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A problem about normal subgroups

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For Karl Gruenberg on his 65th birthday

During a conversation that took place in the early seventies, Philip Hall asked the first author the following rather innocent-sounding question:

Which infinite groups are isomorphic to all their non-trivial normal subgroups?

The obvious examples are the infinite cyclic group $\mathbb{Z}$, simple groups and free groups of infinite rank, and it may be that these are the only examples. Certainly, the only soluble group of this type is the infinite cyclic group, and one should perhaps be able to deduce the same result in the locally soluble case, though we have not succeeded in doing this as yet.

We shall concentrate on the case of finitely generated groups, where $\mathbb{Z}$ and simple groups ought to be the only examples. In partial confirmation of this, we have the following theorem:

Theorem 1. *Let G be a finitely generated infinite group that is isomorphic to all its non-trivial normal subgroups. If G contains a proper normal subgroup of finite index, then $G \cong \mathbb{Z}$.*

Proof. We set $d(G) = r$, and divide the proof into two essentially different parts.

Firstly, suppose that G is perfect. By assumption, there is a proper normal subgroup M_0 of finite index in G such that G/M_0 is a finite non-abelian simple group, S say. We use the fact that there is a largest positive integer m such that

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the mth direct power S^m of S is an r-generator [1] to get a contradiction to the existence of M_0 in this case. Set $N_0 = G$. We shall construct a descending chain

$$N_0 > N_1 > \cdots > N_i > \cdots$$

of normal subgroups of G and, for each i, a normal subgroup M_i of N_i such that $N_i/M_i \cong S$ and N_i/N_{i+1} is a direct product of at most m isomorphic copies of S. Set $N_1 = \mathrm{Core}_G(M_0)$. Suppose, for some $i \geq 1$, that $N_0, \ldots, N_i, M_0, \ldots, M_{i-1}$ have been defined. Choose M_i to be any normal subgroup of N_i such that $N_i/M_i \cong S$, and set $N_{i+1} = \mathrm{Core}_G(M_i)$. Since S is simple, N_i/N_{i+1} is a direct product of at most m isomorphic copies of S, and the construction of the N_i and M_i is complete. For each $i \geq 1$, the centralizer C_i of N_i/N_{i+1} in G has index bounded by a number k independent of i, since the order of N_i/N_{i+1} is bounded in terms of $|S|$ and m. Since G is finitely generated, the intersection $C = \bigcap_{i=0}^{\infty} C_i$ also has finite index in G: there are only finitely many different C_i. But $C \cap N_i \leq \bigcap_{i=0}^{\infty} N_{i+1}$ for all i, so that $C \leq N_i$, which is a contradiction since C has finite index while the N_i descend properly.

Thus G is not perfect, and the argument splits once more.

Suppose first in this part of the proof that G/G' is finite, and let p be a prime dividing its order. Consider the series

$$G = P_0 > P_1 > \cdots > P_i > \cdots$$

of G where, for each $i \geq 0$, P_i/P_{i+1} is the abelian p-image of P_i of maximal order. Then, since the P_i are mutually isomorphic, P_i/P_{i+1} is a maximal normal abelian subgroup of the finite p-group G/P_{i+1}, which means that it is self-centralizing. It follows that the order of G/P_{i+1} is bounded independently of i, and the contradiction is again that the P_i descend properly.

Finally, we have to conclude that G/G' is infinite. Let s be its torsion-free rank. Define a series

$$G = H_0 > H_1 > \cdots > H_i > \cdots$$

of characteristic subgroups of G, as follows. For $i \geq 0$, let H_{i+1}/H_i' be the torsion subgroup of H_i/H_i'. Then H_i/H_{i+1} is free abelian of rank s, for each i. Consider the action of G on H_i/H_{i+1} via conjugation. By a theorem of Zassenhaus [3], there is an integer n such that the nth term $K = G^{(n)}$ of the derived series of G centralizes H_i/H_{i+1}, for each $i \geq 0$. Thus K is nilpotent mod H_i, for each i. For any $i \geq 1$, let AH_i/H_i be a maximal normal abelian subgroup of KH_i/H_i. Then AH_i/H_i is self-centralizing and of torsion-free rank at most s, so again by Zassenhaus's theorem, K has derived length at most $n+1$ mod H_i. Then, for each i, G/H_i has derived length at most $2n+1$ and so it has Hirsch length at most $s(2n+1)$. This means that $H_{2n+1} = 1$, so that G is soluble and therefore cyclic. □

The situation where G is a finitely generated group with no non-trivial finite images has not yielded to attack. Since all powers of G could have the same number of generators as G (see [2]), any argument like that at the beginning of the proof of Theorem 1 is bound to fail. However, since G is finitely generated, it satisfies the maximal condition for normal subgroups, and one would think that it cannot be too far from being simple. As a very modest justification for this comment, we have the following:

Theorem 2. *Let G be a finitely generated infinite group that is isomorphic to all its non-trivial normal subgroups. Then every pair of non-trivial normal subgroups intersect non-trivially.*

Proof. Let M and N be non-trivial normal subgroups of G. If $M \cap N = 1$, then $\langle M, N \rangle = M \times N$, so that $G \cong G \times G$. But then G is non-Hopf and so it cannot satisfy the maximal condition for normal subgroups. □

Note added in proof. See B.H. Neumann [Compositio Math. 13 (1956) 128] for a result that deals with the case of imperfect G in Theorem 1.

References

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