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A numerical computation on the structure of the roots of q-extension of Genocchi polynomials

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Abstract

In this work we observe the behavior of real roots of the q-extension of Genocchi polynomials, $c_{n,q}(x)$, using numerical investigation. By means of numerical experiments, we demonstrate a remarkably regular structure of the complex roots of the $c_{n,q}(x)$ for $-1 < q < 0$. Finally, we give a table for the solutions of the q-extension of Genocchi polynomials.

Keywords: Genocchi numbers; Genocchi polynomials; q-extension of Genocchi numbers; q-extension of Genocchi polynomials; Roots of the q-extension of Genocchi polynomials; Reflection symmetries of the q-extension of Genocchi polynomials

1. Introduction

In the 21st century, the computing environment will make more and more rapid progress. Over the years, there has been increasing interest in solving mathematical problems with the aid of computers. Recently, many mathematicians have studied Genocchi polynomials and Genocchi numbers. Genocchi polynomials and Genocchi numbers possess many interesting properties and arise in many areas of mathematics and physics. In [2], Kim constructed the q-extension of the Genocchi numbers $c_{n,q}$ and polynomials $c_{n,q}(x)$ using generating functions. In order to study the q-extension of Genocchi polynomials $c_{n,q}(x)$, we must understand the structure of the q-extension of Genocchi polynomials $c_{n,q}(x)$. Therefore, using computers, a realistic study for the q-extension of Genocchi polynomials $c_{n,q}(x)$ is very interesting. For related topics the interested reader is referred to [3]. The main purpose of this work is to describe the distribution and structure of the zeros of the q-extension of Genocchi polynomials for values of the index n by using computers. The outline of this work is as follows. We introduce the q-extension of Genocchi polynomials $c_{n,q}(x)$. In Section 2, we describe the beautiful zeros of the q-extension of Genocchi polynomials $c_{n,q}(x)$ using a numerical investigation. We also investigate the roots of the q-extension of Genocchi polynomials $c_{n,q}(x)$. Finally, we consider the reflection symmetries of the q-extension of Genocchi polynomials $c_{n,q}(x)$.

First, we introduce the Genocchi numbers and Genocchi polynomials. The Genocchi numbers G_n are defined by the generating function

$$F(t) = \frac{2t}{\mathrm{e}^t + 1} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!}, \quad (|t| < \pi), \text{ cf. [1,2]} \tag{1}$$

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where we use the technique method notation by replacing G^n by G_n $(n \geq 0)$ symbolically. Here is the list of the first Genocchi numbers:

$$G_1 = 1, \quad G_2 = -1, \quad G_3 = 0, \quad G_4 = -1, \quad G_5 = 0, \quad G_6 = -3,$$
$$G_7 = 0, \quad G_8 = 17, \quad G_9 = 0, \quad G_{10} = -155, \quad G_{11} = 0, \quad G_{12} = 2073,$$
$$G_{14} = -38\,227 \quad G_{16} = 929\,569, \quad G_{18} = -28\,820\,619 \quad G_{20} = 1109\,652\,905, \ldots.$$

In general, it satisfies $G_3 = G_5 = G_7 = \cdots = 0$, and even coefficients are given by $G_n = 2(1-2^{2n})B_{2n} = 2nE_{2n-1}$, where B_n are Bernoulli numbers and E_n are Euler numbers.

For $x \in \mathbb{R}$ (=the field of real numbers), we consider the Genocchi polynomials $G_n(x)$ as follows:

$$F(x,t) = \frac{2t}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!}. \tag{2}$$

Note that $G_n(x) = \sum_{k=0}^{n} \binom{n}{k} G_k x^{n-k}$. In the special case $x = 0$, we define $G_n(0) = G_n$. Here is the list of the first Genocchi polynomials:

$$G_1(x) = 1, \quad G_2(x) = 2x - 1, \quad G_3(x) = 3x^2 - 3x, \quad G_4(x) = 4x^3 + 6x^2 - 1$$
$$G_5(x) = 5x^4 - 10x^3 + 5x, \quad G_6(x) = 6x^5 - 15x^4 + 15x^2 - 3, \ldots.$$

Next, we introduce the q-extension of Genocchi polynomials $c_{n,q}(x)$ (see [1,2]). Some of the following consideration is the same as that of Kim [2] except for obvious modifications. Let q be a complex number with $|q| < 1$. We consider the following generating functions:

$$F_q(t) = \sum_{n=0}^{\infty} c_{n,q} \frac{t^n}{n!} = e^{\frac{t}{1-q}} \sum_{n=0}^{\infty} \frac{(2n+1)(1-q^n)}{1-q^{2n+1}} \left(\frac{1}{1-q}\right)^{n-1} (-1)^{n-1} \frac{t^n}{n!}, \tag{3}$$

and

$$F_q(x,t) = \sum_{n=0}^{\infty} c_{n,q}(x) \frac{t^n}{n!} = e^{\frac{t}{1-q}} \sum_{n=0}^{\infty} \frac{(2n+1)(1-q^n)}{1-q^{2n+1}} \left(\frac{1}{1-q}\right)^{n-1} q^{nx} (-1)^{n-1} \frac{t^n}{n!}. \tag{4}$$

By simple calculation in (4), we obtain

$$\begin{aligned}
F_q(x,t) &= e^{\frac{t}{1-q}} \sum_{i=0}^{\infty} \frac{(2i+1)(1-q^i)}{1-q^{2i+1}} \left(\frac{1}{1-q}\right)^{i-1} (-1)^{i-1} q^{ix} \frac{t^i}{i!} \\
&= \left(\sum_{i=0}^{\infty} \frac{(2i+1)(1-q^i)}{1-q^{2i+1}} \left(\frac{1}{1-q}\right)^{i-1} (-1)^{i-1} q^{ix} \frac{t^i}{i!}\right) \left(\sum_{j=0}^{\infty} \left(\frac{1}{1-q}\right)^j \frac{t^j}{j!}\right) \\
&= \sum_{n=0}^{\infty} \sum_{i=0}^{n} \frac{(2i+1)(1-q^i)}{1-q^{2i+1}} \left(\frac{1}{1-q}\right)^{i-1} (-1)^{i-1} q^{ix} \frac{t^i}{i!} \left(\frac{1}{1-q}\right)^{n-i} \frac{t^{n-i}}{(n-i)!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} \binom{n}{i} \frac{(2i+1)(1-q^i)}{1-q^{2i+1}} \left(\frac{1}{1-q}\right)^{n-1} (-1)^{i-1} q^{ix}\right) \frac{t^n}{n!}.
\end{aligned}$$

For $n \geq 0$, we have

$$c_{n,q}(x) = \sum_{i=0}^{n} \binom{n}{i} \frac{(2i+1)(1-q^i)}{1-q^{2i+1}} \left(\frac{1}{1-q}\right)^{n-1} (-1)^{i-1} q^{ix}.$$

When $x = 0$, we write $c_{n,q} = c_{n,q}(0)$, which are called the q-extension of Genocchi numbers. $c_{n,q}(x)$ is a polynomial of degree n in q^x. By definition of the q-extension of Genocchi polynomials $c_{n,q}(x)$, we obtain

$$c_{n,q} = \sum_{i=0}^{n} \binom{n}{i} \frac{(2i+1)(1-q^i)}{1-q^{2i+1}} \left(\frac{1}{1-q}\right)^{n-1} (-1)^{i-1}.$$

Table 1
q-extension of Genocchi numbers

n	$q=-\frac{1}{2}$	$q=\frac{1}{2}$
1	4	12/7
2	32/11	−192/217
3	1872/473	−32 720/27 559
4	45 312/8987	−2 088 448/2 011 807
5	37 597 248/6 138 121	−137 130 816/588 309 847
6	120 643 876 352/16 763 208 451	49 530 750 358 528/33 731 921 697 439
7	45 808 895 808 768/5 548 621 997 281	21 852 649 304 280 320/5 093 520 176 313 289
8	2 255 428 752 014 581 760/242 424 843 683 204 171	5 541 259 829 023 557 107 712/667 612 783 029 559 102 519

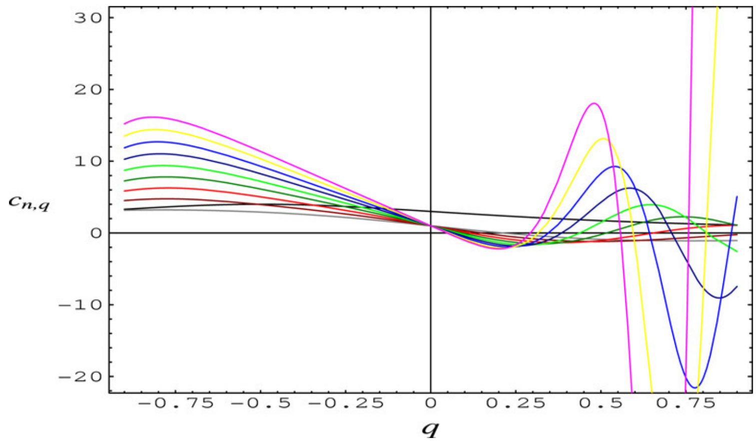

Fig. 1. Curve of $c_{n,q}$, $n = 1, \dots, 10$.

We obtain the first values of the q-extension of Genocchi numbers $c_{n,q}$:

$$c_{0,q} = 0,$$
$$c_{1,q} = \frac{3}{1+q+q^2},$$
$$c_{2,q} = -\frac{-1+3q+7q^2+6q^3}{(1+q+q^2)(1+q+q^2+q^3+q^4)},$$
$$c_{3,q} = \frac{(-1+q)(-1+3q+18q^2+42q^3+57q^4+52q^5+30q^6+9q^7)}{(1+q+q^2)(1+q+q^2+q^3+q^4)(1+q+q^2+q^3+q^4+q^5+q^6)},$$
$$c_{4,q} = -\frac{-1+6q+21q^2+32q^3+15q^4-35q^5-97q^6-132q^7-119q^8-66q^9-9q^{10}+28q^{11}+30q^{12}+12q^{13}}{(1+q+q^2)(1+q+q^2+q^3+q^4)(1+q^3+q^6)(1+q+q^2+q^3+q^4+q^5+q^6)},$$
$$\dots.$$

With $q=-\frac{1}{2}, q=\frac{1}{2}$ we obtain Table 1.

For $n = 1, \dots, 10$, we can draw a plot of the q-extension of Genocchi numbers $c_{n,q}$. This shows the ten plots combined into one. We display the shape of $c_{n,q}$, $-9/10 \le q \le 9/10$ (Fig. 1).

2. Zeros of the q-extension of Genocchi polynomials $c_{n,q}(x)$

In order to study $c_{n,q}(x)$, we must understand the structure of the q-extension of Genocchi polynomials. In this section, by means of numerical investigation, we examine properties of the figures, look for patterns, and give open problems. First, we display the shapes of the q-extension of Genocchi polynomials $c_{n,q}(x)$ and we investigate the zeros of the q-extension of Genocchi polynomials $c_{n,q}(x)$. For $n = 1, \dots, 10$, we can draw a plot of the

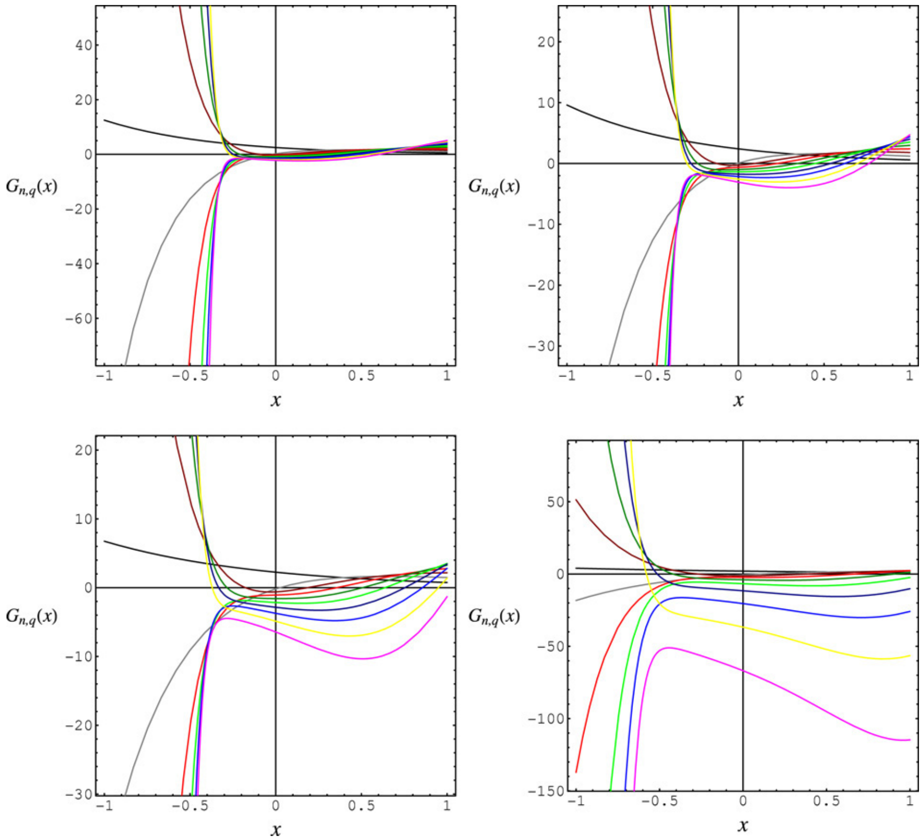

Fig. 2. Curve of $c_{n,q}(x)$, $q = 1/5, 1/4, 1/3, 1/2$.

q-extension of Genocchi polynomials $c_{n,q}(x)$. This shows the ten plots combined into one. We display the shape of $c_{n,q}(x)$, $-1 \le x \le 1$ (Fig. 2).

We investigate the beautiful zeros of the $c_{n,q}(x)$ by using a computer. We plot the zeros of the q-extension of Genocchi polynomials $c_{n,q}(x)$ for $n = 10, 20, 30, 40$, $q = \frac{1}{2}$ and $x \in \mathbb{C}$ (Fig. 3).

We plot the zeros of the q-extension of Genocchi polynomials $c_{n,q}(x)$ for $n = 10, 20, 30, 40$, $q = -\frac{1}{2}$ and $x \in \mathbb{C}$ (Fig. 4).

Since

$$\sum_{n=0}^{\infty} G_n(1-x)\frac{(-t)^n}{n!} = F(1-x,-t) = \frac{-2t}{e^{-t}+1}e^{(1-x)(-t)} = \frac{-2t}{e^{t}+1}e^{xt} = -F(x,t) = -\sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!},$$

we obtain that

$$G_n(x) = (-1)^{n+1} G_n(1-x). \tag{5}$$

We prove that $G_n(x)$, $x \in \mathbb{C}$, has $\mathrm{Re}(x) = \frac{1}{2}$ reflection symmetry in addition to the usual $\mathrm{Im}(x) = 0$ reflection symmetry of analytic complex functions. The question is: What happens with the reflection symmetry (5), when one considers the q-extension of Genocchi polynomials? We are going now to reflection at $\frac{1}{2}$ of x on the q-extension of Genocchi polynomials. Since

$$c_{n,q}(x) = \left(\frac{1}{1-q}\right)^{n-1} \sum_{i=0}^{n} \binom{n}{i} \frac{(2i+1)(1-q^i)}{1-q^{2i+1}} (-1)^{i-1} q^{ix},$$

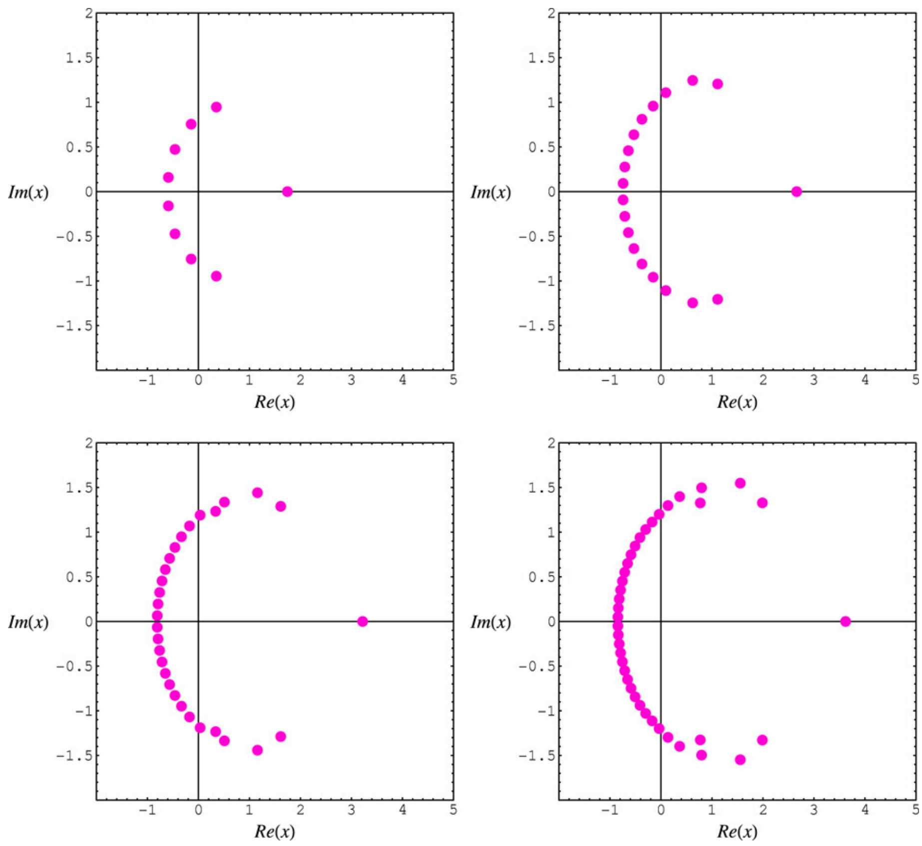

Fig. 3. Zeros of $c_{n,q}(x)$ for $n = 10, 20, 30, 40$, $q = 1/2$.

by simple calculation, we have

$$\begin{aligned}
c_{n,q^{-1}}(1-x) &= \left(\frac{1}{1-q^{-1}}\right)^{n-1} \sum_{i=0}^{n} \binom{n}{i} (-1)^{i-1} (q^{-1})^{i(1-x)} \frac{(2i+1)(1-q^{-i})}{1-q^{-2i-1}} \\
&= \left(\frac{q}{q-1}\right)^{n-1} \sum_{i=0}^{n} \binom{n}{i} (-1)^{i-1} q^{-i} q^{ix} \frac{(2i+1)q^{-i}(q^{i}-1)}{q^{-2i-1}(q^{2i+1}-1)} \\
&= \left(\frac{-q}{1-q}\right)^{n-1} \sum_{i=0}^{n} \binom{n}{i} (-1)^{i-1} q q^{ix} \frac{(2i+1)(q^{i}-1)}{(q^{2i+1}-1)} \\
&= (-1)^{n-1} q^{n} \left(\frac{1}{1-q}\right)^{n-1} \sum_{i=0}^{n} \binom{n}{i} (-1)^{i-1} q^{ix} \frac{(2i+1)(1-q^{i})}{1-q^{2i+1}}.
\end{aligned}$$

Hence we obtain the following theorem.

Theorem 1. *For $n \geq 0$, we have*

$$c_{n,q^{-1}}(1-x) = (-1)^{n-1} q^{n} c_{n,q}(x). \tag{6}$$

(6) is the q-analog of the classical reflection formula (5). $c_{n,q}(x)$ $(q > 0)$ has $\mathrm{Im}(x) = 0$ reflection symmetry analytic complex functions (Fig. 3). $c_{n,q}(x)$ does not have $\mathrm{Re}(x) = 1/2$ reflection symmetry (Fig. 3). The open question is: What happens with the reflection symmetry (6) when one considers the q-extension of Genocchi polynomials for $q < 0$? We have the following corollary.

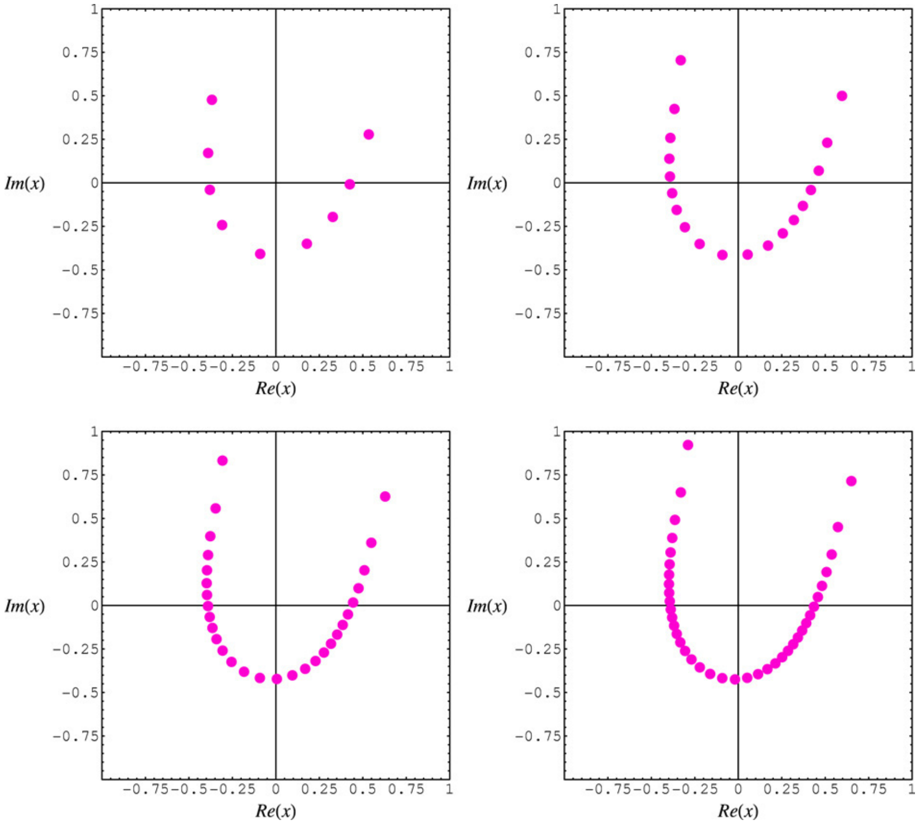

Fig. 4. Zeros of $c_{n,q}(x)$ for $n = 10, 20, 30, 40$, $q = -1/2$.

Table 2
Numbers of real and complex zeros of $c_{n,q}(x)$

Degree n	$q = -\frac{1}{2}$		$q = \frac{1}{2}$	
	Real zeros	Complex zeros	Real zeros	Complex zeros
2	0	1	1	0
4	0	3	1	2
6	0	5	1	4
8	0	7	1	6
10	0	9	1	8
12	0	11	1	10
14	0	13	1	12

Corollary 2. *If* $c_{n,q}(x) = 0$ $(q > 0)$, *then* $c_{n,q^{-1}}(1 - x) = c_{n,q}(x^*) = c_{n,q^{-1}}(1 - x^*) = 0$, *where* $*$ *denotes complex conjugation.*

Corollary 3. *If* $c_{n,q}(x) = 0$ $(q < 0)$, *then* $c_{n,q^{-1}}(1 - x) = 0$.

We observe a remarkably regular structure of the complex roots of the q-extension of Genocchi polynomials. We hope to verify a remarkably regular structure of the complex roots of the q-extension of Genocchi polynomials $c_{n,q}(x)$ (Table 2). Next, we calculate an approximate solution satisfying $c_{n,q}(x)$, $x \in \mathbb{R}$. The results are given in Table 3.

Table 3
Approximate solutions of $c_{n,q}(x) = 0, x \in \mathbb{R}$

Degree n	x
2	0.0995357
3	−0.271627, 0.394484
4	0.670557
5	−0.41925, 0.909775
6	1.11775
7	−0.514195, 1.30079
8	1.4639
9	−0.579197, 1.61083
10	1.74442

3. Directions for further research

Finally, we shall consider more general problems. Prove or disprove: $c_{n,q}(x) = 0$ has $n - 1$ distinct solutions. Find the numbers of complex zeros $C_{c_{n,q}(x)}$ of $c_{n,q}(x)$, $\text{Im}(x) \neq 0$. Prove or disprove: Since $n - 1$ is the degree of the polynomial $c_{n,q}(x)$, the number of real zeros $R_{c_{n,q}(x)}$ lying on the real plane $\text{Im}(x) = 0$ is then $R_{c_{n,q}(x)} = n - 1 - C_{c_{n,q}(x)}$, where $C_{c_{n,q}(x)}$ denotes complex zeros. See Table 2 for tabulated values of $R_{c_{n,q}(x)}$ and $C_{c_{n,q}(x)}$. Find the equation of envelope curves bounding the real zeros lying on the plane. The author has no doubt that investigation along these lines will lead to a new approach employing numerical methods in the field of research into the q-extension of Genocchi polynomials $c_{n,q}(x)$ appearing in mathematics and physics. The reader may refer to [3,4] for the details.

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