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Improved Gagliardo–Nirenberg–Sobolev inequalities on manifolds with positive curvature

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Abstract

We apply the method of [J. Demange, From porous media equation to generalized Sobolev inequalities on a Riemannian manifold, preprint, http://www.lsp.ups-tlse.fr/Fp/Demange/, 2004] and [J. Demange, Porous Media equation and Sobolev inequalities under negative curvature, preprint, http://www.lsp.ups-tlse.fr/Fp/Demange/, 2004], based on the curvature–dimension criterion and the study of Porous Media equation, to the case of a manifold M with strictly positive Ricci curvature. This gives a new way to prove classical Sobolev inequalities on M. Moreover, this enables to improve non-critical Sobolev inequalities as well. As an application, we study the rate of convergence of the solutions of the Porous Media equation to the equilibrium.

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1. Introduction

In this paper, we derive inequalities from a family of nonlinear partial differential equations on a general n-dimensional compact manifold M whose Ricci curvature is bounded below by a positive constant ρ. The cases of nonnegative curvature and of strictly negative curvature have already been discussed in [7,8]. In this work we will follow the lines of [7,8] by differentiating

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functions called *Entropy* and *Information* linked to a nonlinear partial differential equation of the form:

$$\frac{\partial u}{\partial t}(t,x) = \Delta\sigma(u)(t,x), \quad t \geqslant 0,\ x \in M.$$

Here σ is a nondecreasing function mapping $\mathbb{R}_+$ onto $\mathbb{R}$. In this paper we use power functions $\sigma(x) = x^\alpha$, with $1 - 1/n \leqslant \alpha \leqslant 1$, but analog results can still be obtained for more general σ. These equations are analogous to the Porous Media equations studied in [7,8]. So let d be a real number greater than n and consider the case $\sigma(x) = x^{1-1/d}$. Denote by μ the normalized Riemannian measure on M. We define the Information as

$$I(t) = \int_M u(t,x)\left|\nabla(d-1)u(t,x)^{-1/d}\right|^2 d\mu.$$

Following [7,8] we will prove that

$$-I'(t) \geqslant K\frac{d}{dt}\int_M u(t,x)^{1-2/d}, \tag{1}$$

where K depends on n, d and ρ. The key to get the inequality is the use the curvature–dimension criterion in M (see [1,3–6]): denote by Δ the Laplace–Beltrami operator on M, $\Gamma(f,g) = \nabla f \cdot \nabla g$ the carré du champ operator and $\Gamma_2(f,g) = (\Delta\Gamma(f,g) - \Gamma(f,\Delta g) - \Gamma(\Delta f, g))/2$ the *iterated* carré du champ operator (see Section 2 for more details). Then for any smooth function g:

$$\Gamma_2(g,g) \geqslant \rho\Gamma(g,g) + \frac{1}{n}(\Delta g)^2. \tag{2}$$

In [7] an integrated version of (2) has been established, and it will be useful here again. Although the paper deals with the Laplace–Beltrami operator of M, we could replace M by a space X equipped with an operator L satisfying the condition $CD(\rho,n)$ (X does not need to have any dimension). In this case we would define the operators:

$$\Gamma(f,g) = \big(L(f\cdot g) - f\cdot Lg - g\cdot Lf\big)/2 \quad \text{and}$$
$$\Gamma_2(f,g) = \big(L\Gamma(f,g) - \Gamma(f,Lg) - \Gamma(g,Lf)\big)/2,$$

and still assume (2) where Δ is replaced by L. We shall also assume that L is a differential operator in the sense that for all functions ϕ, ψ:

$$\Gamma\big(\phi(f),\psi(g)\big) = \phi'(f)\psi'(g)\Gamma(f,g) \quad \text{and} \quad L\phi(f) = \phi'(f)Lf + \phi''(f)\Gamma(f,f),$$

and that an integration by part formula holds for L with respect to a given measure μ on X (here the Riemannian measure). Basic examples of such operators on the real line are the following one. Let a be a (smooth) function and $Lf = f'' + a\cdot f'$. Then under certain differential conditions on a (see Section 2), L satisfies (2). A basic example is

$$Lf(x) = f''(x) + (n-1)\tan(x)f'(x),$$

which satisfies (2) with $\rho = n - 1$, just like the Laplace–Beltrami operator of the n-dimensional sphere. See [4] for more details. More generally, if X is a manifold of dimension m, the operator $L = \Delta + \nabla h \cdot \nabla$ satisfies (2) if and only if $n \geqslant m$ and

$$(n-m)[Ricci - \operatorname{Hess} h - \rho g] \geqslant \nabla h \otimes \nabla h.$$

A basic study of (1) implies the classical family of Sobolev inequalities on M, and this paper gives in fact a new way of proving them. However a more precise study of (1) leads to an improvement of the inequality in the case when $d > n$, that is when we are below the critical Sobolev exponent. This improvement was already known for the logarithmic Sobolev inequality (see [1,3]), which corresponds to the (limit) case when $d = \infty$. Indeed we get the following inequality for any smooth positive f:

$$\int_M f\left|\nabla(d-1) f^{-1/d}\right|^2 d\mu \geqslant \frac{\psi(\int_M f\, d\mu)^{1-2/d}] - \psi[\int_M f^{1-2/d}\, d\mu]}{\psi'[(\int_M f\, d\mu)^{1-2/d}]},$$

where ψ is not affine (the affine case corresponds to the classical Sobolev inequality). The expression of ψ can be found in Theorem 1 below. An application of those inequalities is the study of the convergence to the equilibrium of the solution to the former PDE. This is the statement of Theorem 2 (see Section 8).

Theorem 1. *Let $n \geqslant 2$ be an integer and M be a compact connected n-dimensional Riemannian manifold with Ricci curvature bounded below by a positive constant ρ. Denote by μ its normalized Riemannian measure. The following inequality holds for $d \in \mathbb{R}$, $d > n$, $d \geqslant 3$, $\alpha = 1/2 - 1/d$ and f, a smooth function mapping M onto $\mathbb{R}_+^*$:*

$$K(n,d)\left\{\psi\left(I^{2\alpha}, I\right) - \psi\left(\int f^{2\alpha}, I\right)\right\} \leqslant \phi\left(\int f^{2\alpha}, I\right)\int \left|\nabla f^{\alpha}\right|^2 d\mu,$$

where $\partial_x \psi(x,y) = \phi(x,y)$, $K(n,d) = \rho(d-2)/(4(1-1/n))$, $I = \int f\, d\mu$, and

$$\phi(x,y) = \exp\left(L(n,d)\frac{x^{d/(4(d-2))}}{y^{1/4}}\right) \quad \text{and} \quad L(n,d) = \frac{d-n}{n+2}\left(4 - 9\frac{d-n}{d(n+2)}\right).$$

Note that Theorem 1 provides a weaker inequality which looks like the Entropy–Logarithmic energy inequality already known for the case $d = \infty$:

Take the notations of Theorem 1. *Then*:

$$\left(\int f\right)^{2\alpha} - \int f^{2\alpha} \leqslant K_1 \log\left(1 + K_2 \int \left|\nabla f^{\alpha}\right|^2\right),$$

where $K_2 = L(n,d)d/(4(d-2)K(n,d)(\int f)^{2\alpha})$ and $K_1^{-1} = K_2 K(n,d)$ (*see Corollary* 2).

In the last section we give extensions to our result by studying some modified Porous Media equation.

2. Notations

Let $n \geqslant 2$ and M be an n-dimensional compact and connected Riemannian manifold whose Ricci curvature is bounded below by a positive constant ρ (this hypothesis implies the compactness of M). Note $\cdot$ the scalar product on M, ∇ the gradient operator, Δ the classical Laplace–Beltrami operator. Then define the following operators respectively called *carré du champs operator* and iterated *carré du champs operator*:

$$\forall f, g, \quad \Gamma(f,g) = \nabla f \cdot \nabla g,$$
$$\Gamma_2(f,g) = \big(\Delta\Gamma(f,g) - \Gamma(f,\Delta g) - \Gamma(\Delta f, g)\big)/2.$$

Moreover we will note $\Gamma_2(f,f) = \Gamma_2(f)$ and $\Gamma(f,f) = \Gamma(f)$. We know from the Bochner–Lichnerowicz formula (classical in Riemannian geometry) that for any smooth ξ:

$$\Gamma_2(\xi) \geqslant \rho\Gamma(\xi) + \frac{1}{n}(\Delta\xi)^2, \tag{3}$$

which we will call curvature–dimension criterion. See [1,3–6] for more information on this criterion. The proofs made in this paper are still valid if we replace Δ by an operator L and M by a space X, which does not need to have any dimension. Three main assumptions are needed:

(1) L satisfies the last curvature–dimension criterion which goes as follows. Define Γ_2 and Γ by

$$\Gamma(f,g) = \big(L(f\cdot g) - f\cdot Lg - g\cdot Lf\big)/2,$$
$$\Gamma_2(f,g) = \big(L\Gamma(f,g) - \Gamma(f,Lg) - \Gamma(g,Lf)\big)/2.$$

Then one must assume that

$$\Gamma_2(\xi,\xi) \geqslant \rho\Gamma(\xi,\xi) + \frac{1}{n}(L\xi)^2.$$

(2) On (X, L), an integration by parts formula holds with respect to a given measure μ (in the last case μ is the Riemannian measure); in other words, for functions f, g:

$$\int_X (Lf)g\,d\mu = -\int \Gamma(f,g)\,d\mu.$$

(3) L should be a differential operator in the sense of the introduction. Some basic examples of such operators for $X = \mathbb{R}$ or an interval in $\mathbb{R}$ are the operators L where $Lf = f'' + af$, a being a function satisfying:

$$a' \geqslant \rho + \frac{a^2}{n-1}. \tag{4}$$

Indeed in this case

$$\Gamma(f,g) = f'g' \quad \text{and} \quad \Gamma_2(f,g) = f''g'' + a'f'g',$$

and $\Gamma_2(f,f) - \rho\Gamma(f,f) - (Lf)^2/n$ equals

$$f''^2(1-1/n) + f'2\big(a' - a^2/n - \rho\big) + (2a/n)f'f''.$$

This polynomial in f' and f'' is nonnegative for any f if and only if its discriminant is nonpositive, which is condition (4). Therefore if $a(x) = \tan(x\sqrt{\rho/(n-1)})\sqrt{\rho(n-1)}$ then L satisfies the curvature–dimension criterion. Note also that those operators give models for the cases $\rho < 0$ and $\rho = 0$ which we do not discuss here. For more explanations see [4]. To avoid dealing with questions on the existence of solutions to the considered PDEs, we shall make the proof only for the Laplace–Beltrami operator of a manifold M (in this case the dimension of the operator Δ coincide with the dimension of the space M unlike above).

Let $d \geqslant n$, $d > 2$, be a real number and consider the following PDE with smooth positive initial data f:

$$\text{(E)}\qquad \frac{\partial u}{\partial t}(t,x) = \Delta u(t,x)^{1-1/d}, \quad t \geqslant 0,\ x \in M.$$

Since M is compact one can find $\varepsilon > 0$ such that $f \geqslant \varepsilon$. Then consider the following PDE starting from $f - \varepsilon$:

$$\text{(F)}\qquad \frac{\partial v}{\partial t}(t,x) = \Delta\sigma_\varepsilon\big(v(t,x)\big), \quad t \geqslant 0,\ x \in M,$$

where $\sigma_\varepsilon(x) = (x+\varepsilon)^{1-1/d} - \varepsilon^{1-1/d}$. Since $\sigma'_\varepsilon(0) > 0$, standard parabolic results show that a smooth nonnegative solution v to (F) exists for $t \geqslant 0$. Remark that $u = v + \varepsilon$ is a smooth solution to (E) and remains greater than ε. This proves the existence of a smooth solution to (E). We can give examples of such solutions on models manifolds: the fundamental solutions to (E) for $d = n$, that is the one that start from a Dirac delta distribution, are explicit: this happens for $\mathbb{R}^n$, for the sphere and for the hyperbolic space. Of course those solutions are smooth only for $t > 0$, but they are smooth for any time if you consider $t = t_0 > 0$ as initial condition. For instance on the sphere S_n of dimension n, denoting $\langle,\rangle$ the Euclidean scalar product,

$$u(t,x) = \left(\frac{\operatorname{sh}((n-1)t)}{\operatorname{ch}((n-1)t) - \langle x_0, x\rangle}\right)^n,$$

is the solution to the porous media equation (E) with critical exponent $d = n$, starting from the Dirac delta mass in $x_0 \in S_n$. More generally, on a space X equipped with an operator satisfying the above conditions, if one can find a function T whose generalized hessian is $-Tg$, that is,

$$\Gamma\big(\xi, \Gamma(T,\xi)\big) - \Gamma\big(T, \Gamma(\xi)\big)/2 = -T\Gamma(\xi),$$

and whose minimum is -1 with $\arg\min(T) = x_0$, then the solution has the same form with $-\langle x, x_0\rangle$ replaced by $\langle x, T(x_0)\rangle$. For the solutions on $\mathbb{R}^n$ (called Barenblatt solutions) we refer to [10] and for the hyperbolic we refer to [8].

Let us finally define the *Entropy* $E_2(t)$ and *Information* $I(t)$ of the solution u as follows:

$$E_2(t) = \int_M u(t,x)^{1-2/d}\, d\mu,$$

$$I(t) = \int_M u(t,x)\Gamma\big((d-1)u(t,x)^{-1/d}\big)\, d\mu.$$

Unlike what usually happens in Entropy–Energy inequalities (for instance: logarithmic-Sobolev inequalities), where the left-hand side is the "Entropy" and the right-hand side is the derivative of the Entropy, called information, here there will not be such a differential relation between both sides of the inequalities that will be proved. Thus we denote the Entropy E_2 instead of E: $E_2(t) \neq C\frac{d}{dt}I(t)$, $C \in \mathbb{R}$. In the rest of the paper we shall write for simplicity u or u_t in place of $u(t,x)$ (u_t is not a derivative) and ξ will stand for $-(d-1)u^{-1/d}$. When writing integrals, we shall write sometimes $\int H$ or $\int_M H$ instead of $\int_M H\, d\mu$.

3. Integrated curvature–dimension criterion

As a special case of an inequality stated in [7] and which follows from (3), we have the

Lemma 3.1. *Let ψ, H be functions of the real variable, ψ being bijective, increasing and $u : M \to \mathbb{R}_+$ be smooth. Let $G(x) = 2x^2\psi'(x)$, $x \geqslant 0$ and $\xi = \psi(u)$. Then*

$$\begin{aligned}\int G(u)\Gamma_2(\xi) \geqslant \int & \frac{n}{n-1}G(u)\rho\Gamma(\xi) \\ & + \left(\frac{3}{2}\frac{d}{d\xi}\left(\frac{G(u)}{n-1}\right) - \frac{n+2}{n-1}G(u)H(u)\right)\Gamma\big(\xi, \Gamma(\xi)\big) \\ & + \int\left(\frac{d^2}{d\xi^2}\left(\frac{G(u)}{n-1}\right) - 2\frac{d}{d\xi}\left(\frac{G(u)H(u)}{n-1}\right) - G(u)H(u)^2\right)\Gamma(\xi)^2\end{aligned}$$

with the natural convention that for any function K, $\frac{d}{d\xi}(K(u)) = \frac{K'(u)}{\psi'(u)}$.

This inequality is obtained by applying formula (3) to $H(\xi)$ instead of ξ and then doing integrations by parts.

4. Differentiation of $I(t)$ and minoration of $-I'(t)$

In the following we differentiate the Information function $I(t)$. Note $\psi(x) = -(d-1)x^{-1/d}$. Recall that u_t is the solution of equation (E) so that for any function $K(t,x)$

$$\int \frac{\partial u}{\partial t}K(t,x) = -\int u_t\Gamma\big(\xi, K(t,\cdot)\big).$$

Therefore we have:

$$I'(t) = \int \big(\partial_t u \Gamma(\xi) + 2u\Gamma\big(\xi, \psi'(u)\partial_t u\big)\big)\, d\mu$$
$$= \int \partial_t u\big[\Gamma(\xi) - 2\psi'(u)\big(\Gamma(u,\xi) + u\Delta\xi\big)\big]\, d\mu,$$
$$-I'(t) = \int u\Gamma\big(\xi, -\Gamma(\xi) - 2\psi'(u)u\Delta\xi\big). \tag{5}$$

Let us first consider the term $u\psi'(u)\Delta\xi$. Note that $-1/d = (x\psi'(x))'\psi'(x)^{-1}$. Therefore

$$\Gamma\big(\xi, u\psi'(u)\Delta\xi\big) = u\psi'(u)\Gamma(\xi,\Delta\xi) - \Delta\xi\,\Gamma(\xi)/d$$
$$= u\psi'(u)\left[\frac{1}{2}\Delta\Gamma(\xi) - \Gamma_2(\xi)\right] - \Delta\xi\,\Gamma(\xi)/d.$$

Firstly, the integration by parts formula implies

$$\int u^2\psi'(u)\Delta\Gamma(\xi)\, d\mu = -\int \Gamma\big(u^2\psi'(u), \Gamma(\xi)\big)\, d\mu$$
$$= -(1-1/d)\int u\Gamma\big(\xi,\Gamma(\xi)\big)\, d\mu. \tag{6}$$

Secondly,

$$-\int u\Delta\xi\,\Gamma(\xi)\, d\mu = \int \Gamma\big(\xi, u\Gamma(\xi)\big)\, d\mu = \int u\Gamma\big(\xi,\Gamma(\xi)\big)\, d\mu + \int \Gamma(\xi)\Gamma(\xi,u)\, d\mu$$
$$= \int u\Gamma\big(\xi,\Gamma(\xi)\big)\, d\mu + \int \frac{1}{\psi'(u)}\Gamma(\xi)^2\, d\mu. \tag{7}$$

Finally, after collecting terms (6) and (7) there remains:

$$-\int 2u\Gamma\big(\xi, u\psi'(u)\Delta\xi\big)\, d\mu = \int 2u^2\psi'(u)\Gamma_2(\xi)\, d\mu$$
$$+ (1-3/d)\int u\Gamma\big(\xi,\Gamma(\xi)\big)\, d\mu - \frac{2}{d}\int \frac{1}{\psi'(u)}\Gamma(\xi)^2\, d\mu. \tag{8}$$

Thanks to (5) this leads to

$$-I'(t) = \int \left[2(1-1/d)u^{1-1/d}\Gamma_2(\xi) - \frac{3}{d}u\Gamma\big(\xi,\Gamma(\xi)\big) - \frac{2}{d-1}u^{1+1/d}\Gamma(\xi)^2\right] d\mu. \tag{9}$$

Now apply Section 3 to $\psi(x) = -(d-1)x^{-1/d}$ and to our solution at time t, $u = u_t$. Then if $G(x) = 2(1-1/d)x^{1-1/d}$, for any function H

$$-I'(t) \geqslant \int \frac{n}{n-1}G(u)\rho\Gamma(\xi) + \int S_1\Gamma\big(\xi,\Gamma(\xi)\big) + \int S_2\Gamma(\xi)^2, \tag{10}$$

where

$$S_1 = \frac{3}{2}\frac{d}{d\xi}\left(\frac{G(u)}{n-1}\right) - \frac{n+2}{n-1}G(u)H(u) - \frac{3}{d}u,$$

$$S_2 = \frac{d^2}{d\xi^2}\left(\frac{G(u)}{n-1}\right) - 2\frac{d}{d\xi}\left(\frac{G(u)H(u)}{n-1}\right) - G(u)H(u)^2 - \frac{2}{d-1}u^{1+1/d}.$$

Now the idea is to choose H such that $S_1 = 0$. A direct computation shows that we shall take

$$H(u) = 3u\frac{n}{(n+2)G(u)}\left(\frac{1}{n} - \frac{1}{d}\right).$$

This leads to $S_2 = C(n,d)u^{1+1/d}$ where

$$C(n,d) = \frac{n(1/n-1/d)}{2(n+2)(1-1/d)}\left(4 - 9\frac{n}{n+2}(1/n-1/d)\right). \tag{11}$$

Since $d \geqslant n$, $C(n,d) \geqslant 0$, with equality only if $d = n$. We finally get the following minoration of $-I'(t)$:

$$-I'(t) \geqslant 2\rho\frac{1-1/d}{1-1/n}\int u^{1-1/d}\Gamma(\xi)\,d\mu + C(n,d)\int u^{1+1/d}\Gamma(\xi)^2\,d\mu. \tag{12}$$

This relation is the key to prove the inequalities of this paper. First abord, we will derive from (12) the classical Sobolev inequality by saying that $C(n,d) \geqslant 0$ and then we will enhance by taking into account all terms of the right-hand side.

5. The classical Sobolev inequality

In this section we give an alternative way to prove Sobolev inequalities. Those inequalities are already known on manifolds with Ricci curvature bounded from below by a positive constant. A good reference is [2,9]. Usually the inequality is obtained by compacity argument, and the constant may not be optimal. Then a discussion on extremal solutions to the inequality gives the best constant. Our method however gives the best known constant directly, without discussion, and proves that it is a consequence of hypothesis about curvature and dimension on the operator Δ. We should also mention that there are other ways to prove that these Sobolev inequalities exist, using curvature–dimension assumptions only. One is explained in [3]: first, the author proves a stronger form of the logarithmic-Sobolev inequality, called "weak Sobolev inequality," and manages to establish a Sobolev inequality, however, the constant might not be optimal. Then again a clever discussion implies that the constant can be chosen the same as in classical Riemannian theory. However in this paper we prove the inequality with the best known constant "directly" by associating to an inequality a partial differential equation.

Lemma 5.1. *Let for* $0 \leqslant i \leqslant 2$:

$$E_i(t) = \int_M u(t,x)^{1-i/d}\,d\mu.$$

Then the following differential inequality holds for $t \geqslant 0$:

$$-E_1''(t) \geqslant AE_2'(t), \tag{13}$$

where the constant A depends on d, n and ρ as follows:

$$A = \rho\frac{(d-1)(1-1/d)}{(1-1/n)(d-2)}.$$

Proof. First let us calculate $E_1'(t)$. Recall that u is the solution to (E). Since M is compact the integrations by parts are valid and

$$\begin{aligned} E_1'(t) &= \int (1-1/d)\frac{\partial u}{\partial t}u^{-1/d} = \int (1-1/d)\Delta u^{1-1/d}u^{-1/d} \\ &= -\int (1-1/d)\Gamma\big(u^{1-1/d}, u^{-1/d}\big)\,d\mu. \end{aligned}$$

Recall that $\xi = -(d-1)u^{-1/d}$. Then

$$E_1'(t) = \frac{1}{d}\int u\Gamma(\xi). \tag{14}$$

In the same way we show that

$$E_2'(t) = 2\frac{d-2}{d(d-1)}\int u^{1-1/d}\Gamma(\xi). \tag{15}$$

Replace $\int u\Gamma(\xi)$ and $\int u^{1-1/d}\Gamma(\xi)$ in formula (12) by their expressions given in formulae (14) and (15) and estimate from below the term with $C(nd)$ by 0. This gives the proposition. □

Corollary 1. *For $\alpha = 1/2 - 1/d$ we have the classical Sobolev inequality*:

$$\left(\int f\right)^{2\alpha} - \int f^{2\alpha} \leqslant K(n,d)^{-1}\int \left|\nabla f^{\alpha}\right|^2,$$

where $K(n,d)$ is defined in Theorem 1. *Letting $d \to n$ gives the Sobolev inequality with critical exponent.*

Proof. First let us study the behavior of $u(t,\cdot)$ as $t \to \infty$. First note that by Hölder inequality $E_1(t) \leqslant E_0^{1-1/d}$ and recall that $E_0 = \int f$ does not depend on t. Hence E_1 is bounded and therefore there exists a diverging sequence t_k such that $\lim_\infty E_1'(t_k) = 0$ (we leave it to the reader). From formula (14), if $\alpha = 1/2 - 1/d$, this implies that

$$\lim_{\infty}\int_M \Gamma\big(u(t_k,\cdot)^{\alpha}\big)\,d\mu = 0,$$

which from the classical Poincaré inequality on M implies that

$$\lim_{\infty} \int_M \left[\left(\int u(t_k, \cdot)^\alpha \right) - u(t_k, \cdot)^\alpha \right]^2 d\mu = 0.$$

Once again the sequence $\int u(t_k, \cdot)^\alpha$ is bounded and therefore, up to a subsequence, we can suppose that it has a limit l^α. Thus, up to a subsequence, $u(t_k, \cdot)$ converges to l a.e. Now thanks to the form of the PDE (E) we also know that $\max u(t_k, \cdot) \leqslant \max f$. Hence the convergence also holds in any L^p, $p > 0$. Hence, for $p = 1$ we see that $l = \int f$ (μ is normalized).

Lemma 5.1 ensures that the function

$$m(t) = E_1'(t) + A E_2(t)$$

is non-increasing. Hence a posteriori $m(0) \geqslant \limsup_\infty m(t_k)$. As we saw $\lim_\infty E_1'(t_k) = 0$, and $\lim_\infty E_2(t_k) = (\int f)^{1-2/d}$ (since $u(t_k, \cdot) \to l$ in $L^{1-2/d}$). Therefore

$$A \left\{ \left[\int f \right]^{2\alpha} - \int f^{2\alpha} \right\} \leqslant E_1'(0). \tag{16}$$

Note that from formula (14),

$$E_1'(0) = \frac{4(d-1)^2}{d(d-2)^2} \int \Gamma(f^\alpha).$$

This proves the proposition. □

Note that we can specify the rate at which the solution converges to $\int f$. A consequence of Lemma 5.1 shows that if $M = 2\rho(1 - 1/d)/(1 - 1/n)$, then

$$-I'(t) \geqslant M \max(f)^{-1/d} I(t),$$

which implies an exponential decay of $I(t)$, and more precisely, thanks to formula (14),

$$\int \left| \nabla u(t, \cdot)^\alpha \right|^2 \leqslant \int \left| \nabla f^\alpha \right|^2 \exp\left(-M \max(f)^{-1/d} t\right).$$

This result can however be slightly enhanced: see Section 8.

6. Differential relation between entropy and information

In this section we prove a differential relation between $E_2(t)$ and $E_1(t)$ (or $I(t)$) which improves Lemma 5.1 and proves Theorem 1. Let us begin with a lemma which is a direct consequence of Cauchy–Schwarz inequality.

Lemma 6.1. *For any smooth $\xi : M \to \mathbb{R}$ and $u : M \to \mathbb{R}_+$, $u \neq 0$, we have*

$$\int_M u^{1+1/d}\Gamma(\xi)^2\,d\mu \geqslant \frac{\int_M u\Gamma(\xi)\,d\mu \int_M u^{1-1/d}\Gamma(\xi)\,d\mu}{\int_M u^{1-1/d}\,d\mu \int_M u^{1-3/d}\,d\mu}.$$

Proof. Note $\theta = 1/2 + 1/(2d)$. Just develop the numerator D of the right-hand side in $M \times M$ and use Cauchy–Schwarz inequality:

$$\begin{aligned} D &= \int\int_{M\times M} u(x)^{1-1/d}u(y)\Gamma(\xi)(x)\Gamma(\xi)(y)\,d\mu(x)\,d\mu(y) \\ &= \int\int_{M\times M} \big[u(x)^{\theta}u(y)^{1-\theta}\Gamma(\xi)(x)\big]\big[u(y)^{\theta}u(x)^{1-1/d-\theta}\Gamma(\xi)(y)\big]\,d\mu(x)\,d\mu(y) \\ &\leqslant \int_M u^{1+1/d}\Gamma(\xi)^2\,d\mu\Big(\int_M u^{1-1/d}\,d\mu\Big)^{1/2}\Big(\int_M u^{1-3/d}\,d\mu\Big)^{1/2}. \qquad \square \end{aligned}$$

Now Lemma 6.1 and inequality (12) bring about the following.

Proposition 6.1. *Let for $0 \leqslant i \leqslant 3$*

$$E_i(t) = \int_M u(t,x)^{1-i/d}\,d\mu.$$

Then the following differential inequality holds for $t \geqslant 0$:

$$-E_1''(t) \geqslant AE_2'(t) + B\frac{E_1'(t)E_2'(t)}{\sqrt{E_1(t)E_3(t)}}, \tag{17}$$

where the constants A and B depend on d, n and ρ as follows:

$$A = \rho\frac{(d-1)(1-1/d)}{(1-1/n)(d-2)} \quad \text{and} \quad B = C(n,d)\frac{d(d-1)}{2(d-2)},$$

and $C(n,d)$ is defined by formula (11).

Proof. Replace $\int u^{1+1/d}\Gamma(\xi)^2\,d\mu$ of formula (12) by the minoration given by Lemma 6.1 and then replace $\int u\Gamma(\xi)$ and $\int u^{1-1/d}\Gamma(\xi)$ by their expressions given in formula (14) and (15). This gives the proposition. □

7. Integration of the differential inequality

First note that E_0 does not depend on t since (E) is a mass-preserving equation. Then since $u(0,x) = f(x)$, $E_0 = \int_M f\,d\mu$. In this section we integrate a weaker version of inequality (17). Indeed, the presence of $E_3(t)$ and $E_1(t)$ lead us to use the following Hölder majorations:

$$E_1(t) \leqslant \sqrt{E_0E_2(t)} \quad \text{and} \quad E_3(t) \leqslant E_2(t)^{\frac{d-3}{d-2}}.$$

Therefore inequality (17) becomes:

$$-E_1''(t) \geqslant AE_2'(t) + \frac{B}{E_0^{1/4}} E_1'(t)E_2'(t)E_2(t)^{-\frac{3d-8}{4(d-2)}}, \tag{18}$$

and can be put into the following form.

Proposition 7.1.

$$\frac{d}{dt}\left(E_1'(t)\phi\left(E_2(t)\right) + A\psi\left(E_2(t)\right)\right) \leqslant 0,$$

where ψ is a primitive of ϕ and

$$\phi(x) = \exp\left[\frac{B}{E_0^{1/4}} \frac{4(d-2)}{d} x^{d/(4(d-2))}\right].$$

Proof. Remark that inequality (18) reads as follows:

$$0 \geqslant E_1''(t) + AE_2'(t) + E_1'(t)E_2'(t)\frac{\phi'(E_2(t))}{\phi(E_2(t))},$$
$$0 \geqslant E_1''(t)\phi\left(E_2(t)\right) + E_1'(t)\frac{d}{dt}\phi\left(E_2(t)\right) + A\frac{d}{dt}\psi\left(E_2(t)\right).$$

This is exactly the result mentioned in the proposition. □

We now are able to prove Theorem 1 stated in the introduction.

Proof. Recall from the proof of Corollary 1, that there is a subsequence t_k diverging such $u(t_k, \cdot)$ converges to $l = \int f$ a.e. and in any L^p, $p > 0$.

Proposition 7.1 ensures that the function

$$m(t) = E_1'(t)\phi\left(E_2(t)\right) + A\psi\left(E_2(t)\right)$$

is non-increasing. Hence a posteriori $m(0) \geqslant \limsup_\infty m(t_k)$. As we saw $\lim_\infty E_1'(t_k) = 0$, and $\lim_\infty E_2(t_k) = (\int f)^{1-2/d}$ (since $u(t_k, \cdot) \to l$ in $L^{1-2/d}$). Therefore

$$A\left\{\psi\left(\left[\int f\right]^{2\alpha}\right) - \psi\left(\int f^{2\alpha}\right)\right\} \leqslant \phi\left(\int f^{2\alpha}\right)E_1'(0). \tag{19}$$

Note that from formula (14),

$$E_1'(0) = \frac{4(d-1)^2}{d(d-2)^2} \int \Gamma\left(f^\alpha\right).$$

Now if we recall the definitions of ϕ in Proposition 7.1 and of $C(n,d)$ in formula (11), then we get exactly the inequality of Theorem 1. □

We will now formulate a weaker inequality. This is an Entropy–Logarithmic energy inequality which looks like the one obtained in [1,3]. In fact, letting $d \to \infty$ gives that inequality (since it makes use of the heat equation). These inequalities imply the classical non-critical Sobolev inequality (which we proved by our method in Section 5) by simple use of inequality $\log(1+x) \leqslant x$, $x > 0$. Other classical proofs can be found for instance in [9], they do not however consider the improved form using log function. Here it appears as a weaker form of Theorem 1.

Corollary 2. *Take the notations of Theorem* 1. *Then*

$$\left(\int f\right)^{2\alpha} - \int f^{2\alpha} \leqslant K_1 \log\left(1 + K_2 \int \left|\nabla\left(f^{\alpha}\right)\right|^2\right),$$

where $K_2 = L(n,d)d/(4(d-2)K(n,d)(\int f)^{2\alpha})$ *and* $K_1^{-1} = K_2 K(n,d)$. *Moreover we have the classical Sobolev inequality*:

$$\left(\int f\right)^{2\alpha} - \int f^{2\alpha} \leqslant K(n,d)^{-1} \int \left|\nabla f^{\alpha}\right|^2.$$

Letting $d \to n$ *gives the Sobolev inequality with critical exponent.*

Proof. The second inequality comes from the first one using inequality $\log(1+x) \leqslant x$. For the first one, observe that if $x \leqslant u \leqslant y$ and $\beta = d/(4(d-2))$, then

$$u^{\beta} - x^{\beta} \geqslant \beta y^{\beta-1}(u - x).$$

Now choose $x = \int f^{2\alpha}$ and $y = (\int f)^{2\alpha}$. Then

$$\frac{\psi(y) - \psi(x)}{\phi(x)} = \int_x^y \exp\left(\frac{L(n,d)}{E_0^{1/4}}\left(u^{\beta} - x^{\beta}\right)\right) du \geqslant K_1\left(\exp\left(K_1^{-1}(x-y)\right) - 1\right).$$

Use this inequality and Theorem 1 to get the first inequality of Corollary 2. □

8. Application to the convergence of the solution

As was shown in the last sections there is a sequence t_k such that $u(t_k, \cdot)$ converges to $\int f$ a.e. as $k \to \infty$. However we can be more precise. What is well known that the logarithmic-Sobolev inequalities give informations on the decay of the solutions to heat equations towards equilibrium. In this section, we give estimates on the convergence of the solutions of porous media equations, using the improved Gagliardo–Nirenberg–Sobolev inequalities of this paper instead. The strategy is similar to Section 5 however. Will give an explicit bound on the H^1-norm of $u(t,\cdot)^{\alpha}$ which is more precise than the one given in Section 5. This is easily obtained thanks to formula (17). Note that the functional inequality itself will not give information on the behaviour of the norm of gradients of the solution but only on Hölder norms (the function S in the following). However, the method used in this paper provides such an information, due to the computation of $E_2''(t)$.

Keep notations of Theorem 1 and note $l = \int f^{1-2/d}$. Let $u(t,\cdot)$ be the smooth solution of the Porous Media equation (E) starting from $f > 0$ smooth, and note $\alpha = 1/2 - 1/d$. Let for $t \geqslant 0$, $S(t,\cdot)$ be the nonnegative function,

$$S(t,\cdot) = \left(\int f\right)^{2\alpha} - u(t,\cdot)^{2\alpha} + 2\alpha\left(\int f\right)^{-2/d}\left(u(t,\cdot) - \int f\right).$$

Let

$$K = 2\rho(d-1)/\big(d^2(1-1/n)\max f^{1/d}\big)$$

and

$$L(t) = \big[l - \psi^{-1}\big(\psi(l) - Z(0)\exp{-Kt}\big)\big]e^{-Kt}.$$

Let $A = 2\rho\frac{1-1/d}{1-1/n}$ and $C(n,d) \geqslant 0$ be given by formula (11) and

$$A' = A\max(f)^{-1/d}, \qquad I = \int \left|\nabla f^{\alpha}\right|^2, \qquad D(n,d) = C(n,d)(d-1)^2/(d-2)^2.$$

We have the following estimates for both Hölder norms and H^1 norms.

Theorem 2.

(i) *For* $t \geqslant 0$

$$\left\|S(t,\cdot)\right\|_{L^1(M)} \leqslant L(t)e^{-Kt}.$$

(ii) *Moreover, for* $t \geqslant 0$ *we have*

$$\int \left|\nabla u(t,\cdot)^{\alpha}\right|^2 \leqslant \frac{AIe^{-A't}}{A + D(n,d)I(1-e^{-A't})}. \tag{20}$$

(iii) *Finally we have the following estimates*:

$$\limsup_{t\to\infty} \frac{\log\|S(t,\cdot)\|_{L^1(M)}}{t} \leqslant -2\frac{\rho(d-1)}{d^2(1-1/n)}\left(\int f\right)^{-1/d}, \tag{21}$$

$$\limsup_{t\to\infty} \frac{\log\|\nabla u^{\alpha}(t,\cdot)\|_2^2}{t} \leqslant -2\rho\frac{1-1/d}{1-1/n}\left(\int f\right)^{-1/d}. \tag{22}$$

Proof. First remark that a simple convexity argument implies that $S(t,\cdot) \geqslant 0$. Let us apply Theorem 1 to $u(t,\cdot)$. Note $l = E_0^{2\alpha}$ and $Z(t) = \psi(l) - \psi(E_2(t))$. Therefore

$$\begin{aligned} Z(t) &\leqslant \phi\big(E_2(t)\big)K(n,d)^{-1}\frac{d(d-2)^2}{4(d-1)^2}\int u\Gamma(\xi) \\ &\leqslant \phi\big(E_2(t)\big)K(n,d)^{-1}\frac{d(d-2)^2}{4(d-1)^2}\max f^{1/d}\int u^{1-1/d}\Gamma(\xi), \end{aligned}$$

since $u(t,\cdot) \leqslant \max f$. By (15)

$$Z(t) \leqslant \phi\big(E_2(t)\big) K^{-1} E_2'(t),$$

where $K = 2\rho(d-1)/(d^2(1-1/n)\max f^{1/d})$. In other words,

$$K Z(t) \leqslant -Z'(t).$$

This yields that $Z(t)\exp(Kt)$ is a non-increasing function of t. Thus

$$Z(t) \leqslant Z(0)\exp(-Kt).$$

Rewriting this in a different form leads to

$$\int S(t,\cdot) = l - E_2(t) \leqslant l - \psi^{-1}\big(\psi(l) - Z(0)\exp(-Kt)\big).$$

With the notations, the right-hand side equals $L(t)\exp(-Kt)$. Then obviously

$$\lim_{t\to\infty} L(t) = \frac{\psi(l) - \psi(E_2(0))}{\phi(l)}.$$

Moreover, a simple convexity argument shows that $L(t) \leqslant l - E_2(0)$.

For the last part of the theorem, by (17), the estimation $u(t,\cdot) \leqslant \max f$ and Hölder inequality we get

$$-I'(t) \geqslant A' I(t) + C(n,d)\max(f)^{-1/d} \frac{I(t)^2}{(\int f)^{1-2/d}}.$$

This autonomous differential inequation is equivalent to

$$\frac{d}{dt}\left(\log\left[\frac{I(t)}{A' + C(n,d)\max(f)^{-1/d} I(t)}\right]\right) \leqslant -A',$$

which leads to formula (20) of the theorem, since

$$I(t) = \frac{(d-1)^2}{(d-2)^2}\int \big|\nabla u(t,\cdot)^\alpha\big|^2.$$

This proves (i) and (ii). (iii) is a direct consequence of (i) and (ii). For instance (i) gives that

$$\limsup_{t\to\infty} \frac{\log\|S(t,\cdot)\|_{L^1(M)}}{t} \leqslant -2\frac{\rho(d-1)}{d^2(1-1/n)}(\max f)^{-1/d}.$$

The semi-group property of the solution, which tells that $u(t+s,\cdot)_{t\geqslant 0}$ is the solution starting from $u(s,\cdot)$ at $t=0$ implies that f can be replaced by $u(s,\cdot)$:

$$\limsup_{t\to\infty} \frac{\log\|S(t,\cdot)\|_{L^1(M)}}{t} \leqslant -2\frac{\rho(d-1)}{d^2(1-1/n)}\big(\max u(s,\cdot)\big)^{-1/d}.$$

The same argument can be derived from (ii). So it is clear that (iii) is obtained as soon as $\lim_{s\to\infty}\max u(s,\cdot)=\int f$. This is done in Lemma 8.1. □

We shall note that the exponential decay of $S(t,\cdot)$ implies the exponential decay of $\|u(t,\cdot)-l)\|_2^2$ since the Taylor formula and the inequality $\min u(t,\cdot)\leqslant \max f$ imply:

$$S(t,x)\geqslant \alpha(1/2+1/d)/2\big(u(t,x)-l\big)^2\max(f)^{-1/2-1/d}.$$

Note also that we can have precise estimates only if we include the term $\max f$ in the formulae (i) and (ii). However, this is not convenient, since this quantity is not conserved along the path, unlike the quantity $\int f$ which appears in (21) and (22) of (iii). In this sense (iii) gives a candidate to be the precise rate of decay of the solution: we suspect that in (i) and (ii), the rate of decay of (iii) is still valid, even if this implies modifying the function in front of the exponential. However we are not able to prove it yet.

Lemma 8.1. *With the notations of Theorem* 1

$$\lim_{t\to\infty}\max\big\{u(t,\cdot)\big\}=\int_M f\,d\mu.$$

Proof. This proof is just an adaptation of the work made in [3] by D. Bakry: here we deal with the semi-group given by the Porous Media equation, which of course is nonlinear, whereas in [3], the author studies the convergence towards equilibrium for linear semi-groups. However, there is no big difficulty in proving Lemma 8.1. Let $\hat m(t)$ and $\hat p(t)$ be two functions and f be the smooth strictly positive starting function of $u(t,\cdot)$. The idea is to differentiate

$$U(t)=e^{-\hat m(t)}\|u_t\|_{\hat p(t)}.$$

As in [4, Théorème 3.3], we have that

$$U'(t)=\frac{U(t)\hat p'(t)}{\hat p^2(t)\int u_t^{\hat p(t)}}\left(E_{\hat p(t)}(u_t)-\frac{\hat p(t)^2}{\hat p'(t)}\left(\mathcal{E}_{\hat p(t)}(u_t)+\hat m'(t)\int u_t^{\hat p(t)}\right)\right),$$

where for any function g, and real number p,

$$E_p(g)=\int_M g^p\log g^p\,d\mu-\left(\int_M g^p\,d\mu\right)\log\int_M g^p\,d\mu,$$

$$\mathcal{E}_p(g)=-\int_M \Delta g^{1-1/n}g^{p-1}.$$

Of course, the generator of the semi-group is $\Delta g^{1-1/n}$ instead of Δg. Therefore, using the inequality

$$\forall g,p\quad \mathcal{E}_p(g)\geqslant(\max g)^{-1/n}(1-1/n)\int_M \Gamma\big(g,g^{p-1}\big)\,d\mu,$$

and reasoning as in [3, Théorème 3.3], we see that $U(t)$ is non-increasing as soon as

$$\frac{\hat{p}(t)^2(1-1/n)}{(\max f)^{1/n}c(\hat{p}(t))} = \hat{p}'(t) \quad \text{and} \quad m\big(\hat{p}(t)\big)(\max f)^{-1/n}(1-1/n) = \hat{m}'(t),$$

and a logarithmic-Sobolev inequality with constants $c(p)$ and $m(p)$ holds for given functions c and m and p lying in an interval $[a,b]$:

$$E_p(g) \leqslant c(p)\left(\mathcal{E}_p(g) + m(p)\int_M g^p\, d\mu\right).$$

Thus, Théorème 3.3 of [3] is true with a change of time $\nu = (\max f)^{1/n}/(1-1/n)$:

$$\big\|u(t\times\nu,\cdot)\big\|_b \leqslant e^{\hat{m}}\|f\|_a,$$

with

$$t = \int_a^b \frac{c(u)}{u^2}\,du \quad \text{and} \quad \hat{m} = \int_a^b m(u)\frac{c(u)}{u^2}\,du.$$

As a result, the subsequent propositions of [3] still hold up to the correct change of time. Hence

$$\limsup_{t\to\infty}\|u_t\|_\infty \leqslant \int_M f\,d\mu.$$

On the other hand, it is obvious that $\|u_t\|_\infty \geqslant \int u_t = \int f$. The lemma then follows. □

Note that by adapting the argument of [3] to our semi-group as in the proof of the previous lemma, we can get the uniform convergence towards equilibrium, and give precise estimates.

9. Another Sobolev inequality

In this section we will not detail the proof, since we use the same arguments as in the previous sections and in [7]. We present a Sobolev inequality, still valid under positive curvature, that generalizes the one of the n-dimensional sphere $\mathbb{S}^n$. The idea is to consider the modified Porous Media equation, as in [7]:

$$\partial_t u = \operatorname{div}\big\{u\nabla\big(-(n-1)u^{-1/n} + T\big)\big\},$$

starting from a smooth function f. Here $T > 0$ is a given function (we will give the assumptions later). Once again the idea is to differentiate the energy function:

$$I(t) = \int_M u_t\Gamma\big(-(n-1)u^{-1/n} + T\big)\,d\mu.$$

The calculus is similar to the one of the previous sections, except that we must consider the terms in which T appears. So if $\xi = -(n-1)u^{-1/n} + T$, and $S = \xi - T$ we get that

$$-I'(t)=2\frac{n-1}{n}\int_M u_t^{1-1/n}\Gamma_2(\xi_t)\,d\mu+2\frac{n+1}{n}\int u_t\,\mathrm{Hess}\,T(\nabla\xi,\nabla\xi)\,d\mu+R,$$

where

$$R=-\frac{2}{n}\int_M u_t\left\{\mathrm{Hess}\,\xi(3\nabla\xi-2\nabla T,\nabla\xi)+\frac{n}{n-1}u^{1/n}\Gamma(S,\xi)^2\right\}d\mu.$$

Now we must use the general curvature–dimension inequality of [7] to give a minoration of the term with Γ_2. This leads to

$$-I'(t)\geqslant 2\int u\,\mathrm{Hess}\,T(\nabla\xi,\nabla\xi)+2\rho\int u^{1-1/n}\Gamma(\xi).$$

The reader sees that if we have conditions on T such as

$$\mathrm{Hess}\,T\geqslant\left(-\frac{\rho}{n-1}T+\alpha\right)g,\quad \alpha\in\mathbb{R},$$

then if $T=-(n-1)v^{-1/n}$,

$$\begin{aligned}-I'(t)&\geqslant\int u\Gamma\left(\xi,2\alpha\xi-\frac{\rho}{n-1}\xi^2\right)d\mu=\int\partial_t u\left(-2\alpha\xi+\frac{\rho}{n-1}\xi^2\right)d\mu\\&=\frac{d}{dt}\int_M\left(\int_{v(x)}^{u_t(x)}\left[-2\alpha\big(-(n-1)s^{-1/n}+T(x)\big)\right.\right.\\&\quad\left.\left.+\frac{\rho}{n-1}\big(-(n-1)s^{-1/n}+T(x)\big)^2\right]ds\right)d\mu(x).\end{aligned}$$

Now we integrate the last inequality between 0 and ∞, assuming that v has the same integral as f (and hence $\lim_\infty u(t;\cdot)=v(\cdot)$). This leads to the theorem below.

Theorem 3. *Let $n>2$ and M be an n-dimensional Riemannian manifold, with a curvature bounded below by a positive constant ρ, a metric g, and equipped with a function $T>0$ satisfying*:

$$\mathrm{Hess}\,T\geqslant\left(-\frac{\rho}{n-1}T+\alpha\right)g,\quad \alpha\in\mathbb{R}.$$

Put $T=(n-1)v^{-1/n}$. Then for any smooth positive function f such that $\int f=\int v$, we have

$$E(f)\leqslant I(f),\quad \textit{where}$$
$$I(f)=\int_M f\Gamma\big(-(n-1)f^{-1/n}+T\big)\,d\mu,$$

$$E(f) = \int_M \left(\int_{f(x)}^{v(x)} \left[-2\alpha\big(-(n-1)s^{-1/n} + T(x)\big) + \frac{\rho}{n-1}\big(-(n-1)s^{-1/n} + T(x)\big)^2 \right] ds \right) d\mu(x).$$

On the sphere, the reader can verify that this gives the optimal Sobolev inequality with T being any first eigenvalue of the Laplacian (modulo a constant). Indeed, in this case, the terms in which T appears in I and E cancel each other.

10. Conclusion

This paper, together with [7,8], concludes the study of Sobolev inequalities in manifolds M with Ricci curvatures bounded from below and eventually equipped with special functions as in [7,8], with the help of the porous media equation:

$$\frac{\partial u}{\partial t} = \Delta u^\alpha + \operatorname{div}(u \nabla T),$$

where $1 - 1/\dim(M) \leqslant \alpha \leqslant 1$ and T is a special function. Now since Sobolev inequalities can be derived on $\mathbb{R}^n$ through mass-transportation techniques (see [11]), we can wonder if this is the case on manifolds such as the sphere or the hyperbolic space, or more general Riemannian structures.

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