcal{E}(n, \overline{\mathcal{D}}) = \prod_{p \in \mathbb{P},\, p \mid n} \mathcal{E}(p^{e_p(n)}, \mathcal{D}_p) + n - 2\,\Phi(n, \mathcal{D}) \tag{23}$$

with $\mathcal{D}_p$ as defined in Section 2.

Proof. Obviously, we have for $1 \leqslant k \leqslant n$ that

$$\lambda_k(n, \mathcal{D}) = \lambda_k(n, \mathcal{D}^*) + c(k, 1) = \lambda_k(n, \mathcal{D}^*) + 1. \tag{24}$$

It is well known [9, Lemma 8.5.1] that

$$-1 - \lambda_1(n, \mathcal{D}^*), -1 - \lambda_2(n, \mathcal{D}^*), \ldots, -1 - \lambda_{n-1}(n, \mathcal{D}^*), n - 1 - \Phi(n, \mathcal{D}^*)$$

are the eigenvalues of $\overline{\mathrm{ICG}(n, \mathcal{D}^*)} = \mathrm{ICG}(n, \overline{\mathcal{D}})$. Thus, by (24), $\mathrm{ICG}(n, \overline{\mathcal{D}})$ has the eigenvalues $-\lambda_k(n, \mathcal{D})$ $(1 \leqslant k \leqslant n-1)$ and $n - 1 - \Phi(n, \mathcal{D}^*) = n - \Phi(n, \mathcal{D})$. The application of [14, Theorem 4.2] (with $g = 1$) yields

$$\begin{aligned}
\mathcal{E}(n, \overline{\mathcal{D}}) &= \sum_{k=1}^{n-1} |\lambda_k(n, \overline{\mathcal{D}})| + |\lambda_n(n, \overline{\mathcal{D}})| \\
&= \sum_{k=1}^{n-1} |\lambda_k(n, \mathcal{D})| + (n - \Phi(n, \mathcal{D})) \\
&= \mathcal{E}(n, \mathcal{D}) - |\lambda_n(n, \mathcal{D})| + (n - \Phi(n, \mathcal{D})) \\
&= \prod_{p \in \mathbb{P},\, p|n} \mathcal{E}(p^{e_p(n)}, \mathcal{D}_p) + n - 2\,\Phi(n, \mathcal{D}). \qquad \square
\end{aligned}$$

Using [18, Theorem 2.1] or [14, Theorem 5.1] in (23) immediately implies the desired explicit formula. It would be very desirable to enlarge this class of integral circulant graphs with non-multiplicative divisor sets and obtain formulae for their energies.

We confined our study of the energies of integral circulant graphs to the rather restricted class having multiplicative divisor sets. Yet, somewhat unexpectedly, this led to good bounds for $\mathcal{E}_{\min}(n)$ and $\mathcal{E}_{\max}(n)$. On top of that, the results obtained by the study of multiplicative divisor sets, combined with some numerical evidence, encourage us to make the following two conjectures.

Conjecture 6.1. *For each integer $n \geqslant 2$, we have $\mathcal{E}_{min}(n) = 2n\left(1 - \frac{1}{p_1}\right)$, where p_1 denotes the smallest prime factor of n.*

Conjecture 6.2. *Let $n \geqslant 2$ be an arbitrary integer. Then $\mathcal{E}(n, \mathcal{D}) = \mathcal{E}_{min}(n)$ implies that $\mathcal{D}$ is a multiplicative divisor set.*

Observe that Conjecture 6.1 is a consequence of Conjecture 6.2 by Theorem 2.2.

In 2005 it was conjectured by So that, given a positive integer n, two graphs $\mathrm{ICG}(n, \mathcal{D}_1)$ and $\mathrm{ICG}(n, \mathcal{D}_2)$ are *cospectral*, that is $Spec(\mathrm{ICG}(n, \mathcal{D}_1)) = Spec(\mathrm{ICG}(n, \mathcal{D}_2))$, if and only if $\mathcal{D}_1 = \mathcal{D}_2$. For $n = p^s$ this follows immediately from (11) by straightforward comparison of the largest eigenvalues $\Phi(p^s, \mathcal{D}_1)$ and $\Phi(p^s, \mathcal{D}_2)$ of the two graphs. So's conjecture was also confirmed for the slightly more general case $n = p^s q^t$ with primes $p \neq q$ and $t \in \{0, 1\}$ (cf. [6] for details), but its proof required the study of eigenvalues other than the largest one as well as their multiplicities. For arbitrary positive integers n and arbitrary divisor sets the problem is still open. Therefore, we suggest to study the following weaker conjecture, which might be more accessible.

Conjecture 6.3. *Let n be a positive integer, and let $\mathcal{D}_1, \mathcal{D}_2 \subseteq D^*(n)$ be two multiplicative sets. If $ICG(n, \mathcal{D}_1)$ and $ICG(n, \mathcal{D}_2)$ are cospectral, then $\mathcal{D}_1 = \mathcal{D}_2$.*

Considering the second largest modulus of the eigenvalues of $\mathrm{ICG}(n, \mathcal{D})$ was the key tool in the proof of Propositions 4.1 and 5.1, and thus Theorem 2.2. Incidentally, the second largest eigenvalue is the decisive parameter for the Ramanujan property of graphs (cf. [16]). As a by-product to our investigations on minimal energies, we would be able to characterise Ramanujan integral circulant graphs of prime power order, thus complementing a result of Droll [8] for Ramanujan unitary Cayley graphs, i.e. graphs of type ICG$(n, \{1\})$. In principle, our method would even suffice to characterise the Ramanujan ICG$(n, \mathcal{D})$ for arbitrary n in case of multiplicative $\mathcal{D}$, but harsh technical complications are to be expected along the way. Therefore, this will be dealt with in a separate publication.

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