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Journal of Algebra 373 (2013) 392–413

Contents lists available at SciVerse ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

The primary components of positive critical binomial ideals

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ARTICLE INFO

Article history:
Received 2 July 2012
Available online 6 November 2012
Communicated by Luchezar L. Avramov

MSC:
primary 13A05, 13A15
secondary 13P05, 13P15

Keywords:
Herzog–Northcott ideal
Positive critical binomial ideal
Smith Normal Form

ABSTRACT

A natural candidate for a generating set of the (necessarily prime) defining ideal of an n-dimensional monomial curve, when the ideal is an almost complete intersection, is a full set of n critical binomials. In a somewhat modified and more tractable context, we prove that, when the exponents are all positive, critical binomial ideals in our sense are not even unmixed for $n \geqslant 4$, whereas for $n \leqslant 3$ they are unmixed. We further give a complete description of their isolated primary components as the defining ideals of monomial curves with coefficients. This answers an open question on the number of primary components of Herzog–Northcott ideals, which comprise the case $n = 3$. Moreover, we find an explicit, concrete description of the irredundant embedded component (for $n \geqslant 4$) and characterize when the hull of the ideal, i.e., the intersection of its isolated primary components, is prime. Note that these last results are independent of the characteristic of the ground field. Our techniques involve the Eisenbud–Sturmfels theory of binomial ideals and Laurent polynomial rings, together with theory of Smith Normal Form and of Fitting ideals. This gives a more transparent and completely general approach, replacing the theory of multiplicities used previously to treat the particular case $n = 3$.

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1 The authors gratefully acknowledge financial support from the research grant MTM2010-20279-C02-01 and the Universitat Politècnica de Catalunya during the development of this research.

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http://dx.doi.org/10.1016/j.jalgebra.2012.10.014

1. Introduction

Let $A = k[x_1, \ldots, x_n]$ be the polynomial ring in n variables $\underline{x} = x_1, \ldots, x_n$ over a field k, $n \geqslant 2$. Set $\mathfrak{m} = (\underline{x})$, the maximal ideal generated by the x_i. Let $a_{i,j} \in \mathbb{N}$, $i, j = 1, \ldots, n$, with $a_{i,i} = \sum_{j \neq i} a_{i,j}$, and let L be the $n \times n$ integer matrix defined as follows:

$$L = \begin{pmatrix} a_{1,1} & -a_{1,2} & \ldots & -a_{1,n} \\ -a_{2,1} & a_{2,2} & \ldots & -a_{2,n} \\ \vdots & \vdots & \ldots & \vdots \\ -a_{n,1} & -a_{n,2} & \ldots & a_{n,n} \end{pmatrix},$$

where the sum of the entries of each row is zero. (Here $\mathbb{N}$ denotes the set of positive integers.) For ease of reference, L will be called a *positive critical binomial matrix* (PCB matrix, for short). Set $d \in \mathbb{N}$ to be the greatest common divisor of the $(n-1) \times (n-1)$ minors of L. (We shall see below that these minors are non-zero.) Let $\underline{f} = f_1, \ldots, f_n$ be the binomials defined by the columns of L:

$$f_1 = x_1^{a_{1,1}} - x_2^{a_{2,1}} \cdots x_n^{a_{n,1}}, \qquad f_2 = x_2^{a_{2,2}} - x_1^{a_{1,2}} x_3^{a_{3,2}} \cdots x_n^{a_{n,2}}, \qquad \ldots, \qquad f_n = x_n^{a_{n,n}} - x_1^{a_{1,n}} \cdots x_{n-1}^{a_{n-1,n}}.$$

Let $I = (\underline{f})$ be the binomial ideal generated by the f_j. We will call I the *positive critical binomial ideal* (PCB ideal, for short) associated to L.

The purpose of this paper is to investigate the primary decomposition of PCB ideals and to contrast this theory with analogous results in [OP2] concerning ideals of Herzog–Northcott type, which comprise the case $n = 3$. We first prove that, if $n \geqslant 4$ (respectively, $n \leqslant 3$), I has at most $d + 1$ (respectively, d) primary components. This answers a question posed in [OP2, Remark 8.6].

We will observe that I is contained in a unique toric ideal $\mathfrak{p}_m$ associated to the monomial curve $\Gamma_m = \{(\lambda^{m_1}, \ldots, \lambda^{m_n}) \in \mathbb{A}_k^n \mid \lambda \in k\}$, where $m = (m_1, \ldots, m_n) = m(I) \in \mathbb{N}^n$ is determined by I. That is, $\mathfrak{p}_m$ (referred to as the *Herzog* ideal associated to m) is the kernel of the natural homomorphism $A \to k[t]$, t a variable over k, that sends each x_i to t^{m_i}.

In somewhat more detail, if k contains the d-th roots of unity and the characteristic of k, $\operatorname{char}(k)$, is zero or $\operatorname{char}(k) = p$, p a prime with $p \nmid d$, we give a full description of a minimal primary decomposition of I. Namely, the intersection of the isolated primary components of I, $\operatorname{Hull}(I)$, is equal to the intersection of d prime toric ideals of "monomial curves with coefficients", i.e., kernels of natural homomorphisms $A \to k[t]$ that send each x_i to $\lambda_i t^{m_i}$, $\lambda_i \in k$. This will explain the "intrinsic" role of the Herzog ideal $\mathfrak{p}_{m(I)}$ among the other minimal primes of I as the instance where each of the "coefficients" λ_i equals 1.

Furthermore, if $n \leqslant 3$, I is unmixed and $I = \operatorname{Hull}(I)$. But if $n \geqslant 4$, I has one irredundant embedded $\mathfrak{m}$-primary component. This provides a very striking contrast between the cases $n \leqslant 3$ and $n \geqslant 4$. In each case we give a concrete description of these primary components (cf. Theorems 4.10 and 7.1).

We now recall briefly from [OP2] some relevant parts of the theory of ideals of Herzog–Northcott type (or HN ideals, as they are referred to). The study of HN ideals had their origin in work of Herzog [Her] on the defining ideals $\mathfrak{p}_m$ of monomial space curves Γ_m, $m \in \mathbb{N}^3$, $\gcd(m) = 1$. The ideals $\mathfrak{p}_m$, which are Cohen–Macaulay almost complete intersection ideals of height two, proved useful in work of the authors in settling a long-standing open question on an aspect of the uniform Artin–Rees property (cf. [OP1]); this work built on the observation that these ideals $\mathfrak{p}_m$ were a particular case of a class of ideals studied by Northcott [Nor].

In [OP2] we defined an HN ideal I as the determinantal ideal generated by the 2×2 minors of a certain matrix. One can easily check that HN ideals and PCB ideals are two notions that coincide when $A = k[\underline{x}]$ with $n = 3$. In [OP2, Definition 7.1 and Remark 7.2] we introduced an integer vector $m(I) = (m_1, m_2, m_3) \in \mathbb{N}^3$ associated to I. We showed that I is prime if and only if the greatest common divisor of $m(I)$ is equal to 1 [OP2, Theorem 7.8]. Further, using techniques from the theory of multiplicities, we gave upper bounds for the number of prime components of I in terms of the m_i and $\gcd(m(I))$. Finally, using a Jacobian criterion, we showed that I is radical if the characteristic of k

is zero or sufficiently large. More particularly, in [OP2, Remark 8.6] we posed the open question as to whether the number of prime components of I was at most $\gcd(m(I))$.

We now return to these matters using the Eisenbud–Sturmfels theory of binomial ideals (see [ES]), and in particular their investigation of so-called Laurent binomial ideals, to obtain a detailed positive answer to this conjecture. The Eisenbud–Sturmfels theory used here provides a more transparent approach that works for general n, and not just when $n=3$. Note that, since a PCB ideal I is binomial, so also are its isolated primary components, their intersection $\mathrm{Hull}(I)$ and, for $n \geqslant 4$, even for a suitable choice for its $\mathfrak{m}$-primary irredundant embedded component. This approach also enables us to give an analogous criterion for $\mathrm{Hull}(I)$ of a general PCB ideal I to be prime. Observe that when $n \geqslant 4$, I cannot be radical since it is not even unmixed. However, we show that, for suitable coefficient fields k, $\mathrm{Hull}(I)$ is radical; as stated above, recall that when $n \leqslant 3$, $\mathrm{Hull}(I)$ coincides with I, since I is then unmixed.

Notice also that for $n=4$, I is somewhat related to the notion of an "ideal generated by a full set of critical binomials" introduced by Alcántar and Villarreal in [AV, §3, p. 3039], although the definitions have essential differences.

For further background and recent related work from a similar perspective to ours, see [Wal,Eto] and especially [Gas], and also [Oje] and [KO].

For an alternative combinatorial approach, see the recent paper [LV] (and Remark 5.8 below) and the survey papers [KM] and [Mil]. Specifically, at the end of Section 3 of [Mil], a general programme is set out whereby binomial primary decompositions can be calculated. Substantial difficulties could present themselves as to how this programme plays out as regards particular binomial ideals and especially as regards abstractly defined classes of binomial ideals. Our point of view in the present paper is to use constructive Commutative Algebra to give explicit, concrete descriptions of binomial primary decompositions of PCB ideals. In particular, we present in the case of PCB ideals an explicit solution to the 'problem' mentioned in [DMM, Remark 3.5], in that Theorem 4.10 below provides a concrete description of the single embedded component of an irredundant binomial primary decomposition of a PCB ideal in the case $n \geqslant 4$ where this ideal is not unmixed.

The paper is organized as follows. In Section 2 we first observe that all the rows of $\mathrm{adj}(L)$, the adjoint matrix of L, are equal and lie in $\mathbb{N}^n$. Then we define the integer vector $m(I) \in \mathbb{N}^n$ associated to a PCB binomial I as the last row of $\mathrm{adj}(L)$ (see Definition 2.2). We see that this definition extends the one given in [OP2, Definition 7.1 and Remark 7.2]. Moreover, this vector $m(I)$ helps define a grading on A in which I becomes homogeneous.

In Section 3, we recover and extend properties of HN ideals, namely we show in Propositions 3.3 and 3.5 that a PCB ideal I is contained in a unique Herzog ideal, specifically $\mathfrak{p}_{m(I)}$, and that, if $n \geqslant 3$, I is an almost complete intersection. Section 4 is devoted to study the (un)mixedness property of PCB ideals. The result, stated above, is surprising: while for $n \leqslant 3$, I is unmixed and $I = \mathrm{Hull}(I)$, for $n \geqslant 4$ we find that I is never unmixed (see Remark 4.7 and Proposition 4.8). We also provide an explicit, concrete description of $\mathrm{Hull}(I)$ and, when $n \geqslant 4$ (in which case I is never unmixed), of a choice for the irredundant embedded component of I, each of these descriptions being independent of the characteristic of k (cf. Proposition 4.4 and Theorem 4.10). This gives a comprehensive and concrete solution to [ES, Problem 6.3] in the case of our binomial ideals.

In Section 5, we review the normal decomposition of an integer matrix (also called the Smith Normal Form). This will lead, on the one hand, to a change of variables that will greatly simplify the description of I. On the other hand, it relates the greatest common divisor of $m(I)$ (i.e., d, the greatest common divisor of the entries of $\mathrm{adj}(L)$) with the cardinality of the torsion group of the abelian group generated by the columns of L (Proposition 5.6).

In Section 6 we pass to the Laurent polynomial ring, apply the change of variables given by the normal decomposition of L, get a better description of I in the Laurent ring, and then contract back to the original polynomial ring (cf. Theorem 6.5). This approach also enables us in Corollary 6.6 to characterize when $\mathrm{Hull}(I)$ is prime. In turn, we can use Corollary 6.6 to show that the class of PCB ideals has minimal overlap with the class of binomial ideals, namely so-called lattice basis ideals for saturated lattices, considered by Hoşten and Shapiro in [HS] (cf. Proposition 6.7).

Finally, in the last section, we use the expression obtained in Theorem 6.5 to prove the main result of the paper: Theorem 7.1. We end by giving some illustrative examples.

Throughout the paper we will use the following notations: $A = k[\underline{x}] = k[x_1, \dots, x_n]$ will be the polynomial ring in n variables $\underline{x} = x_1, \dots, x_n$ over a field k, $n \geqslant 2$. The maximal ideal generated by $\underline{x}$ will be denoted $\mathfrak{m} = (\underline{x})$. The multiplicatively closed set in A generated by $x = x_1 \cdots x_n$, the product of the variables $x_1, \dots, x_n$, will be denoted by S, and $B = S^{-1}A = k[\underline{x}^{\pm}] = k[x_1, \dots, x_n, x_1^{-1}, \dots, x_n^{-1}]$ will be the corresponding *Laurent polynomial ring*.

We will use the following multi-index notation: for $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$, or more generally, α a row or a column of a matrix with ordered entries $\alpha_1, \dots, \alpha_n \in \mathbb{Z}$, set $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ in B. Given such an $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$, let $\alpha_+ = \max(\alpha, 0) \in \mathbb{N}_0^n$ and $\alpha_- = -\min(\alpha, 0) \in \mathbb{N}_0^n$, where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, so that $\alpha = \alpha_+ - \alpha_-$.

By a binomial in A we understand a polynomial of A with at most two terms, say $\lambda x^\alpha - \mu x^\beta$, where $\lambda, \mu \in k$ and $\alpha, \beta \in \mathbb{N}_0^n$. A binomial ideal of A is an ideal of A generated by binomials.

Unless stated otherwise, L will always be a PCB matrix, i.e., an $n \times n$ integer matrix defined as above, $\underline{f} = f_1, \dots, f_n$ will be the binomials defined by the columns of L and $I = (\underline{f})$ will the PCB ideal of A associated to L.

Given an $n \times s$ integer matrix M, we will denote by $m_{i,*}$ and $m_{*,j}$ its i-th row and j-th column, respectively. Then $f_{m_{*,j}} = x^{(m_{*,j})_+} - x^{(m_{*,j})_-}$ will denote the binomial defined by the j-th column of M. The ideal $I(M) = (f_{m_{*,j}} \mid j = 1, \dots, s)$, generated by the binomials $f_{m_{*,j}}$, will be called the binomial ideal associated to the matrix M. For instance, a PCB ideal I is the binomial ideal $I = I(L)$ associated to a PCB matrix L.

For an $n \times s$ integer matrix M, we will denote by $\mathcal{M} \subset \mathbb{Z}^n$ the subgroup spanned by the columns of M ($\mathcal{M}$ is often called a lattice of $\mathbb{Z}^n$). In other words, $\mathcal{M} = \mathbb{Z}m_{*,1} + \cdots + \mathbb{Z}m_{*,s} = \varphi(\mathbb{Z}^s) \subseteq \mathbb{Z}^n$, where $\varphi : \mathbb{Z}^s \to \mathbb{Z}^n$ is the homomorphism defined by the matrix M. The binomial ideal $I(\mathcal{M}) = (x^{m_+} - x^{m_-} \mid m \in \mathcal{M})$ is usually called the lattice ideal of A associated to $\mathcal{M}$ (see, e.g., [MS, Definition 7.2] or alternatively [ES, just before Corollary 2.5], where $I(\mathcal{M})$ is denoted by $I_+(\rho)$, $\rho : \mathcal{M} \to k^*$ being the trivial partial character on the lattice $\mathcal{M}$; see also [Vil, Corollary 7.1.4]).

By an $n \times n$ invertible integer matrix, we will understand an $n \times n$ matrix P with entries in $\mathbb{Z}$ whose determinant is ± 1. Thus its inverse matrix P^{-1} is also an integer matrix.

2. Endowing A with a grading that makes I homogeneous

Let $L_{i,j}$ be the (i,j)-cofactor of an $n \times n$ PCB matrix L, i.e., the $(n-1) \times (n-1)$ matrix obtained from L by eliminating the i-th row and the j-th column of L. Let $h_{i,j} = (-1)^{i+j} \det(L_{j,i})$ and set $H = (h_{i,j}) = \operatorname{adj}(L)$, the adjoint matrix of L. In the next result, all the computations are thought of in $\mathbb{Z}$ or $\mathbb{Q}$ (i.e., in characteristic zero) and the ranks are taken over $\mathbb{Q}$.

Lemma 2.1. *With the notations above*:

(a) $\det(L_{i,i}) > 0$, *for all* $i = 1, \dots, n$. *In particular,* $\operatorname{rank}(L) = n - 1$;
(b) $\det(L_{i,n}) = (-1)^{n-i} \det(L_{i,i})$, *for all* $i = 1, \dots, n$;
(c) $\det(L_{i,j}) = (-1)^{n-j} \det(L_{i,n})$, *for all* $i, j = 1, \dots, n$.

Moreover,

(d) $h_{i,j} > 0$, *for all* $i, j = 1, \dots, n$;
(e) $h_{i,*} = h_{n,*}$, *for all* $i = 1, \dots, n$. *In particular* $\operatorname{rank}(\operatorname{adj}(L)) = 1$;
(f) $\operatorname{Nullspace}(L^\top)$ *is generated as a* $\mathbb{Q}$*-linear subspace by* $h_{n,*}^\top$, *the transpose of the last row of* $\operatorname{adj}(L)$.

Proof. The proof of (a) follows easily from standard facts about so-called strictly diagonally dominant matrices (cf., e.g., an easy adaptation of the statement and proof of [Gas, Bemerkung 6.1, pp. 37–38] where one employs induction based on the number of rows, using row reduction). We present here another proof based on a general fact about the eigenvalues of such matrices. Fix $i \in \{1, \dots, n\}$. By the Gershgorin Circle Theorem, every (possibly complex) eigenvalue λ of $L_{i,i}$ lies within at least one of the discs $\{z \in \mathbb{C} \mid |z - a_{j,j}| \leqslant R_j\}$, $j \neq i$, where $R_j = \sum_{u \neq i,j} |-a_{j,u}| < a_{j,j}$ since $L_{i,i}$ is a strictly diagonally dominant matrix. If $\lambda \in \mathbb{R}$, then $\lambda > 0$. If $\lambda \notin \mathbb{R}$, then since $L_{i,i}$ is a real matrix, its conjugate $\bar{\lambda}$ must also be an eigenvalue of $L_{i,i}$. By means of the Jordan canonical form of $L_{i,i}$, one deduces that $\det(L_{i,i}) > 0$. Clearly $(1,\dots,1)^\top$ is in the nullspace of L and (a) holds.

Fix $i \in \{1,\dots,n-1\}$. By performing $n-1-i$ permutations, the i-th column of $L_{i,n}$ may be taken to the outer right-hand side. Add to this new right hand column the sum of the other columns and change the sign. Using that the sum of the entries of each row of L is zero, one gets in the outer right hand column the n-th column of $L_{i,i}$. Therefore $\det(L_{i,n}) = (-1)^{n-i}\det(L_{i,i})$. This proves (b).

Let $j \in \{1,\dots,n-1\}$. Since the sum of the entries of each row of L is zero, to calculate $\det(L_{i,j})$ one can substitute the last column of $L_{i,j}$ by the corresponding j-th column of $L_{i,n}$ with the sign changed. By performing $n-1-j$ permutations, one gets the matrix $L_{i,n}$. Therefore $\det(L_{i,j}) = (-1)^{n-j}\det(L_{i,n})$. This proves (c).

For $i, j = 1,\dots,n$, using (c), (b) and (a), respectively, we have

$$h_{i,j} = (-1)^{i+j}\det(L_{j,i}) = (-1)^{i+j+n-i}\det(L_{j,n}) = (-1)^{i+j+n-i+n-j}\det(L_{j,j}) = \det(L_{j,j}).$$

This proves (d).

For $j \in \{1,\dots,n-1\}$, using (c),

$$h_{i,j} = (-1)^{i+j}\det(L_{j,i}) = (-1)^{i+j}(-1)^{n-i}\det(L_{j,n}) = (-1)^{n+j}\det(L_{j,n}) = h_{n,j}.$$

Therefore all the rows of $\operatorname{adj}(L)$ are equal. In particular, since $\operatorname{adj}(L) \neq 0$ by (d), we see that $\operatorname{rank}(\operatorname{adj}(L)) = 1$. This proves (e).

Since $\operatorname{rank}(L^\top) = \operatorname{rank}(L) = n-1$, we have that $\dim \operatorname{Nullspace}(L^\top) = 1$. Furthermore, since $\operatorname{adj}(L)L = 0$, the transpose of the (non-zero) last row of $\operatorname{adj}(L)$ generates the $\mathbb{Q}$-linear subspace $\operatorname{Nullspace}(L^\top)$. $\square$

As before, set $A = k[\underline{x}]$, the polynomial ring in n variables $\underline{x} = x_1,\dots,x_n$ over a field k, $n \geqslant 2$.

Definition 2.2. Let $I = (\underline{f})$ be the PCB ideal associated to L. Let $m = m(I) = (m_1,\dots,m_n)$ be the n-th row of $\operatorname{adj}(L)$; this will be called the *integer vector associated to* I. By the previous lemma, $m(I) \in \mathbb{N}^n$ and $m(I)^\top$ is a basis of the $\mathbb{Q}$-linear subspace $\operatorname{Nullspace}(L^\top)$. We will denote by d the greatest common divisor of the coefficients of $m(I)$, $d := \gcd(m(I))$, and set $\nu(I) = m(I)/d = (\nu_1,\dots,\nu_n) \in \mathbb{N}^n$. From now on, given a PCB ideal I of A, we will endow A with the natural grading induced by giving x_i weight ν_i. Then A is graded by $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, and $\underline{x}$ and $\underline{f}$ are homogeneous elements of positive degree. In particular, I is homogeneous. Hence so are its isolated primary components and its associated primes, and an irredundant embedded primary component may be chosen homogeneous (see, e.g., [ZS, Ch. VII, §2, Theorem 9 and Corollary, pp. 153–154]).

Remark 2.3. For $n = 2$ we have $I = (f_1, f_2)$, where $f_1 = x_1^{a_{1,1}} - x_2^{a_{2,1}}$ and $f_2 = x_2^{a_{2,2}} - x_1^{a_{1,2}}$, with $a_{1,1} = a_{1,2}$ and $a_{2,2} = a_{2,1}$. Thus $f_2 = -f_1$ and $I = (f_1)$ is a complete intersection. In particular, I is unmixed. Here, $m(I) = (a_{2,2}, a_{1,1}) \in \mathbb{N}^2$ and $d = \gcd(m(I)) = \gcd(a_{1,1}, a_{2,2})$.

Remark 2.4. For $n = 3$ we have $I = (f_1, f_2, f_3)$, where $f_1 = x_1^{a_{1,1}} - x_2^{a_{2,1}}x_3^{a_{3,1}}$, $f_2 = x_2^{a_{2,2}} - x_1^{a_{1,2}}x_3^{a_{3,2}}$ and $f_3 = x_3^{a_{3,3}} - x_1^{a_{1,3}}x_2^{a_{2,3}}$, with $a_{1,1} = a_{1,2} + a_{1,3}$, $a_{2,2} = a_{2,1} + a_{2,3}$ and $a_{3,3} = a_{3,1} + a_{3,2}$. Observe that f_1, f_2, f_3 are, up to sign, the 2×2 minors of the matrix

$$\begin{pmatrix} x_1^{a_{1,2}} & x_2^{a_{2,3}} & x_3^{a_{3,1}} \\ x_2^{a_{2,1}} & x_3^{a_{3,2}} & x_1^{a_{1,3}} \end{pmatrix}.$$

It follows that when $n = 3$, PCB ideals are precisely the ideals of Herzog–Northcott type, or HN ideals for short, considered in [OP2]. In the proof of [OP2, Remark 4.4], there appear positive integers m_1, m_2, m_3 presented as the 2×2 minors of the matrix defining the exponents of f_1 and f_2.

Subsequently, in [OP2, Definition 7.1 and Remark 7.2], (m_1, m_2, m_3) is defined as the integer vector associated to the Herzog–Northcott ideal I. In conclusion, one can easily check that, when $n = 3$, the present definition of $m(I)$ coincides with the one given in [OP2, Definition 7.1 and Remark 7.2].

Remark 2.5. It is a long-standing open problem to find a minimal generating set for the defining ideals $\mathfrak{p}_m$ of monomial curves Γ_m, $m \in \mathbb{N}^n$, $\gcd(m) = 1$, and to decide whether the $\mathfrak{p}_m$ are set theoretically complete intersections. For $n = 3$, the problem was completely solved by Herzog in [Her]. For $n = 4$, and provided that $\mathfrak{p}_m$ is an almost complete intersection, Gastinger in [Gas] and Eto in [Eto] gave a definitive answer. In an attempt to study this problem for $n = 4$, Alcántar and Villarreal defined in [AV] what they called a *full set of critical binomials* as a set of four binomials $f_1, f_2, f_3, f_4 \in \mathfrak{p}_m$, where $m = (m_1, m_2, m_3, m_4) \in \mathbb{N}^4$, $m_1 < m_2 < m_3 < m_4$ and $\gcd(m) = 1$. The f_i were respectively defined as in our introduction, namely

$$x_1^{a_{1,1}} - x_2^{a_{2,1}} x_3^{a_{3,1}} x_4^{a_{4,1}}, \qquad x_2^{a_{2,2}} - x_1^{a_{1,2}} x_3^{a_{3,2}} x_4^{a_{4,2}}, \qquad x_3^{a_{3,3}} - x_1^{a_{1,3}} x_2^{a_{2,3}} x_4^{a_{4,3}}, \qquad x_4^{a_{4,4}} - x_1^{a_{1,4}} x_2^{a_{2,4}} x_3^{a_{3,4}},$$

but with $a_{i,i} > 0$ and $a_{i,j} \in \mathbb{N}_0$, and such that $a_{i,i}$ is minimal with respect to the condition $a_{i,i} m_i \in \sum_{j \neq i} m_j \mathbb{N}_0$. They then studied when the ideal generated by the f_i is the whole of $\mathfrak{p}_m$. As is clear, our definition of PCB ideal for $n = 4$ does not exactly match their definition. On the one hand, we do not allow zero exponents, and on the other hand we do not ask for the above minimal condition or for restrictions on the m_i.

3. First properties of PCB ideals

Set $A = k[\underline{x}]$ to be the polynomial ring in n variables $\underline{x} = x_1, \ldots, x_n$ over a field k, $n \geqslant 2$. We start this section by recovering a definition from [OP2].

Definition 3.1. Let $u = (u_1, \ldots, u_n) \in \mathbb{N}^n$ be an integer vector with greatest common divisor not necessarily equal to 1. The *Herzog ideal associated to* u is the prime ideal $\mathfrak{p}_u$ defined as the kernel of the natural homomorphism $\varphi_u : A \to k[t]$ that sends x_i to t^{u_i}, for each $i = 1, \ldots, n$.

The following is a list of well-known properties of Herzog ideals, with a sketched proof for the sake of completeness.

Remark 3.2. Let $u \in \mathbb{N}^n$. The extension $k[t^{u_1}, \ldots, t^{u_n}] \subset k[t]$ is integral. Hence $A/\mathfrak{p}_u \cong k[t^{u_1}, \ldots, t^{u_n}]$ has Krull dimension 1 and $\mathfrak{p}_u$ is a prime ideal of height $n - 1$. Since $0 \in V(\mathfrak{p}_u) \subseteq \mathbb{A}_k^n$, where $V(\mathfrak{p}_u)$ denotes the affine set of zeros over k of $\mathfrak{p}_u$, $\mathfrak{m} = I(\{0\}) \supseteq I(V(\mathfrak{p}_u)) \supseteq \mathfrak{p}_u$ and $\mathfrak{p}_u \subsetneq \mathfrak{m}$. Moreover, if $v \in \mathbb{N}^n$ is such that $u = dv$ for some $d \in \mathbb{N}$, clearly $\mathfrak{p}_u \supseteq \mathfrak{p}_v$ and, by the equality of heights, $\mathfrak{p}_u = \mathfrak{p}_v$.

We claim that if $\gcd(u) = 1$, then $V(\mathfrak{p}_u) = \Gamma_u := \{(\lambda^{u_1}, \ldots, \lambda^{u_n}) \in \mathbb{A}_k^n \mid \lambda \in k\}$ (see [RVZ, Proposition 2.9]). Clearly $\Gamma_u \subseteq V(\mathfrak{p}_u)$. Note that for $i = 2, \ldots, n$, $x_1^{u_i} - x_i^{u_1}$ is in $\mathfrak{p}_u$. Hence if $(\lambda_1, \ldots, \lambda_n) \in V(\mathfrak{p}_u) \setminus \{0\}$, then each $\lambda_i \neq 0$ and, taking $\lambda := \lambda_1^{c_1} \cdots \lambda_n^{c_n}$ where $c_1 u_1 + \cdots + c_n u_n = 1$ with $c_i \in \mathbb{Z}$, one has $\lambda^{u_i} = \lambda_i$ and hence $(\lambda_1, \ldots, \lambda_n) \in \Gamma_u$.

Moreover, $\mathfrak{p}_u = (x_1 - t^{u_1}, \ldots, x_n - t^{u_n}) \cap A$, where the ideal $(x_1 - t^{u_1}, \ldots, x_n - t^{u_n})$ is considered in $A[t] = k[x_1, \ldots, x_n, t]$. Indeed, if $f \in \mathfrak{p}_u$,

$$\begin{aligned} f &= \sum a_\alpha x^\alpha = \sum a_\alpha \left(x_1 - t^{u_1} + t^{u_1}\right)^{\alpha_1} \cdots \left(x_n - t^{u_n} + t^{u_n}\right)^{\alpha_n} \\ &= g + \sum a_\alpha \left(t^{u_1}\right)^{\alpha_1} \cdots \left(t^{u_n}\right)^{\alpha_n} = g + f\left(t^{u_1}, \ldots, t^{u_n}\right) = g + \varphi_u(f) = g, \end{aligned}$$

where $g \in (x_1 - t^{u_1}, \ldots, x_n - t^{u_n})$. Thus, $f = g \in (x_1 - t^{u_1}, \ldots, x_n - t^{u_n}) \cap A$. The other inclusion follows easily. In particular, by [ES, Corollary 1.3], $\mathfrak{p}_u$ is a binomial ideal.

Finally, if k is infinite and $\gcd(u) = 1$, we claim that $\mathfrak{p}_u = I(\Gamma_u)$, the vanishing ideal of Γ_u. On the one hand, since $\Gamma_u = V(\mathfrak{p}_u)$, $I(\Gamma_u) = I(V(\mathfrak{p}_u)) \supseteq \mathfrak{p}_u$. On the other hand, let $f \in I(\Gamma_u) \subset A \subset A[t]$. The argument above shows that $f(\underline{x}) = g(\underline{x}, t) + r(t)$, with $g \in (x_1 - t^{u_1}, \dots, x_n - t^{u_n}) \subset A[t]$ and $r \in k[t]$. For any $\lambda \in k$, evaluate x_i in λ^{u_i} and t in λ. Then $0 = f(\lambda^{u_1}, \dots, \lambda^{u_n}) = g(\lambda^{u_1}, \dots, \lambda^{u_n}, \lambda) + r(\lambda) = r(\lambda)$. Thus $r(\lambda) = 0$ for all $\lambda \in k$. Since k is infinite, $r = 0$ and $f(\underline{x}) = g(\underline{x}, t) \in (x_1 - t^{u_1}, \dots, x_n - t^{u_n}) \cap A = \mathfrak{p}_u$.

The next result gives us the first properties of a PCB ideal.

Proposition 3.3. *Let $I = (\underline{f})$ be the PCB ideal associated to L. Then the following hold.*

(a) *Any subset of $n-1$ elements of $\underline{f}$ is a regular sequence in A.*
(b) *$\mathfrak{p}_{m(I)}$ is the unique Herzog ideal containing I and is a minimal prime over I. In particular,* $\operatorname{height}(I) = n-1$.
(c) *If $n = 2$, I is principal. If $n \geqslant 3$, $f_1, \dots, f_n$ is a minimal (homogeneous) system of generators of I and every (non-necessarily homogeneous) system of generators of I has at least n elements.*

Proof. Since $(f_1, \dots, f_{n-1}, x_n) = (x_1^{a_{1,1}}, \dots, x_{n-1}^{a_{n-1,n-1}}, x_n)$, the grades of these ideals are equal and coincide with $\operatorname{grade}(x_1, \dots, x_n) = n$ (see, e.g., [Kap, Exercise 3.1.12(c)]). Using that A is graded and that $f_1, \dots, f_{n-1}, x_n$ are homogeneous, we deduce that these elements form a regular sequence in any order (see, e.g., [OP2, Theorem 4.1]) (and similarly for the possible variations on this argument). This proves (a).

Given $v \in \mathbb{N}^n$, clearly $I \subseteq \mathfrak{p}_v$ if and only if v satisfies the system of equations $vL = 0$, i.e., if and only if $v^\top$ is in the nullspace of $L^\top$, which by Lemma 2.1 and Definition 2.2 is the $\mathbb{Q}$-linear subspace generated by $m(I)^\top$. Therefore $I \subseteq \mathfrak{p}_{m(I)}$. Since $n-1 \leqslant \operatorname{grade}(I) = \operatorname{height}(I) \leqslant \operatorname{height}(\mathfrak{p}_{m(I)}) = n-1$, $\mathfrak{p}_{m(I)}$ is a minimal prime over I and $\operatorname{height}(I) = n-1$. On the other hand, if $I \subseteq \mathfrak{p}_v$, for some $v \in \mathbb{N}^n$, then $vL = 0$ and $rv = sm(I)$, with $r, s \in \mathbb{N}$. Hence $\mathfrak{p}_v = \mathfrak{p}_{rv} = \mathfrak{p}_{sm(I)} = \mathfrak{p}_{m(I)}$.

Suppose that $n \geqslant 3$. We see first that $f_1, \dots, f_n$ is a minimal homogeneous system of generators of I in the sense that none of them is irredundant. For, if f_n were redundant, say, since $n \geqslant 3$, $I = (f_1, \dots, f_{n-1}) \subseteq (x_1, \dots, x_{n-1})$ and $f_n = g_1 x_1 + \cdots + g_{n-1} x_{n-1}$, for some $g_i \in A$. Substituting $x_1, \dots, x_{n-1}$ by 0 and x_n by 1, one would get a contradiction. By [BH, Proposition 1.5.15], every minimal homogeneous system of generators of I has exactly $\mu(I_\mathfrak{m})$ elements. Hence $n = \mu(I_\mathfrak{m})$. Finally, if $h_1, \dots, h_r$ is a minimal (non-necessarily homogeneous) system of generators of I, $h_1, \dots, h_r$ certainly lie in $\mathfrak{m}$, and $h_1, \dots, h_r$ in $A_\mathfrak{m}$ still generate $I_\mathfrak{m}$. Thus $r \geqslant \mu(I_\mathfrak{m}) = n$. $\square$

Remark 3.4. Similarly to [OP2, Remark 6.2], we can show a relation among $f_1, \dots, f_n$. Concretely, $x^{b(1)} f_1 + \cdots + x^{b(n)} f_n = 0$, where the $b(i) \in \mathbb{N}_0^n$ are defined as follows:

$$b(1) = (0, 0, a_{3,3} - a_{3,4} - \cdots - a_{3,n} - a_{3,1}, a_{4,4} - a_{4,5} - \cdots - a_{4,n} - a_{4,1}, \dots, a_{n,n} - a_{n,1}),$$

$$b(2) = (a_{1,1} - a_{1,2}, 0, 0, a_{4,4} - a_{4,5} - \cdots - a_{4,n} - a_{4,1} - a_{4,2}, \dots, a_{n,n} - a_{n,1} - a_{n,2}),$$

$$b(3) = (a_{1,1} - a_{1,2} - a_{1,3}, a_{2,2} - a_{2,3}, 0, 0, \dots, a_{n,n} - a_{n,1} - a_{n,2} - a_{n,3}), \dots,$$

$$b(n-1) = (a_{1,1} - a_{1,2} - \cdots - a_{1,n-1}, \dots, a_{n-2,n-2} - a_{n-2,n-1}, 0, 0) \quad \text{and}$$

$$b(n) = (0, a_{2,2} - a_{2,3} - \cdots - a_{2,n}, a_{3,3} - a_{3,4} - \cdots - a_{3,n}, \dots, a_{n-1,n-1} - a_{n-1,n}, 0).$$

For instance, when $n = 2$, $b(1) = b(2) = (0, 0)$ and $x^{b(1)} f_1 + x^{b(2)} f_2 = f_1 + f_2$, which is certainly zero. For $n = 3$, since the sum of the entries of each row is zero, $b(1) = (0, 0, a_{3,2})$, $b(2) = (a_{1,3}, 0, 0)$ and $b(3) = (0, a_{2,1}, 0)$. Thus $x^{b(1)} f_1 + x^{b(2)} f_2 + x^{b(3)} f_3 = x_3^{a_{3,2}} f_1 + x_1^{a_{1,3}} f_2 + x_2^{a_{2,1}} f_3 = 0$, which is (up to sign) the second syzygy in [OP2, Remark 6.2]. For $n = 4$, we have

$$x_3^{a_{3,2}} x_4^{a_{4,2} + a_{4,3}} f_1 + x_4^{a_{4,3}} x_1^{a_{1,3} + a_{1,4}} f_2 + x_1^{a_{1,4}} x_2^{a_{2,4} + a_{2,1}} f_3 + x_2^{a_{2,1}} x_3^{a_{3,1} + a_{3,2}} f_4 = 0.$$

With respect to the property of being an almost complete intersection (in the sense of Herrmann, Moonen and Villamayor [HMV]), we have a result similar to that of [OP2, Proposition 6.3].

Proposition 3.5. *Let $I = (\underline{f})$ be the PCB ideal associated to L. Then the following hold.*

(a) *For any associated prime $\mathfrak{p}$ of I, either* $\operatorname{height}(\mathfrak{p}) = n-1$ *and $x_i \notin \mathfrak{p}$, for all $i = 1, \ldots, n$, or else $\mathfrak{p} = \mathfrak{m}$.*
(b) *For any minimal prime ideal $\mathfrak{p}$ over I, $IA_{\mathfrak{p}}$ is a complete intersection.*
(c) *If $n = 2$, I is a complete intersection. If $n \geqslant 3$, I is an almost complete intersection.*

Proof. Let $\mathfrak{p}$ an associated prime of I. Since I is homogeneous, $\mathfrak{p}$ is homogeneous too and hence $\mathfrak{p} \subseteq \mathfrak{m}$ (see, e.g., [BH, §1.5]). If $\mathfrak{p} \subsetneq \mathfrak{m}$, since $\operatorname{height}(I) = n-1$, then $\operatorname{height}(\mathfrak{p}) = n-1$ too. Moreover, for each i, $x_i \notin \mathfrak{p}$, otherwise $(\underline{f}, x_i) \subseteq \mathfrak{p}$ and $\mathfrak{p} = \mathfrak{m}$.

Let $\mathfrak{p}$ be a minimal prime over I, so in particular $\mathfrak{p} \neq \mathfrak{m}$ (because $I \subseteq \mathfrak{p}_{m(I)} \subsetneq \mathfrak{m}$). Thus $x_i \notin \mathfrak{p}$, for all $i = 1, \ldots, n$. Using Remark 3.4, and with $x = x_1 \cdots x_n$ as before, $IA_x = (f_1, \ldots, f_{n-1})A_x$ and $IA_{\mathfrak{p}} = (IA_x)A_{\mathfrak{p}} = (f_1, \ldots, f_{n-1})A_{\mathfrak{p}}$, where $f_1, \ldots, f_{n-1}$ is a regular sequence in $A_{\mathfrak{p}}$.

Finally, if $n = 2$, I is a complete intersection (cf. Remark 2.3). If $n \geqslant 3$, by Proposition 3.3(a), (c), I has height $n-1$ and is minimally generated by n elements. Since I is locally a complete intersection at its minimal primes, I is an almost complete intersection. $\square$

4. On the (un)mixedness property of PCB ideals

Let S be the multiplicatively closed set in $A = k[\underline{x}]$ generated by $x = x_1 \cdots x_n$. Let $B = S^{-1}A = k[\underline{x}^{\pm}] = k[x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}]$ be the Laurent polynomial ring. As usual, if I is an ideal of A, IB will denote its extension in B, and, if J is an ideal of B, $J \cap A = J^c$ will denote its contraction in A. We will also use the notation $S(I) = IB \cap A$ for the contraction of the extension of an ideal I of A.

Following the notation in [ES, p. 31], we write $\operatorname{Hull}(I)$ for the intersection of the isolated primary components of I.

Note that, if $\alpha \in \mathbb{N}_0^n$, and according to our multi-index notation, x^α is not normally a power of $x = x_1 \cdots x_n$ but rather is a monomial in $x_1, \ldots, x_n$. This monomial x^α is indeed a unit in the localized ring A_x, since A_x equals the Laurent ring $B = A[x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}]$.

The next (standard) result helps to describe the associated primes of I in terms of the associated primes of IB, its extension in B.

Proposition 4.1. *Let I be a PCB ideal of A. Then the following hold.*

(a) $S(I) = \operatorname{Hull}(I)$.
(b) *Either I is unmixed and $I = S(I)$, or else $I = S(I) \cap \mathfrak{Q}$, where $\mathfrak{Q}$ is $\mathfrak{m}$-primary and this intersection is irredundant.*
(c) *If $\alpha \in \mathbb{N}_0^n \setminus \{0\}$, $S(I) = I : (x^\alpha)^\infty := \{f \in A \mid f x^{N\alpha} \in I,$ for some $N \gg 0\}$.*
(d) *Suppose that $IB = \mathfrak{b}_1 \cap \cdots \cap \mathfrak{b}_r$ is a minimal primary decomposition of IB in B. Then $S(I) = \mathfrak{b}_1^c \cap \cdots \cap \mathfrak{b}_r^c$ is a minimal primary decomposition of $S(I)$ in A and* $\operatorname{rad}(\mathfrak{b}_i^c) = \operatorname{rad}(\mathfrak{b}_i)^c$.

Proof. By Proposition 3.5(a), I has a minimal primary decomposition either of the form $I = \mathfrak{a}_1 \cap \cdots \cap \mathfrak{a}_r$, or else $I = \mathfrak{a}_1 \cap \cdots \cap \mathfrak{a}_r \cap \mathfrak{Q}$, where the $\mathfrak{a}_j$ are $\mathfrak{p}_j$-primary ideals with $\operatorname{height}(\mathfrak{p}_j) = n-1$, and $\mathfrak{Q}$ is $\mathfrak{m}$-primary. In particular, $x_i \notin \mathfrak{p}_j$ for each i. Therefore $IB = \mathfrak{a}_1 B \cap \cdots \cap \mathfrak{a}_r B$ is a minimal primary decomposition of IB in B and $S(I) = \mathfrak{a}_1 \cap \cdots \cap \mathfrak{a}_r$, which is precisely equal to $\operatorname{Hull}(I)$.

Moreover, either I is unmixed and $I = S(I)$, or else $I = S(I) \cap \mathfrak{Q}$, where $\mathfrak{Q}$ is $\mathfrak{m}$-primary and this intersection is irredundant. This proves (b).

If $I = \mathfrak{a}_1 \cap \cdots \cap \mathfrak{a}_r$, $\alpha \in \mathbb{N}_0^n \setminus \{0\}$ and $N \gg 0$, $I : x^{N\alpha} = \bigcap_{j=1}^r (\mathfrak{a}_j : x^{N\alpha}) = \bigcap_{j=1}^r \mathfrak{a}_j$, because for all i, j, $x_i \notin \mathfrak{p}_j = \operatorname{rad}(\mathfrak{a}_j)$ and $\mathfrak{a}_j$ is $\mathfrak{p}_j$-primary. On the other hand, if $I = \mathfrak{a}_1 \cap \cdots \cap \mathfrak{a}_r \cap \mathfrak{Q}$ and $N \gg 0$, then $I : x^{N\alpha} = \bigcap_{j=1}^r (\mathfrak{a}_j : x^{N\alpha}) \cap (\mathfrak{Q} : x^{N\alpha}) = \bigcap_{j=1}^r \mathfrak{a}_j$ again, because $\operatorname{rad}(\mathfrak{Q}) = \mathfrak{m}$ and $\mathfrak{Q} : x^{N\alpha} = A$, for $N \gg 0$. Thus, in both cases, $I : x^{N\alpha} = \mathfrak{a}_1 \cap \cdots \cap \mathfrak{a}_r = S(I)$ when $N \gg 0$.

Finally, if $IB = \mathfrak{b}_1 \cap \cdots \cap \mathfrak{b}_r$ is a minimal primary decomposition of IB in B, then $S(I) = \mathfrak{b}_1^c \cap \cdots \cap \mathfrak{b}_r^c$ is a primary decomposition of $S(I)$ in A, where $\operatorname{rad}(\mathfrak{b}_i^c) = \operatorname{rad}(\mathfrak{b}_i)^c$. Moreover, if $\mathfrak{b}_1^c \supseteq \mathfrak{b}_2^c \cap \cdots \cap \mathfrak{b}_r^c$, say, then, since $S^{-1}A$ is a flat extension of A, $\mathfrak{b}_1 = \mathfrak{b}_1^{ce} \supseteq \mathfrak{b}_2^{ce} \cap \cdots \cap \mathfrak{b}_r^{ce} = \mathfrak{b}_2 \cap \cdots \cap \mathfrak{b}_r$, a contradiction. Therefore $S(I) = \mathfrak{b}_1^c \cap \cdots \cap \mathfrak{b}_r^c$ is a minimal primary decomposition. $\square$

Before proceeding we state, for the sake of reference, a list of well-known properties of lattice ideals.

Proposition 4.2. *Let M be an $n \times s$ integer matrix and let $\mathcal{M} \subseteq \mathbb{Z}^n$ be the lattice spanned by the columns of M. Let $I(M) = (x^{(m_{*,j})_+} - x^{(m_{*,j})_-} \mid j = 1, \ldots, s)$ be the ideal of A generated by the binomials defined by the columns of M and let $I(\mathcal{M}) = (x^u - x^v \mid u, v \in \mathbb{N}_0^n,\ u - v \in \mathcal{M})$ be the lattice ideal of A associated to $\mathcal{M}$. The following hold:*

(a) *$I(M) \subseteq I(\mathcal{M})$ and $I(\mathcal{M}) = I(M) : x^\infty$. In particular, $I(\mathcal{M})B \cap A = I(\mathcal{M})$;*
(b) *$I(M)B \equiv (x^{m_{*,j}} - 1 \mid j = 1, \ldots, s)B$ coincides with $I(\mathcal{M})B \equiv (x^\alpha - 1 \mid \alpha \in \mathcal{M})B$;*
(c) *Given $\alpha \in \mathbb{Z}^n$, $\alpha \in \mathcal{M}$ if and only if $x^\alpha - 1 \in I(M)B$;*
(d) *If N is an $n \times r$ integer matrix with $I(M) = I(N)$, then $\mathcal{M} = \mathcal{N}$;*
(e) *Let Q be an $s \times s$ invertible integer matrix. If $MQ = T$, then $I(M)B = I(T)B$.*

Proof. The containment at the beginning of (a) is clear and the first equality is [MS, Lemma 7.6]. In particular, $I(\mathcal{M})B \cap A = I(\mathcal{M})$, because for any ideal J of A, $JB \cap A = J : x^\infty$. Since the x_i are invertible in the Laurent polynomial ring $B = S^{-1}A$, which is a flat A-module, $I(\mathcal{M})B = (I(M) : x^\infty)B = I(M)B$. This proves (b). If $\alpha \in \mathcal{M}$, then $x^\alpha - 1 \in I(\mathcal{M})B = I(M)B$, by item (b). Conversely, take $x^\alpha - 1 \in I(M)B = I(\mathcal{M})B$. Let $\rho : \mathcal{M} \to k^*$ be the trivial character and $L_\rho = \mathcal{M}$. Following the notation in [ES, §2], $I(\mathcal{M})B$ is the Laurent binomial ideal $I(\rho)$. The argument in the second paragraph of [ES, Theorem 2.1(a), p. 13, last line] shows that $\alpha \in \mathcal{M}$. This proves (c). Suppose now $I(M) = I(N)$ and take $\alpha \in \mathcal{M}$. Then, by (c), $x^\alpha - 1 \in I(M)B = I(N)B$. By (c) again, this implies that $\alpha \in \mathcal{N}$, so that $\mathcal{M} \subseteq \mathcal{N}$. Analogously, $\mathcal{N} \subseteq \mathcal{M}$. This proves (d). Finally, if $MQ = T$ with Q invertible, then $\mathcal{M} = \mathcal{T}$ and, by (b), $I(M)B = I(\mathcal{M})B = I(\mathcal{T})B = I(T)B$. $\square$

With this terminology, we see that Proposition 4.1(c) says that the hull of a PCB ideal is the lattice ideal of the lattice spanned by the columns of the PCB matrix. That is, in concrete terms, we have the following.

Corollary 4.3. *Let I the PCB ideal of A associated to L. Then $S(I) = I(\mathcal{L})$, where $\mathcal{L} \subseteq \mathbb{Z}^n$ is the lattice spanned by the columns of L.*

Proof. By Proposition 4.1(c), with $\alpha = (1, \ldots, 1)$, and Proposition 4.2(a), $S(I) = I(L) : x^\infty = I(\mathcal{L})$. $\square$

We give now an explicit description of $S(I)$ and thus of $\mathrm{Hull}(I)$ (see [ES, Problem 6.3]).

Proposition 4.4. *Let $I = (\underline{f}) = (f_1, \ldots, f_n)$ be a PCB ideal and set $J = (f_1, \ldots, f_{n-1})$. Set $b(n) = (0, a_{2,2} - a_{2,3} - \cdots - a_{2,n}, \ldots, a_{n-1,n-1} - a_{n-1,n}, 0)$. Then $S(I) = I : x^{b(n)} = J : x^{b(n)}$.*

Proof. By Proposition 3.3(a), $f_1, \ldots, f_{n-1}$ is a regular sequence in A. Hence J is a complete intersection and an unmixed ideal of height $n-1$.

If $n = 2$, I is principal and unmixed, and $J = I = S(I)$. Moreover, $b(n) = (0, 0)$ and $J : x^{b(n)} = I : x^{b(n)} = S(I)$.

Set $n \geqslant 3$, so $b(n) \neq 0$. By Remark 3.4, $x^{b(n)} f_n \in J$. Hence $x^{b(n)} I \subseteq J$. By Proposition 4.1(c),

$$I \subseteq J : x^{b(n)} \subseteq I : x^{b(n)} \subseteq I : \left(x^{b(n)}\right)^\infty = S(I).$$

In particular, $J : x^{b(n)}$ is a proper ideal. By the properties of the colon operation vis-à-vis intersection of ideals, since J is unmixed, it follows that $J : x^{b(n)}$ is unmixed with associated primes a (non-empty) subset of the primes associated to J, and hence each of height $n-1$.

Moreover, if $\mathfrak{p}$ is an associated prime of $J : x^{b(n)}$, since $I \subseteq J : x^{b(n)}$, then $I \subseteq \mathfrak{p}$ and, since $\mathrm{height}(\mathfrak{p}) = n-1$, $\mathfrak{p}$ is a minimal prime over I. In particular, $x^{b(n)} \notin \mathfrak{p}$ and $(J : x^{b(n)})_\mathfrak{p} = J_\mathfrak{p}$.

Therefore, for any associated prime $\mathfrak{p}$ of $J : x^{b(n)}$ (so that $\mathfrak{p}$ is a minimal prime over I),

$$I_{\mathfrak{p}} \subseteq \left(J : x^{b(n)}\right)_{\mathfrak{p}} = J_{\mathfrak{p}} \subseteq I_{\mathfrak{p}} = S(I)_{\mathfrak{p}}.$$

Hence $(J : x^{b(n)})_{\mathfrak{p}} = S(I)_{\mathfrak{p}}$ for all associated primes $\mathfrak{p}$ of $J : x^{b(n)}$, so $J : x^{b(n)} = I : x^{b(n)} = S(I)$. $\square$

The next result is a kind of ad hoc "unmixedness test". For a more general result, see the Unmixedness Test of W.V. Vasconcelos in [Vas, p. 76].

Corollary 4.5. *Let I be a PCB ideal of A. Then the following conditions are equivalent.*

(i) *I is unmixed*;
(ii) *Each of $x_1, \dots, x_n$ is regular modulo I*;
(iii) *$I = I : x_1$.*

Proof. If I is unmixed and if x_i were a zero-divisor modulo I, then x_i would be in an associated prime $\mathfrak{p}$ of I and $\mathfrak{p}$ would be equal to $\mathfrak{m}$, a contradiction. If $I = I : x_1$, then clearly $I = I : x_1^{\infty}$. By Proposition 4.1(c), $S(I) = I : x_1^{\infty}$. Thus $I = S(I)$ and, by Proposition 4.1(b), I is unmixed. $\square$

Let us state the last result in terms of lattice ideals (cf. also [ES, Corollary 2.5] or [LV, Theorem 3.2]).

Corollary 4.6. *Let I be a PCB ideal of A. Then I is unmixed if and only if I is a lattice ideal.*

Proof. If I is unmixed, by Proposition 4.1(b), $I = S(I)$ and, by Corollary 4.3, $S(I)$ is a lattice ideal. Conversely, if $I = I(\mathcal{M})$ is a lattice ideal, then $S(I) = IB \cap A = I(\mathcal{M})B \cap A$, which, by Proposition 4.2, is equal to $I(\mathcal{M}) = I$. Hence, $S(I) = I$ and, by Proposition 4.1(b), I is unmixed. $\square$

Remark 4.7. Let I be a PCB ideal of A. If $n \leqslant 3$, I is unmixed. This follows from Remark 2.3 for the case $n = 2$, and the fact that, for $n = 3$, PCB ideals are ideals of Herzog–Northcott type (cf. [OP2, Proposition 2.2(b)]).

Proposition 4.8. *Let $I = (\underline{f})$ be a PCB ideal of A, $n \geqslant 4$. Set $g_1 = x_2^{a_{2,1}} x_3^{a_{3,1}} \cdots x_{n-1}^{a_{n-1,1}}$ and $g_2 = x_2^{a_{2,n}} x_3^{a_{3,n}} \cdots x_{n-1}^{a_{n-1,n}}$. Let $g = x_1^{a_{1,1}-1} x_n^{a_{n,n}-a_{n,1}} - x_1^{a_{1,n}-1} g_1 g_2$. Then $g \in (I : x_1) \setminus I$. In particular, I is not unmixed.*

Proof. It is easy to check that $x_1 g = x_n^{a_{n,n}-a_{n,1}} f_1 + g_1 f_n$. Moreover, if $g \in I$, setting $x_i = 0$ for $i = 2, \dots, n-1$, it would follow that $x_1^{a_{1,1}-1} x_n^{a_{n,n}-a_{n,1}}$ lies in $(x_1^{a_{1,1}}, x_n^{a_{n,n}})k[x_1, x_n]$, a contradiction. Thus $g \in (I : x_1) \setminus I$. By Corollary 4.5, I is not unmixed. (Observe that the condition $n \geqslant 4$ is essential, for if $n = 3$, the ideal obtained from I when substituting x_2 by 0 is $(x_1^{a_{1,1}}, x_1^{a_{1,2}} x_3^{a_{3,2}}, x_3^{a_{3,3}})k[x_1, x_3]$.) $\square$

Example 4.9. Let $I = (x_1^3 - x_2x_3x_4, x_2^3 - x_1x_3x_4, x_3^3 - x_1x_2x_4, x_4^3 - x_1x_2x_3) \subset A$ be the "simplest" PCB ideal in dimension 4. By Proposition 4.8, I is not unmixed. In fact, the element $g \in (I : x_1) \setminus I$ built in the proof there is $x_1^2x_4^2 - x_2^2x_3^2$. A computation with Singular (see [GPS]) shows that $I : x_1 = I + (x_1^2x_2^2 - x_3^2x_4^2, x_1^2x_3^2 - x_2^2x_4^2, x_1^2x_4^2 - x_2^2x_3^2)$ and that $I : x_1 = I : x_1^2$. In particular, by Proposition 4.1(c), $S(I) = I : x_1$. Alternatively, from Proposition 4.4, we get another description of $S(I)$, namely, since $b(4) = (0, 1, 2, 0)$, $S(I) = I : (x_2x_3^2)$.

On the other hand, clearly $m(I) = (16, 16, 16, 16)$ and so $d = \gcd(m(I)) = 16$. We will see (cf. Theorem 7.1 below) that, provided $k = \mathbb{C}$, I has exactly sixteen isolated primary components and one irredundant embedded primary component. The next result says that $\mathfrak{Q} = I + (x_1) = (x_1, x_2x_3x_4, x_2^3, x_3^3, x_4^3)$ is an embedded primary component of I. Alternatively, $I + (x_2x_3^2)$ is another embedded primary component of I.

We now give an explicit description of an irredundant embedded component of I, provided that $n \geqslant 4$, that is independent of the characteristic of k. Note that in this case, each irredundant primary decomposition of I has precisely one embedded component.

Theorem 4.10. *Let $I = (\underline{f})$ be a PCB ideal of A, $n \geqslant 4$. Suppose that $I : x^\alpha = I : (x^\alpha)^\infty$ for some $\alpha \in \mathbb{N}_0^n \setminus \{0\}$. Then the following hold.*

(a) *$I + (x^\alpha)$ is an irredundant $\mathfrak{m}$-primary component of I;*
(b) *In particular, for $b(n) = (0, a_{2,2} - a_{2,3} - \cdots - a_{2,n}, \ldots, a_{n-1,n-1} - a_{n-1,n}, 0)$, $I + (x^{b(n)})$ is an irredundant $\mathfrak{m}$-primary component of I.*

Proof. By Proposition 4.1, $S(I) = \mathrm{Hull}(I)$ and $S(I) = I : (x^\alpha)^\infty = I : x^\alpha$. Moreover, since $n \geqslant 4$, by Proposition 4.8, I is not unmixed.

Since $I : x^\alpha = I : (x^\alpha)^\infty$, by [ES, Proposition 7.2(a)], $I = (I : x^\alpha) \cap (I + (x^\alpha))$, so $I = S(I) \cap (I + (x^\alpha))$, where $S(I) = \mathrm{Hull}(I)$ is the intersection of the isolated primary components of I. Since I is not unmixed, $I + (x^\alpha)$ is not redundant.

Clearly, $\mathrm{rad}(I, x^\alpha) = \mathfrak{m}$. Thus $I + (x^\alpha)$ is $\mathfrak{m}$-primary. One deduces that $I + (x^\alpha)$ is an irredundant $\mathfrak{m}$-primary component of I.

By Proposition 4.4, $S(I) = I : x^{b(n)}$, i.e., $I : x^{b(n)} = I : (x^{b(n)})^\infty$. It follows, that $I + (x^{b(n)})$ is an irredundant $\mathfrak{m}$-primary component of I. $\square$

Example 4.11. Let $I = (\underline{f}) = (x_1^4 - x_2x_3x_4, x_2^4 - x_1^2x_3x_4, x_3^3 - x_1x_2^2x_4, x_4^3 - x_1x_2x_3) \subset A$. Again, by Proposition 4.8, I is not unmixed. Since $b(n) = (0, 1, 2, 0)$, by Theorem 4.10, $I + (x_2x_3^2)$ is an irredundant $\mathfrak{m}$-primary component of I. On the other hand, the integer vector associated to I is $m(I) = (20, 24, 31, 25)$ and its greatest common divisor is $d = \gcd(m(I)) = 1$. By Proposition 3.3, $\mathfrak{p}_{m(I)} = \ker(\varphi_{m(I)})$ is the unique Herzog ideal containing I. Recall that the natural map $\varphi_{m(I)} : A \to k[t]$ sends x_1, x_2, x_3 and x_4 to t^{20}, t^{24}, t^{31} and t^{25}, respectively. Therefore $I \subseteq \mathfrak{p}_{m(I)} \cap \mathfrak{Q}$. We will see (cf. Corollary 6.6 below) that, since $d = 1$, $S(I) = \mathfrak{p}_{m(I)}$, so $I = S(I) \cap \mathfrak{Q} = \mathfrak{p}_{m(I)} \cap \mathfrak{Q}$, an irredundant intersection, and the previous inclusion is an equality.

5. A review of the normal decomposition of an integer matrix

In this section we review some well-known facts about linear algebra over $\mathbb{Z}$ or, more generally, over a Principal Ideal Domain. Our general reference is [Jac, Chapter 3]. As before, $A = k[\underline{x}]$ is the polynomial ring in n variables $\underline{x} = x_1, \ldots, x_n$ over a field k, $n \geqslant 2$.

Definition 5.1. Let M be a non-zero $n \times s$ integer matrix. Then there exist an $n \times n$ invertible integer matrix P and an $s \times s$ invertible integer matrix Q such that $PMQ = D$, where D is an $n \times s$ integer diagonal matrix $D = \mathrm{diag}(d_1, d_2, \ldots, d_r, 0, \ldots, 0)$, with $d_i \in \mathbb{N}$ and $d_i \mid d_j$ if $i \leqslant j$, and $r = \mathrm{rank}(M)$. The matrix D is called a *normal form* of M and the expression $PMQ = D$ a *normal decomposition* of M (D is also called the Smith Normal Form of M, see, e.g., [GPS]).

Remark 5.2. The non-zero diagonal elements of a normal form D of M, referred to as the *invariant factors* of M, are unique. Indeed, let $I_t(M)$ be the ideal of $\mathbb{Z}$ generated by the $t \times t$-minors of the matrix M, $I_t(M) := \mathbb{Z}$ for $t \leqslant 0$ and $I_t(M) := 0$ for $t > \min(n, s)$. Then $I_t(M) = I_t(PMQ)$ for all invertible integer matrices P and Q (see, e.g., [CLO, Chapter 5, Lemma 4.8 and Exercise 10, pp. 232–233]). In particular, $I_t(M) = I_t(D)$ and $\gcd(I_t(M)) = \gcd(I_t(D))$, understanding by the $\gcd(J)$ of a non-zero ideal J of $\mathbb{Z}$ its non-negative generator (and setting 0 to be the gcd of the zero ideal). Therefore, setting $\Delta_t = \gcd(I_t(M))$, one has that $d_1 = \Delta_1$, $d_2 = \Delta_2\Delta_1^{-1}, \ldots, d_r = \Delta_r\Delta_{r-1}^{-1}$, where $r = \mathrm{rank}(M)$ (see, e.g., [Jac, Theorems 3.8 and 3.9]). Observe that, in particular, $d_1 = \Delta_1$, $d_1d_2 = \Delta_2$, $\ldots$, $d_1 \cdots d_r = \Delta_r$.

Lemma 5.3. *Let I be the PCB ideal of A associated to L. Let $m(I)$ be the integer vector associated to I, $d = \gcd(m(I))$ and $\nu(I) = m(I)/d$. Let $PLQ = D$ be a normal decomposition of L and $d_1, \dots, d_{n-1}$ the invariant factors of L. Then $d = d_1 \cdots d_{n-1}$. Moreover, the last row of P is $\pm\nu(I)$.*

Proof. Observe that, by Lemma 2.1, $\operatorname{rank}(L) = n-1$ and hence there are $n-1$ (non-zero) invariant factors $d_1, \dots, d_{n-1}$. By Remark 5.2, $d_1 \cdots d_{n-1} = \Delta_{n-1} = \gcd(I_{n-1}(L))$. But the $(n-1)\times(n-1)$ minors of L are precisely the entries of the matrix $\operatorname{adj}(L)$, each of whose rows is equal to the last one, denoted by $m(I)$ (see Lemma 2.1 and Definition 2.2). Thus $d_1 \cdots d_{n-1} = \Delta_{n-1} = \gcd(I_{n-1}(L)) = \gcd(m(I)) = d$.

Since $PL = DQ^{-1}$ and the last row of DQ^{-1} is zero, $p_{n,*}L = 0$ and $p_{n,*}^{\top} \in \operatorname{Nullspace}(L^{\top})$. By Lemma 2.1 and Definition 2.2, $\operatorname{Nullspace}(L^{\top})$ is generated, as a $\mathbb{Q}$-linear subspace, by the vector $m(I)^{\top}$, or equivalently, by the vector $\nu(I)^{\top} = m(I)^{\top}/d$. Therefore there exist $r, s \in \mathbb{Z} \setminus \{0\}$ such that $rp_{n,*} = s\nu(I)$. Observe that, since $\det(P) = \pm 1$, then $\gcd(p_{n,*}) = 1$. Now, taking the greatest common divisor, we get $r = \pm s$ and hence $p_{n,*} = \pm\nu(I)$. $\square$

Example 5.4. Let I be the PCB ideal of A associated to L. Suppose that $n = 2$. Then $a_{1,1} = a_{1,2}$ and $a_{2,1} = a_{2,2}$. Moreover $m(I) = (a_{2,2,}, a_{1,1})$. Let $d = \gcd(m(I))$ and write $a_{i,i} = da'_{i,i}$ and $d = b_1a_{1,1} + b_2a_{2,2}$ for some $b_1, b_2 \in \mathbb{Z}$. The invariant factor of L is $d_1 = d$ and

$$\begin{pmatrix} b_1 & -b_2 \\ a'_{2,2} & a'_{1,1} \end{pmatrix} \begin{pmatrix} a_{1,1} & -a_{1,2} \\ -a_{2,1} & a_{2,2} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} d & 0 \\ 0 & 0 \end{pmatrix}$$

is a normal decomposition of L.

In order to describe the isolated components of the PCB ideal I associated to L, it will be convenient to know the entries of a matrix P in a normal decomposition $PLQ = D$ of L (see Theorem 7.1).

If $n = 3$ and if the entries of a row of L are relatively prime, or more generally if their greatest common divisor equals the first invariant factor d_1, we see next that obtaining P explicitly is almost immediate. Observe that the example also covers the situation where $\gcd(a_{2,1}, a_{2,3}) = d_1$ or $\gcd(a_{1,2}, a_{1,3}) = d_1$, via an appropriate relabelling of the suffices. However, calculating an explicit normal decomposition of a general matrix L, even for $n = 3$, is technical and unilluminating. For concrete instances of the matrix L, a normal decomposition of L can be obtained for example in Singular (see 'smithNormalForm' [GPS]).

Example 5.5. Let I be the PCB ideal of A associated to L. Suppose that $n = 3$. Let $m(I)$ be the integer vector associated to I, $d = \gcd(m(I))$ and $\nu(I) = m(I)/d$. Let d_1, d_2 be the invariant factors of L. In particular, $d_1 = \gcd(I_1(L))$ and $d_1d_2 = d$. Set $d'_2 = d_2/d_1$. Let $b = \gcd(a_{3,1}, a_{3,2}) = b'd_1$. Let $c_1, c_2 \in \mathbb{Z}$ with $b = c_1a_{3,1} + c_2a_{3,2}$. Set $\alpha_1 = -c_1a_{1,1} + c_2a_{1,2} = d_1\alpha'_1$ and $\alpha_2 = c_1a_{2,1} - c_2a_{2,2} = d_1\alpha'_2$, for some $\alpha'_1, \alpha'_2 \in \mathbb{Z}$. The following conditions are equivalent:

(a) $\gcd(a_{3,1}, a_{3,2}) = d_1$;
(b) $\gcd(\nu_1, \nu_2) = 1$, $b' \mid c$ and $(b')^2 \mid d'_2$, where $c = s_2\alpha'_1 - s_1\alpha'_2$ and $s_1, s_2 \in \mathbb{Z}$ are such that $s_1\nu_1 + s_2\nu_2 = 1$;
(c) There exists a normal decomposition $PLQ = D$ of L with the first row of P equal to $(0, 0, 1)$.

Moreover, in this particular case, the second row of P is given by $(s_2, -s_1, -c)$, while the third row is given by $\nu(I) = (\nu_1, \nu_2, \nu_3)$.

Proof. Set $a_{3,1} = b\tilde{a}_{3,1}$, $a_{3,2} = b\tilde{a}_{3,2}$, with $\tilde{a}_{3,1}, \tilde{a}_{3,2} \in \mathbb{N}$. Let $Q = \left(\begin{smallmatrix} -c_1 & \tilde{a}_{3,2} & 1 \\ -c_2 & -\tilde{a}_{3,1} & 1 \\ 0 & 0 & 1 \end{smallmatrix}\right)$. Then $\det(Q) = 1$ and

$$LQ = \begin{pmatrix} a_{1,1} & -a_{1,2} & -a_{1,3} \\ -a_{2,1} & a_{2,2} & -a_{2,3} \\ -a_{3,1} & -a_{3,2} & a_{3,3} \end{pmatrix} \begin{pmatrix} -c_1 & \tilde{a}_{3,2} & 1 \\ -c_2 & -\tilde{a}_{3,1} & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha_1 & m_2/b & 0 \\ \alpha_2 & -m_1/b & 0 \\ b & 0 & 0 \end{pmatrix}.$$

Since $\nu(I)^\top$ is a $\mathbb{Q}$-basis of Nullspace$(L^\top)$ (cf. Definition 2.2), $\nu(I)LQ=0$ and

$$\alpha_1\nu_1+\alpha_2\nu_2+b\nu_3=0. \tag{1}$$

For $P_1=\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}$, $\det(P_1)=-1$ and $P_1LQ=\begin{pmatrix}b&0&0\\\alpha_2&-m_1/b&0\\\alpha_1&m_2/b&0\end{pmatrix}$.
For $P_2=\begin{pmatrix}1&0&0\\-\alpha'_2&1&0\\-\alpha'_1&0&1\end{pmatrix}$, $\det(P_2)=1$ and $P_2P_1LQ=\begin{pmatrix}b&0&0\\\alpha'_2(d_1-b)&-m_1/b&0\\\alpha'_1(d_1-b)&m_2/b&0\end{pmatrix}$.

The unique non-zero 2×2 minor of P_2P_1LQ defined by the last two rows is, up to sign, equal to $((d_1-b)d_2/b)[\alpha_1\nu_1+\alpha_2\nu_2]$, which, by the equality (1) above, is equal to $-(d_1-b)d_2\nu_3$. Since $I_2(D)=I_2(L)=I_2(P_2P_1LQ)$ (cf. Remark 5.2),

$$d=d_1d_2=\gcd\bigl(I_2(D)\bigr)=\gcd\bigl(I_2(L)\bigr)=\gcd\bigl(I_2(P_2P_1LQ)\bigr)=\gcd\bigl(\nu_1d,\nu_2d,(b-d_1)d_2\nu_3\bigr).$$

Since $b=d_1b'$, then $d=d\cdot\gcd(\nu_1,\nu_2,(b'-1)\nu_3)$. Therefore, $\gcd(\nu_1,\nu_2,(b'-1)\nu_3)=1$ and $\gcd(\nu_1,\nu_2,(b'-1))=1$.

Observe that until now we have not used any of the hypotheses (a), (b) or (c). Now suppose that $\gcd(a_{3,1},a_{3,2})=d_1$. Then $b'=1$ and $\gcd(\nu_1,\nu_2)=1$. Thus (a) implies (b).

Suppose now that $\gcd(\nu_1,\nu_2)=1$ (where b is not assumed a priori to be equal to d_1). Let $s_1,s_2\in\mathbb{Z}$ with $s_1\nu_1+s_2\nu_2=1$. Set $c=s_2\alpha'_1-s_1\alpha'_2$. Let $P_3=\begin{pmatrix}1&0&0\\c&-s_1&s_2\\(1-b')\nu_3&\nu_2&\nu_1\end{pmatrix}$. Then $\det(P_3)=-1$ and $P_3P_2P_1LQ=\begin{pmatrix}d_1b'&0&0\\d_1c&d_2/b'&0\\0&0&0\end{pmatrix}$, using the equality (1) above.

Suppose that $b'\mid c$. Set $\tilde{c}=c/b'$ and $P_4=\begin{pmatrix}1&0&0\\-\tilde{c}&1&0\\0&0&1\end{pmatrix}$. Then $\det(P_4)=1$. Set $P=P_4P_3P_2P_1$. Then $P=\begin{pmatrix}0&0&1\\s_2&-s_1&-\tilde{c}\\\nu_1&\nu_2&\nu_3\end{pmatrix}$ and $PLQ=\begin{pmatrix}d_1b'&0&0\\0&d_2/b'&0\\0&0&0\end{pmatrix}$. If $(b')^2\mid d'_2$, then d_1b' and d_2/b' are positive integers with $d_1b'\mid(d_2/b')$. By the unicity of the normal form of L, $b'=1$. Therefore $PLQ=D$ is a normal decomposition of L and (b) implies (c).

Finally, suppose that there exists a normal decomposition $PLQ=D$ of L, with the first row of P equal to $(0,0,1)$. Equating the first rows of the identity $PL=DQ^{-1}$, one has that, if $Q^{-1}=(u_{i,j})$, $(-a_{3,1},-a_{3,2},a_{3,3})=(d_1u_{1,1},d_1u_{1,2},d_1u_{1,3})$. Therefore

$$\gcd(a_{3,1},a_{3,2},a_{3,3})=d_1\cdot\gcd(u_{1,1},u_{1,2},u_{1,3})=d_1$$

and (c) implies (a). $\square$

We finish the section with the answer to a question posed by Josep Àlvarez Montaner. Denote by $\mathrm{Fitt}_i(\mathcal{M})$ the i-th Fitting ideal of a $\mathbb{Z}$-module $\mathcal{M}$ (see, e.g., [CLO, Definition 5.4.9]).

Proposition 5.6. *Let I be the PCB ideal of A associated to L. Let $m(I)$ be the integer vector associated to I, $d=\gcd(m(I))$. Let $\mathcal{L}$ be the lattice of $\mathbb{Z}^n$ spanned by the columns of L. Then the following hold.*

(a) $\mathrm{Fitt}_1(\mathbb{Z}^n/\mathcal{L})=d\mathbb{Z}$ *and* $\mathrm{Fitt}_0(\mathbb{Z}^n/\mathcal{L})=0$.
(b) $\mathbb{Z}^n/\mathcal{L}\cong\mathbb{Z}\oplus\mathbb{Z}/d_1\mathbb{Z}\oplus\cdots\oplus\mathbb{Z}/d_{n-1}\mathbb{Z}$, *with $d_1,\dots,d_{n-1}$ the invariant factors of L.*
(c) *The cardinality of the torsion group of $\mathbb{Z}^n/\mathcal{L}$ is d.*
(d) *$\mathcal{L}$ is a direct summand of $\mathbb{Z}^n$ if and only if $d=1$.*

Proof. Let $PLQ=D$ be a normal decomposition of L and $d_1,\dots,d_{n-1}$ the invariant factors of L. By Lemma 5.3, $d=d_1\cdots d_{n-1}$. By definition, $\mathrm{Fitt}_1(\mathbb{Z}^n/\mathcal{L})=I_{n-1}(L)=I_{n-1}(D)=(d_1\cdots d_{n-1})\mathbb{Z}=d\mathbb{Z}$ and $\mathrm{Fitt}_0(\mathbb{Z}^n/\mathcal{L})=I_n(L)=I_n(D)=0$. Since $PLQ=D$ is a normal decomposition of $\mathcal{L}$, the $\mathbb{Z}$-module $\mathbb{Z}^n/\mathcal{L}$

admits a decomposition $\mathbb{Z} \oplus \mathcal{T}$, where $\mathcal{T} = \mathbb{Z}/d_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_{n-1}\mathbb{Z}$ is the torsion module (see, e.g., [Jac, Chapter 3]). Clearly $d = d_1 \cdots d_{n-1}$ is the cardinality of the torsion group of $\mathbb{Z}^n/\mathcal{L}$. Finally, since $\operatorname{rank}(L) = n-1$, then $\operatorname{rank}(\mathcal{L}) = n-1$ and $\operatorname{rank}(\mathbb{Z}^n/\mathcal{L}) = 1$ (see, e.g., [BH, §1.4]). Hence $\mathcal{L}$ is a direct summand of $\mathbb{Z}^n$ if and only if $\mathbb{Z}^n/\mathcal{L}$ is a free $\mathbb{Z}$-module of rank 1. By [Eis, Proposition 20.8], the latter holds if and only if $\operatorname{Fitt}_1(\mathbb{Z}^n/\mathcal{L}) = \mathbb{Z}$ and $\operatorname{Fitt}_0(\mathbb{Z}^n/\mathcal{L}) = 0$, i.e., if and only if $d = 1$. □

Remark 5.7. Note that an obvious analogue of Proposition 5.6 holds for any $n \times n$ matrix M of rank $n-1$, with invariant factors $d_1, \ldots, d_{n-1}$, with d now defined merely as the product $d_1 \cdots d_{n-1}$. Note also that the transpose $M^\top$ of M again has $d_1, \ldots, d_{n-1}$ as its invariant factors: indeed, any integer matrix and its transpose have the same invariants.

Remark 5.8. There is an overlap between the results of Section 5 and the results of [LV, §3]. (Recall that an integer matrix and its transpose have the same invariant factors.) The latter results were obtained using Gröbner Basis Theory rather than the theory of Fitting ideals, and with different objectives in mind. For example, one can contrast the statement of [LV, Corollary 3.19] with the situation that obtains for the ideal I considered in Example 4.11 above. In Example 4.11, $d = 1$ and $I(\mathcal{L}) = S(I) = \mathfrak{p}_{m(I)}$, which is the kernel of the natural map $A \to k[t]$ sending x_1, x_2, x_3 and x_4 to t^{20}, t^{24}, t^{31} and t^{25}, respectively, whereas in López and Villarreal's theory, $d = 1$ and $I(\mathcal{L}) = (x_1 - x_2, x_1 - x_3, x_1 - x_4)$. Observe that the ideals in [LV] have to be homogeneous in the standard grading, hence this simpler form. See also Remark 7.2.

6. Applying the Eisenbud–Sturmfels theory of Laurent binomial ideals

In this section we apply the theory of Laurent binomial ideals developed in [ES]. Recall that for an $n \times s$ integer matrix M, we denote by $m_{i,*}$ and $m_{*,j}$ its i-th row and j-th column, respectively. The abelian group generated by the columns of M is denoted $\mathcal{M} = \mathbb{Z}m_{*,1} + \cdots + \mathbb{Z}m_{*,s} = \varphi(\mathbb{Z}^s) \subseteq \mathbb{Z}^n$, where $\varphi : \mathbb{Z}^s \to \mathbb{Z}^n$ is the homomorphism defined by the matrix M. By an $n \times n$ invertible integer matrix, we understand an $n \times n$ matrix P with entries in $\mathbb{Z}$ whose determinant is ± 1. Thus its inverse matrix P^{-1} is also an integer matrix. Set $A = k[\underline{x}]$ to be the polynomial ring in n variables $\underline{x} = x_1, \ldots, x_n$ over a field k, $n \geqslant 2$, and $B = k[\underline{x}^{\pm}] = k[x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}]$, the Laurent polynomial ring.

Remark 6.1. Let $I = (\underline{f}) = (f_1, \ldots, f_n)$ be the PCB ideal of A associated to L. Then

$$IB = (f_1, \ldots, f_{n-1})B = \left(x^{l_{*,1}} - 1, \ldots, x^{l_{*,n-1}} - 1\right)B.$$

In particular, IB is a complete intersection.

Proof. Clearly $IB = (f_1, \ldots, f_n)B = (x^{l_{*,1}} - 1, \ldots, x^{l_{*,n}} - 1)B$. Since the sum of the entries of each row is zero, the last column of L, $l_{*,n}$, is a $\mathbb{Z}$-linear combination of the first $n-1$ columns of L. Thus, by Remark 4.2(c), $x^{l_{*,n}} - 1 \in (x^{l_{*,1}} - 1, \ldots, x^{l_{*,n-1}} - 1)B$. An alternative proof would follow from Remark 3.4 (see the proof of Proposition 3.5(b)). □

We now make explicit the change of variables we will use.

Lemma 6.2. *Let $P = (p_{i,j})$ be an $n \times n$ invertible integer matrix. Set $R = (r_{i,j})$, its inverse. Let $y_1 = x^{r_{*,1}}, \ldots, y_n = x^{r_{*,n}}$ in $B = k[\underline{x}^{\pm}] = k[x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}]$. Then*

(a) $x_1 = y^{p_{*,1}}, \ldots, x_n = y^{p_{*,n}}$;
(b) $B = k[x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}] = k[y_1, \ldots, y_n, y_1^{-1}, \ldots, y_n^{-1}]$;
(c) *$y_1, \ldots, y_n$ are algebraically independent over k.*

Proof. Since RP is the identity matrix, $y^{p_{*,i}} = y_1^{p_{1,i}} \cdots y_n^{p_{n,i}} = x^{r_{*,1}p_{1,i}} \cdots x^{r_{*,n}p_{n,i}} = x^{Rp_{*,i}} = x_i$. Clearly $k[\underline{y}^{\pm}] \subseteq B$ and the equality follows by part (a). Writing $Q(R)$ to denote the quotient field of a domain R, we have $Q(A) = Q(B) = Q(k[\underline{y}^{\pm}]) = Q(k[\underline{y}])$. Thus $\dim A = \operatorname{trdeg}_k(Q(A)) = \operatorname{trdeg}_k(Q(k[\underline{y}]))$ and the transcendence degree of $k[y_1, \dots, y_n]$ over k is n. It follows, e.g., using Noether's Normalization Lemma, that $y_1, \dots, y_n$ are algebraically independent over k. □

The next result expresses IB in terms of the new variables.

Lemma 6.3. *Let I be the PCB ideal of A associated to L. Let $PLQ = D$ be a normal decomposition of L and $d_1, \dots, d_{n-1}$ the invariant factors of L. Let $R = (r_{i,j})$ be the inverse of P. Set $y_1 = x^{r_{*,1}}, \dots, y_n = x^{r_{*,n}}$ in $B = k[\underline{x}^{\pm}]$. Then $IB = (y_1^{d_1} - 1, \dots, y_{n-1}^{d_{n-1}} - 1)B$.*

Proof. By Remark 6.1, $IB = (x^{l_{*,1}} - 1, \dots, x^{l_{*,n-1}} - 1)B$. Using Lemma 6.2(a) and substituting x_i by $y^{p_{*,i}}$, we get $x^{l_{*,i}} = x_1^{-a_{1,i}} \cdots x_i^{a_{i,i}} \cdots x_n^{-a_{n,i}} = y^{p_{*,1}(-a_{1,i})} \cdots y^{p_{*,i}a_{i,i}} \cdots y^{p_{*,n}(-a_{n,i})} = y^{Pl_{*,i}}$. Therefore the ideal $(x^{l_{*,1}} - 1, \dots, x^{l_{*,n-1}} - 1)B$ is equal to $(y^{Pl_{*,1}} - 1, \dots, y^{Pl_{*,n-1}} - 1)B$, and so equal to $(y^{(DQ^{-1})_{*,1}} - 1, \dots, y^{(DQ^{-1})_{*,n-1}} - 1)B$. By Remark 4.2(e), the latter is equal to the ideal $(y^{D_{*,1}} - 1, \dots, y^{D_{*,n-1}} - 1)B = (y_1^{d_1} - 1, \dots, y_{n-1}^{d_{n-1}} - 1)B$. □

Our aim now is to give a minimal primary decomposition of IB in terms of these new variables. Before this, we introduce some notation.

Notation 6.4. Let I be the PCB ideal associated to L, $m(I)$ its associated integer vector, $d = \gcd(m(I))$ and $\nu(I) = m(I)/d$. Let $PLQ = D$ be a normal decomposition of L with $p_{n,*} = \nu(I)$ (see Lemma 5.3). Let $d_1, \dots, d_{n-1}$ be the invariant factors of L. Let $R = (r_{i,j})$ be the inverse of P and set $y_1 = x^{r_{*,1}}, \dots, y_n = x^{r_{*,n}}$ in $B = k[\underline{x}^{\pm}]$. For any $\lambda = (\lambda_1, \dots, \lambda_{n-1}) \in (k^*)^{n-1}$, $k^* := k \setminus \{0\}$, set $\mathfrak{b}_\lambda = (y_1 - \lambda_1, \dots, y_{n-1} - \lambda_{n-1})B$. Clearly $\mathfrak{b}_\lambda$ is a prime ideal of B of height $n-1$. In particular, $\mathfrak{b}_\lambda^c$ is a prime ideal of A of height $n-1$.

Suppose that k contains the d_{n-1}-th roots of unity. (Note that then k will also contain the d_1-th, $\dots, d_{n-2}$-th roots of unity, respectively, since $d_i \mid d_{i+1}$ for $i = 1, \dots, n-2$.) We will write $\{\xi_{i,1}, \dots, \xi_{i,d_i}\}$ to denote the set of d_i-th roots of unity in k when these exist and are distinct, and set $\Lambda(D) = \prod_{i=1}^{n-1} \{\xi_{i,1}, \dots, \xi_{i,d_i}\} \subset (k^*)^{n-1}$. Clearly, if the characteristic of k, $\operatorname{char}(k)$, is zero or $\operatorname{char}(k) = p$, p a prime with $p \nmid d_{n-1}$, then the cardinality of $\Lambda(D)$ is $d_1 \cdots d_{n-1}$, which, by Lemma 5.3, is equal to d.

Theorem 6.5. *Let I be the PCB ideal associated to L, $m(I)$ its associated integer vector, $d = \gcd(m(I))$ and $\nu(I) = m(I)/d$. Let $d_1, \dots, d_{n-1}$ be the invariant factors of L. With the notations above*:

(a) *Suppose that k contains the d_{n-1}-th roots of unity and that the characteristic of k, $\operatorname{char}(k)$, is zero or $\operatorname{char}(k) = p$, p a prime with $p \nmid d_{n-1}$. Then $IB = \bigcap_{\lambda \in \Lambda(D)} \mathfrak{b}_\lambda$ and $S(I) = \bigcap_{\lambda \in \Lambda(D)} \mathfrak{b}_\lambda^c$ are minimal primary decompositions. In particular, IB and $S(I)$ are unmixed, radical and have exactly d distinct associated primes.*

(b) *If k is an arbitrary field, then IB and $S(I)$ have at most d distinct associated primes.*

Proof. As in Notation 6.4, set $y_1 = x^{r_{*,1}}, \dots, y_n = x^{r_{*,n}}$ in $B = k[\underline{x}^{\pm}]$. Lemma 6.2 says that $\underline{y} = y_1, \dots, y_n$ are algebraically independent over k and $B = k[\underline{x}^{\pm}] = k[\underline{y}^{\pm}]$. By Remark 6.1, IB is a complete intersection and, by Lemma 6.3, $IB = (y_1^{d_1} - 1, \dots, y_{n-1}^{d_{n-1}} - 1)B$.

Let us prove (a). Consider $\Lambda(D) = \prod_{i=1}^{n-1} \{\xi_{i,1}, \dots, \xi_{i,d_i}\} \subset (k^*)^{n-1}$ and, for any $\lambda \in \Lambda(D)$, $\mathfrak{b}_\lambda = (y_1 - \lambda_1, \dots, y_{n-1} - \lambda_{n-1})B$, as in Notation 6.4. If $\mathfrak{p}$ is any prime ideal containing $IB = (y_1^{d_1} - 1, \dots, y_{n-1}^{d_{n-1}} - 1)B$ then an immediate argument shows that $\mathfrak{b}_\lambda \subseteq \mathfrak{p}$ for some $\lambda \in \Lambda(D)$. Hence the

$\mathfrak{b}_\lambda$, for $\lambda \in \Lambda(D)$, are the minimal primes containing IB. Since IB is a complete intersection, the $\mathfrak{b}_\lambda$ are also the associated primes of IB.

Consider the inclusion $IB \subseteq \bigcap_{\lambda\in\Lambda(D)} \mathfrak{b}_\lambda$ and localize at a particular associated prime $\mathfrak{b}_\mu = (y_1 - \mu_1, \dots, y_{n-1} - \mu_{n-1})B$, with $\mu \in \Lambda(D)$, say. We see that this inclusion becomes an equality, since, if for any $\lambda \in \Lambda(D)$ and $i \in \{1, \dots, n-1\}$ we have $\lambda_i \neq \mu_i$, then $y_i - \lambda_i$ becomes a unit in the localization and, as a result, each $y_i^{d_i} - 1$ becomes an associate of $y_i - \mu_i$. Therefore, by Proposition 4.1(d), $IB = \bigcap_{\lambda\in\Lambda(D)} \mathfrak{b}_\lambda$ and $S(I) = \bigcap_{\lambda\in\Lambda(D)} \mathfrak{b}_\lambda^c$ are minimal primary decompositions of IB and $S(I)$, respectively. Recall that the cardinality of $\Lambda(D)$ is $d_1 \cdots d_{n-1} = d$ by Lemma 5.3.

Now suppose that k is an arbitrary field; let $\bar{k}$ denote the algebraic closure of k. Note that, by Base Change, the extension $k[\underline{y}^{\pm}] \hookrightarrow \bar{k}[\underline{y}^{\pm}]$ is faithfully flat and integral. In particular, by the former property, $J := I\bar{k}[\underline{y}^{\pm}]$ is again a complete intersection.

There are at most d_i distinct d_i-th roots of unity in $\bar{k}$. The above argument yields that there are then at most d distinct minimal and so associated primes of the ideal J. By standard properties of integral extensions, the number of minimal primes containing IB is at most the number of minimal primes containing J, and the result follows. $\square$

As an immediate corollary, we have the following analogue of [OP2, Theorem 7.8]. Note that the next result can be interpreted as giving equivalent conditions for $S(I)$ (which is a lattice ideal by Corollary 4.3) to be a toric ideal (in the terminology of [MS, Chapter 7]).

Corollary 6.6. *Let I be the PCB ideal associated to L, $m(I)$ its associated integer vector and $d = \gcd(m(I))$. The following conditions are equivalent*:

(i) $S(I)$ *is prime*;
(ii) $S(I) = \mathfrak{p}_{m(I)}$;
(iii) $d = 1$.

Proof. On one hand, by Proposition 3.3, $\mathfrak{p}_{m(I)}$ is a minimal prime of I. Moreover, by Proposition 4.1(a), $S(I) = \mathrm{Hull}(I)$, the intersection of the isolated primary components of I. Therefore $I \subseteq S(I) \subseteq \mathfrak{p}_{m(I)}$ and $\mathfrak{p}_{m(I)}$ is a minimal prime of $S(I)$ too. Therefore, $S(I)$ is prime if and only if $S(I) = \mathfrak{p}_{m(I)}$.

Suppose that $d = 1$. Then $d_{n-1} = 1$ and k fulfills the hypotheses of (a) in Theorem 6.5. Thus $S(I)$ is prime.

Conversely, suppose that $S(I)$ is prime and $d > 1$. We will derive a contradiction.

Now IB equals the localization $S(I)_x$, so IB is also prime. Recall Lemmas 6.2 and 6.3, and set $A' = k[y_1, \dots, y_n]$. Note that $d_{n-1} > 1$ and that $B = A'_y$, where $y = y_1 \cdots y_n$. Hence $y_{n-1}^{d_{n-1}} - 1 \in IB \cap A' = (y_1^{d_1} - 1, \dots, y_{n-1}^{d_{n-1}} - 1)A' :_{A'} y^\infty$. This last ideal is a prime ideal in A' since IB is prime in B. Now for some $t \in \mathbb{N}_0$, either $y^t(y_{n-1} - 1)$ or $y^t(y_{n-1}^{d_{n-1}-1} + \dots + 1)$ lies in $(y_1^{d_1} - 1, \dots, y_{n-1}^{d_{n-1}} - 1)A'$.

On setting each of $y_1, \dots, y_{n-2}$ equal to 1, we deduce that either $y_{n-1}^t(y_{n-1} - 1)$ or $y_{n-1}^t(y_{n-1}^{d_{n-1}-1} + \dots + 1)$ lies in $(y_{n-1}^{d_{n-1}} - 1)k[y_{n-1}]$. Since $k[y_{n-1}]$ is a UFD, it follows in fact that either $y_{n-1} - 1 \in (y_{n-1}^{d_{n-1}} - 1)k[y_{n-1}]$ or else $y_{n-1}^{d_{n-1}-1} + \dots + 1 \in (y_{n-1}^{d_{n-1}} - 1)k[y_{n-1}]$. Since $d_{n-1} > 1$, this is impossible. $\square$

Before proceeding to study the explicit description of $\mathfrak{b}_\lambda^c$, we take advantage of this result to examine how the class of PCB ideals plays off against the class of binomial ideals considered by Hoşten and Shapiro [HS]. We find that the overlap in the two classes is a trivial one (recall the notations at the end of Section 1; see also Proposition 4.2).

Proposition 6.7. *Let $A = k[\underline{x}]$ be the polynomial ring in n variables $\underline{x} = x_1, \dots, x_n$. Let $\mathfrak{A}$ be the class of binomial ideals of A defined by the columns of $n \times r$ integer matrices M of rank r, $r \leqslant n$, with $\mathcal{M} \cap \mathbb{N}_0^n = \{0\}$ and such that, as in [HS], the lattice ideal $I(\mathcal{M})$ is prime. Let $\mathfrak{B}$ be the class of PCB ideals of A.*

(a) *If $n > 2$, the intersection of $\mathfrak{A}$ and $\mathfrak{B}$ is empty.*
(b) *If $n = 2$, the intersection of $\mathfrak{A}$ and $\mathfrak{B}$ is the class of principal ideals generated by an irreducible pure binomial.*

Proof. Suppose that J is an ideal in the intersection of $\mathfrak{A}$ and $\mathfrak{B}$. Then $J = I(M)$, for some $n \times r$ matrix M of rank r, $r \leqslant n$, with $\mathcal{M} \cap \mathbb{N}_0^n = \{0\}$ and with $I(\mathcal{M})$ prime, and $J = I(L)$ for an $n \times n$ PCB matrix L. By Proposition 4.2(d), $\mathcal{M} = \mathcal{L}$. Hence, by Lemma 2.1(a), $r = \operatorname{rank}(\mathcal{M}) = \operatorname{rank}(\mathcal{L}) = n - 1$. Thus J, which is generated by r binomials defined by the r columns of M, can be generated by $r = n - 1$ elements. If follows by Proposition 3.3(c) that n must be equal to 2. Hence, if $n > 2$, the intersection of $\mathfrak{A}$ and $\mathfrak{B}$ is empty.

Suppose now that $n = 2$ and that J is as above. By Proposition 4.2(b), $JB = I(M)B = I(\mathcal{M})B$, which, by hypothesis, is prime. Therefore $S(J) = JB \cap A$ is also prime. By Corollary 6.6, $\gcd(m(J)) = 1$. By Remark 2.3, $J = (x_1^{a_{1,1}} - x_2^{a_{2,2}})$, with $m(J) = (a_{2,2}, a_{1,1})$. Thus $\gcd(a_{1,1}, a_{2,2}) = 1$. In particular, $x_1^{a_{1,1}} - x_2^{a_{2,2}}$ is irreducible (see [Fos, Corollary 10.15] or [OP2, Lemma 8.2]; cf. also Example 7.3).

Conversely, suppose that $n = 2$ and let $I = (f_1)$ be a principal ideal generated by $f_1 = x_1^a - x_2^b$, an irreducible binomial, i.e., $\gcd(a, b) = 1$. In particular, I is prime. Set $M = (a\ \ {-b})^\top$ and complete M to the obvious 2×2 PCB matrix L. Clearly M is a 2×1 matrix of rank 1 with $\mathcal{M} \cap \mathbb{N}_0^2 = \{0\}$ and $\mathcal{M} = \mathcal{L}$. Moreover, $I = I(M) = I(L)$. Thus I is the PCB ideal associated to L. By Corollary 4.3, $I(\mathcal{L}) = S(I)$. But, by Remark 4.7 and Proposition 4.1(b), $I = S(I)$. Therefore, $I = S(I) = I(\mathcal{L}) = I(\mathcal{M})$ and $I(\mathcal{M})$ is prime. Thus I is in the intersection of $\mathfrak{A}$ and $\mathfrak{B}$. $\square$

We now express $\mathfrak{b}_\lambda^c$ in terms of $\underline{x}$, the original set of variables. We see that these prime ideals can be expressed as the vanishing ideals of monomials curves "with coefficients".

Lemma 6.8. *Let $PLQ = D$ be a normal decomposition of L with $p_{n,*} = (\nu_1, \dots, \nu_n) \in \mathbb{N}^n$. Let $d_1, \dots, d_{n-1}$ be the invariant factors of L. Let $R = (r_{i,j})$ be the inverse of P and set $y_1 = x^{r_{*,1}}, \dots, y_n = x^{r_{*,n}}$ in $B = k[\underline{x}^{\pm}]$. For any $\lambda = (\lambda_1, \dots, \lambda_{n-1}) \in (k^*)^{n-1}$, set $\mathfrak{a}_\lambda = \ker(\varphi_\lambda)$, where $\varphi_\lambda : A \to k[t]$ is the natural map defined by the rule $\varphi_\lambda(x_i) = \lambda_1^{p_{1,i}} \cdots \lambda_{n-1}^{p_{n-1,i}} t^{\nu_i}$, for $i = 1, \dots, n$. Then the following hold.*

(a) *$\mathfrak{a}_\lambda$ is a prime ideal of A of height $n - 1$;*
(b) *φ_λ induces the morphism $\tilde{\varphi}_\lambda : B \to k[t, t^{-1}]$ that sends y_i to λ_i, for $i = 1, \dots, n - 1$, and y_n to t;*
(c) *$\mathfrak{a}_\lambda = \mathfrak{b}_\lambda \cap A = \mathfrak{b}_\lambda^c$, where $\mathfrak{b}_\lambda = (y_1 - \lambda_1, \dots, y_{n-1} - \lambda_{n-1})B$.*

Proof. Set $\theta_i = \lambda_1^{p_{1,i}} \cdots \lambda_{n-1}^{p_{n-1,i}} \in k^*$. Since $k[\theta_1 t^{\nu_1}, \dots, \theta_n t^{\nu_n}] \subset k[t]$ is an integral extension, $A/\mathfrak{a}_\lambda \cong k[\theta_1 t^{\nu_1}, \dots, \theta_n t^{\nu_n}]$ has Krull dimension 1, and (a) follows. Notice that

$$
\begin{aligned}
\tilde{\varphi}_\lambda(y_i) = \tilde{\varphi}_\lambda\left(x^{r_{*,i}}\right) &= \left(\tilde{\varphi}_\lambda(x_1)\right)^{r_{1,i}} \cdots \left(\tilde{\varphi}_\lambda(x_n)\right)^{r_{n,i}} \\
&= \lambda_1^{p_{1,1}r_{1,i} + \cdots + p_{1,n}r_{n,i}} \cdots \lambda_{n-1}^{p_{n-1,1}r_{1,i} + \cdots + p_{n-1,n}r_{n,i}} t^{\nu_1 r_{1,i} + \cdots + \nu_n r_{n,i}}.
\end{aligned}
$$

Since PR is the identity matrix, the latter is equal to λ_i, for $i = 1, \dots, n - 1$, and to t, for $i = n$. This proves (b). Finally, since $\mathfrak{b}_\lambda^c$ is a prime ideal of A of height $n - 1$, to prove (c) is enough to show $\mathfrak{b}_\lambda^c \subseteq \mathfrak{a}_\lambda$. Let $\sigma : A \to B = S^{-1}A$ and $\rho : k[t] \to k[t, t^{-1}]$ be the canonical morphisms, so $\tilde{\varphi}_\lambda \circ \sigma = \rho \circ \varphi_\lambda$. Now, take $f \in \mathfrak{b}_\lambda^c$. Then $f \in \mathfrak{a}_\lambda$ if and only if $\varphi_\lambda(f) = 0$, and since ρ is injective, if and only if $\sigma(f) \in \ker(\tilde{\varphi}_\lambda)$. Since $\sigma(f) \in \mathfrak{b}_\lambda$, it follows that $\sigma(f) = \sum_{i=1}^{n-1} g_i(y_i - \lambda_i)$, for some $g_i \in B$. Thus $\tilde{\varphi}_\lambda(\sigma(f)) = \sum_{i=1}^{n-1} \tilde{\varphi}_\lambda(g_i)\tilde{\varphi}_\lambda(y_i - \lambda_i) = 0$ by (b). $\square$

We finish the section by stating the "intrinsic" role of the minimal prime component $\mathfrak{p}_{m(I)}$, the unique Herzog ideal containing the PCB binomial ideal I, among the other minimal primes of I. As was to be expected, $\mathfrak{p}_{m(I)}$ is the minimal prime ideal picked out by the element $(1, \dots, 1) \in \Lambda(D)$, which exists for an arbitrary coefficient field k.

Remark 6.9. Let I be the PCB ideal associated to L, $m(I)$ its associated integer vector, $d = \gcd(m(I))$ and $\nu(I) = m(I)/d$. Let $PLQ = D$ be a normal decomposition of L with $p_{n,*} = \nu(I)$. Let $d_1, \ldots, d_{n-1}$ be the invariant factors of L. Even if k does not contain the d_{n-1}-th roots of unity, we may write $\{\xi_{i,1}, \ldots, \xi_{i,d_i}\}$ for the set of d_i-th roots of unity in a field extension $\tilde{k}$ of k (allowing possible repetitions, by abuse of notation) and set $\Lambda(D) = \prod_{i=1}^{n-1}\{\xi_{i,1}, \ldots, \xi_{i,d_i}\} \subset (\tilde{k}^*)^{n-1}$. However, there is always one $\lambda \in \Lambda(D) \cap (k^*)^{n-1}$, namely $\lambda = (1, \ldots, 1)$. For this especial λ, $\varphi_\lambda : A \to k[t]$ sends x_i to t^{ν_i}. Therefore, according to Lemma 6.8 and Definition 2.2, $\mathfrak{a}_\lambda = \ker(\varphi_\lambda) = \mathfrak{p}_{\nu(I)} = \mathfrak{p}_{m(I)}$.

7. Main theorem

We are now in position to state the main result of the paper, recalling Theorem 4.10 for this purpose. As always, $A = k[\underline{x}]$ is the polynomial ring in n variables $\underline{x} = x_1, \ldots, x_n$ over a field k, $n \geqslant 2$, $\mathfrak{m} = (\underline{x})$ is the maximal ideal generated by $\underline{x}$, S is the multiplicatively closed set generated by $x = x_1 \cdots x_n$ and $B = S^{-1}A = k[\underline{x}^{\pm}] = k[x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}]$ is the Laurent polynomial ring. If I is an ideal of A, IB denotes the extension of I in B and $S(I) = IB \cap A$ the contraction of IB in A.

Theorem 7.1. *Let I be the PCB ideal associated to L, $m(I)$ its associated integer vector, $d = \gcd(m(I))$ and $\nu(I) = m(I)/d$. Let $PLQ = D$ be a normal decomposition of L with $p_{n,*} = \nu(I)$. Let $d_1, \ldots, d_{n-1}$ be the invariant factors of L.*

(a) *Suppose that k contains the d_{n-1}-th roots of unity and that the characteristic of k, $\mathrm{char}(k)$, is zero or $\mathrm{char}(k) = p$, p a prime with $p \nmid d_{n-1}$. Write $\{\xi_{i,1}, \ldots, \xi_{i,d_i}\}$ to denote the set of d_i-th roots of unity in k and $\Lambda(D) = \prod_{i=1}^{n-1}\{\xi_{i,1}, \ldots, \xi_{i,d_i}\}$. For any $\lambda \in \Lambda(D)$, set $\mathfrak{a}_\lambda = \ker(\varphi_\lambda)$, where $\varphi_\lambda : A \to k[t]$ is the natural map defined by the rule $\varphi_\lambda(x_i) = \lambda_1^{p_{1,i}} \cdots \lambda_{n-1}^{p_{n-1,i}} t^{\nu_i}$, for $i = 1, \ldots, n$. If $n \geqslant 4$, $I = \bigcap_{\lambda \in \Lambda(D)} \mathfrak{a}_\lambda \cap \mathfrak{Q}$, with $\mathfrak{Q}$ an irredundant $\mathfrak{m}$-primary ideal, is a minimal primary decomposition of I in A and I has exactly $d+1$ primary components. If $n \leqslant 3$, $I = \bigcap_{\lambda \in \Lambda(D)} \mathfrak{a}_\lambda$ is a minimal primary decomposition of I in A, I is radical and has exactly d primary components.*
(b) *Suppose that k is an arbitrary field. If $n \geqslant 4$, I is not unmixed and has at most $d+1$ primary components, only one of them embedded. If $n \leqslant 3$, I is unmixed and has at most d primary components.*

Proof. Let us show (a). Let $R = (r_{i,j})$ be the inverse of P and set $y_1 = x^{r_{*,1}}, \ldots, y_n = x^{r_{*,n}}$ in $B = k[\underline{x}^{\pm}]$. By Theorem 6.5(a), $IB = \bigcap_{\lambda \in \Lambda(D)} \mathfrak{b}_\lambda$ is a minimal primary decomposition of IB in B, where $\mathfrak{b}_\lambda = (y_1 - \lambda_1, \ldots, y_{n-1} - \lambda_{n-1})B$, and $S(I) = \bigcap_{\lambda \in \Lambda(D)} \mathfrak{b}_\lambda^c$ is a minimal primary decomposition of $S(I)$ in A. By Lemma 6.8, $\mathfrak{b}_\lambda^c = \mathfrak{a}_\lambda$. Thus $S(I) = \bigcap_{\lambda \in \Lambda(D)} \mathfrak{a}_\lambda$ is a minimal primary decomposition of $S(I)$ in A. If $n \geqslant 4$, by Proposition 4.8, I is not unmixed and, by Proposition 4.1(b), $I = S(I) \cap \mathfrak{Q}$, with $\mathfrak{Q}$ an $\mathfrak{m}$-primary ideal, and this intersection is irredundant. Therefore $I = \bigcap_{\lambda \in \Lambda(D)} \mathfrak{a}_\lambda \cap \mathfrak{Q}$ is a minimal primary decomposition of I in A and I has exactly $d+1$ primary components. On the other hand, if $n \leqslant 3$, by Remark 4.7, I is unmixed and, by Proposition 4.1(b), $I = S(I)$. Therefore $I = \bigcap_{\lambda \in \Lambda(D)} \mathfrak{a}_\lambda$ is a minimal primary decomposition of $S(I)$ in A, I is radical and has exactly d primary components. Finally, (b) follows from Theorem 6.5(b) and Proposition 4.1(d). $\square$

Remark 7.2. The thrust of [LV] is to calculate the degree of a lattice ideal that is homogeneous in the standard grading, whereas ours (cf. Theorem 7.1 above) is to calculate the number of primary components in a minimal binomial primary decomposition of a PCB ideal, as well as to describe such components explicitly. The two enterprises are of course linked to an extent by the Associativity Law of Multiplicities.

Example 7.3. Let $I = (f_1, f_2)$ be the PCB ideal of A associated to L, $n = 2$. Then $I = (x_1^{a_{1,1}} - x_2^{a_{2,2}})$, $m(I) = (a_{2,2}, a_{1,1})$ and $d = \gcd(m(I)) = \gcd(a_{1,1}, a_{2,2})$. Set $a_{i,i} = da'_{i,i}$ and $d = b_1 a_{1,1} + b_2 a_{2,2}$, for some $b_1, b_2 \in \mathbb{Z}$. Suppose that k contains the d-th roots of unity and that the characteristic of k, $\mathrm{char}(k)$, is zero or $\mathrm{char}(k) = p$, p a prime with $p \nmid d$. Write $\Lambda(D) = \{\xi_1, \ldots, \xi_d\}$ to denote the set of d-th roots of unity in k. For any $i = 1, \ldots, d$, set $\mathfrak{a}_i = \ker(\varphi_i)$, where $\varphi_i : A \to k[t]$ is the natural map defined by the rule $\varphi_i(x_1)=\xi_i^{b_1}t^{a'_{2,2}}$ and $\varphi_i(x_2)=\xi_i^{-b_2}t^{a'_{1,1}}$. By Example 5.4 and Theorem 7.1, $I=\mathfrak{a}_1\cap\cdots\cap\mathfrak{a}_d$ is a minimal primary decomposition of I in A, I is radical and has exactly d primary components.

Observe that each $\mathfrak{a}_i$ is a prime ideal of A of height 1, hence principal (see Lemma 6.8). Clearly, $x_1^{a'_{1,1}}-\xi_i x_2^{a'_{2,2}}$ is in $\mathfrak{a}_i$. A variation of the argument in [Fos, Corollary 10.15] proves that $x_1^{a'_{1,1}}-\xi_i x_2^{a'_{2,2}}$ is irreducible. Alternatively, let $\tilde{k}$ be a field extension of k containing an $a'_{2,2}$-th root η of ξ_i. Set $y_1=x_1$ and $y_2=\eta x_2$ and $A=k[x_1,x_2]\to\tilde{k}[x_1,x_2]=\tilde{k}[y_1,y_2]=:C$, a flat extension. Set $J=(x_1^{a'_{1,1}}-\xi_i x_2^{a'_{2,2}})C=(y_1^{a'_{1,1}}-y_2^{a'_{2,2}})$, a PCB ideal in $C=\tilde{k}[y_1,y_2]$, with $\gcd(a'_{1,1},a'_{2,2})=1$. Applying Theorem 7.1 or Corollary 6.6 to J, one deduces that J is prime, hence $JC\cap A=(x_1^{a'_{1,1}}-\xi_i x_2^{a'_{2,2}})$ is prime. Therefore $\mathfrak{a}_i=(x_1^{a'_{1,1}}-\xi_i x_2^{a'_{2,2}})$. Since $I=\mathfrak{a}_1\cap\cdots\cap\mathfrak{a}_d=\mathfrak{a}_1\cdots\mathfrak{a}_d$, the binomial $x_1^{a_{1,1}}-x_2^{a_{2,2}}$ admits the decomposition $\prod_{i=1}^d(x_1^{a'_{1,1}}-\xi_i x_2^{a'_{2,2}})$ as a product of irreducibles. In particular, any factor of $x_1^{a_{1,1}}-x_2^{a_{2,2}}$ is of the form $\prod_{j=1}^r(x_1^{a'_{1,1}}-\xi_{i_j}x_2^{a'_{2,2}})$, where $1\leqslant r\leqslant d$, i.e., $x_1^{ra'_{1,1}}+\eta_1 x_1^{(r-1)a'_{1,1}}x_2^{a'_{2,2}}+\cdots+\eta_{r-1}x_1^{a'_{1,1}}x_2^{(r-1)a'_{2,2}}+\eta_r x_2^{ra'_{2,2}}$, for some $\eta_j\in\tilde{k}$, which may or not be in k. This result recovers [OP2, Lemma 8.2].

Example 7.4. Let $I=(x_1^3-x_2x_3x_4,x_2^3-x_1x_3x_4,x_3^3-x_1x_2x_4,x_4^3-x_1x_2x_3)\subset A$, be the PCB ideal of Example 4.9. We know that $m(I)=(16,16,16,16)$ and $d=\gcd(m(I))=16$. Thus, by Theorem 7.1, I has at most seventeen primary components, one of them embedded. By Theorem 4.10, $\mathfrak{Q}=I+(x_2x_3^2)$ is an irredundant embedded $\mathfrak{m}$-primary component of I.

One can check that

$$\begin{pmatrix}1&0&0&0\\-1&1&0&0\\0&-1&1&0\\1&1&1&1\end{pmatrix}\begin{pmatrix}3&-1&-1&-1\\-1&3&-1&-1\\-1&-1&3&-1\\-1&-1&-1&3\end{pmatrix}\begin{pmatrix}1&2&1&1\\1&3&1&1\\1&3&2&1\\0&0&0&1\end{pmatrix}=\begin{pmatrix}1&0&0&0\\0&4&0&0\\0&0&4&0\\0&0&0&0\end{pmatrix}$$

is a normal decomposition of L. In particular, the invariant factors of L are $d_1=1$, $d_2=4$, $d_3=4$. Suppose that $k=\mathbb{C}$. Then $\Lambda(D)=\{1\}\times\{1,i,-1,-i\}\times\{1,i,-1,-i\}\subset\mathbb{C}^3$. According to Theorem 7.1, for a $\lambda\in\Lambda(D)$, the natural morphism $\varphi_\lambda:k[x_1,x_2,x_3,x_4]\to k[t]$ is defined by setting $\varphi_\lambda(x_1)=\lambda_1\lambda_2^{-1}t$, $\varphi_\lambda(x_2)=\lambda_2\lambda_3^{-1}t$, $\varphi_\lambda(x_3)=\lambda_3 t$ and $\varphi_\lambda(x_4)=t$. To simplify notations we just write the ordered 4-tuple $(\lambda_1\lambda_2^{-1}t,\lambda_2\lambda_3^{-1}t,\lambda_3 t,t)$ to describe the morphism φ_λ. Using this notation, the sixteen morphisms are the following:

$$(t,t,t,t),(t,-it,it,t),(t,-t,-t,t),(t,it,-it,t),(-it,it,t,t),(-it,t,it,t),$$
$$(-it,-it,-t,t),(-it,-t,-it,t),(-t,-t,t,t),(-t,it,it,t),(-t,t,-t,t),$$
$$(-t,-it,-it,t),(it,-it,t,t),(it,-t,it,t),(it,it,-t,t),(it,t,-it,t).$$

Therefore, $I=\bigcap_{i=1}^{16}\mathfrak{a}_i\cap\mathfrak{Q}$, where the sixteen minimal primary components $\mathfrak{a}_i$ are the kernels of the preceding morphisms:

$$\mathfrak{a}_1=(x_1-x_4,x_2-x_4,x_3-x_4),\qquad \mathfrak{a}_2=(x_1-x_4,x_2+ix_4,x_3-ix_4),$$
$$\mathfrak{a}_3=(x_1-x_4,x_2+x_4,x_3+x_4),\qquad \mathfrak{a}_4=(x_1-x_4,x_2-ix_4,x_3+ix_4),$$
$$\mathfrak{a}_5=(x_1+ix_4,x_2-ix_4,x_3-x_4),\qquad \mathfrak{a}_6=(x_1+ix_4,x_2-x_4,x_3-ix_4),$$
$$\mathfrak{a}_7=(x_1+ix_4,x_2+ix_4,x_3+x_4),\qquad \mathfrak{a}_8=(x_1+ix_4,x_2+x_4,x_3+ix_4),$$
$$\mathfrak{a}_9=(x_1+x_4,x_2+x_4,x_3-x_4),\qquad \mathfrak{a}_{10}=(x_1+x_4,x_2-ix_4,x_3-ix_4),$$

$$\mathfrak{a}_{11}=(x_1+x_4,x_2-x_4,x_3+x_4),\qquad \mathfrak{a}_{12}=(x_1+x_4,x_2+ix_4,x_3+ix_4),$$
$$\mathfrak{a}_{13}=(x_1-ix_4,x_2+ix_4,x_3-x_4),\qquad \mathfrak{a}_{14}=(x_1-ix_4,x_2+x_4,x_3-ix_4),$$
$$\mathfrak{a}_{15}=(x_1-ix_4,x_2-ix_4,x_3+x_4),\qquad \mathfrak{a}_{16}=(x_1-ix_4,x_2-x_4,x_3+ix_4).$$

Let us obtain the minimal primary components of I over $\mathbb{R}$ (and similarly over $\mathbb{Q}$). Consider the ideal $I_{2,4}:=(x_1-x_4,x_2+x_3,x_3^2+x_4^2)$ in $A=\mathbb{C}[\underline{x}]$. Clearly, $A/I_{2,4}\cong\mathbb{C}[x_3,x_4]/(x_3^2+x_4^2)$, so $I_{2,4}$ is a complete intersection of height 3, in particular, unmixed. Moreover, $I_{2,4}\subseteq\mathfrak{a}_2\cap\mathfrak{a}_4$, and if $\mathfrak{p}$ is a prime over $I_{2,4}$ we see that $\mathfrak{p}$ contains $\mathfrak{a}_2$ or $\mathfrak{a}_4$. Thus $\mathfrak{a}_2$ and $\mathfrak{a}_4$ are the associated primes of $I_{2,4}$. Since $x_3+ix_4\notin\mathfrak{a}_2$, x_3-ix_4 and $x_3^2+x_4^2$ are associated in $A_{\mathfrak{a}_2}$ and $(I_{2,4})_{\mathfrak{a}_2}=(\mathfrak{a}_2\cap\mathfrak{a}_4)_{\mathfrak{a}_2}$. Analogously $(I_{2,4})_{\mathfrak{a}_4}=(\mathfrak{a}_2\cap\mathfrak{a}_4)_{\mathfrak{a}_4}$. Therefore $I_{2,4}=\mathfrak{a}_2\cap\mathfrak{a}_4$. Similarly, we have

$$\begin{aligned}
I_{5,13}&:=\big(x_1+x_2,x_2^2+x_4^2,x_3-x_4\big)=\mathfrak{a}_5\cap\mathfrak{a}_{13},\\
I_{6,16}&:=\big(x_1+x_3,x_2-x_4,x_3^2+x_4^2\big)=\mathfrak{a}_6\cap\mathfrak{a}_{16},\\
I_{7,15}&:=\big(x_1-x_2,x_2^2+x_4^2,x_3+x_4\big)=\mathfrak{a}_7\cap\mathfrak{a}_{15},\\
I_{8,14}&:=\big(x_1-x_3,x_2+x_4,x_3^2+x_4^2\big)=\mathfrak{a}_8\cap\mathfrak{a}_{14}\quad\text{and}\\
I_{10,12}&:=\big(x_1+x_4,x_2-x_3,x_3^2+x_4^2\big)=\mathfrak{a}_{10}\cap\mathfrak{a}_{12}.
\end{aligned}$$

Therefore, $I=\mathfrak{a}_1\cap\mathfrak{a}_3\cap\mathfrak{a}_9\cap\mathfrak{a}_{11}\cap I_{2,4}\cap I_{5,13}\cap I_{6,16}\cap I_{7,15}\cap I_{8,14}\cap I_{10,12}\cap\mathfrak{Q}$. Note that the ideals appearing in this expression are generated by binomials with coefficients in $\mathbb{R}$. Let us momentarily denote by $\tilde{I}$, $\tilde{\mathfrak{a}}_i$, $\tilde{I}_{i,j}$ and $\tilde{\mathfrak{Q}}$ the corresponding ideals considered in $R=\mathbb{R}[\underline{x}]$, i.e., $\tilde{I}=(x_1^3-x_2x_3x_4,x_2^3-x_1x_3x_4,x_3^3-x_1x_2x_4,x_4^3-x_1x_2x_3)R$, $\tilde{\mathfrak{a}}_1=(x_1-x_4,x_2-x_4,x_3-x_4)R$ and so on. Clearly, their extension in A are the original ideals, i.e., $\tilde{I}A=I$, $\tilde{\mathfrak{a}}_iA=\mathfrak{a}_i$, $\tilde{I}_{i,j}A=I_{i,j}$ and $\tilde{\mathfrak{Q}}A=\mathfrak{Q}$. Moreover, since $R\to A$ is faithfully flat, $\tilde{I}=\tilde{I}A\cap R=I\cap R$, $\tilde{\mathfrak{a}}_i=\tilde{\mathfrak{a}}_iA\cap R=\mathfrak{a}_i\cap R$, $\tilde{I}_{i,j}=\tilde{I}_{i,j}A\cap R=I_{i,j}\cap R$ and $\tilde{\mathfrak{Q}}=\tilde{\mathfrak{Q}}A\cap R=\mathfrak{Q}\cap R$.

Hence $\tilde{I}=\tilde{\mathfrak{a}}_1\cap\tilde{\mathfrak{a}}_3\cap\tilde{\mathfrak{a}}_9\cap\tilde{\mathfrak{a}}_{11}\cap\tilde{I}_{2,4}\cap\tilde{I}_{5,13}\cap\tilde{I}_{6,16}\cap\tilde{I}_{7,15}\cap\tilde{I}_{8,14}\cap\tilde{I}_{10,12}\cap\tilde{\mathfrak{Q}}$ is a minimal primary decomposition of $\tilde{I}$ in $R=\mathbb{R}[\underline{x}]$. Indeed, $\tilde{\mathfrak{a}}_i$ is a prime ideal of R for $i=1,3,9$ and 11. Moreover, $R/\tilde{I}_{2,4}\cong\mathbb{R}[x_3,x_4]/(x_3^2+x_4^2)$, a domain, so $\tilde{I}_{2,4}=(x_1-x_4,x_2+x_3,x_3^2+x_4^2)$ is a prime ideal of R. Analogously, $I_{5,13}$, $I_{6,16}$ $I_{7,15}$, $I_{8,14}$ and $I_{10,12}$ are prime ideals of R. Moreover, applying Theorem 4.10 to the PCB ideal $\tilde{I}$ of $R=\mathbb{R}[\underline{x}]$, one obtains $\tilde{\mathfrak{Q}}$ as an irredundant embedded primary component of $\tilde{I}$. Finally, the full decomposition is irredundant because all the primes $\tilde{\mathfrak{a}}_i$ and $\tilde{I}_{i,j}$ appearing are different and of the same height.

Now suppose that $k=\mathbb{Z}/2\mathbb{Z}$. As before, I is not unmixed, $\mathfrak{Q}=I+(x_2x_3^2)$ is an irredundant embedded component of I and $d_1=1$, $d_2=4$ and $d_3=4$ are the invariant factors of L. By Lemma 6.3, $IB=(y_1-1,y_2^4-1,y_3^4-1)B=(y_1-1,(y_2-1)^4,(y_3-1)^4)B$. For $\lambda=(1,1,1)$, we clearly have $\mathfrak{b}_\lambda^4\subsetneq IB\subsetneq\mathfrak{b}_\lambda$, where $\mathfrak{b}_\lambda=(y_1-1,y_2-1,y_3-1)$ (see Notation 6.4). By Lemma 6.8, $\mathfrak{b}_\lambda^c=\mathfrak{a}_\lambda$ and $\mathfrak{a}_\lambda=(x_1-x_4,x_2-x_4,x_3-x_4)$. Hence $\mathfrak{a}_\lambda^4\subseteq(\mathfrak{b}_\lambda^4)^c\subseteq S(I)\subseteq\mathfrak{a}_\lambda$. Since $S(I)$ is unmixed (cf. Proposition 4.1), $S(I)$ is an $\mathfrak{a}_\lambda$-primary ideal. Therefore I has exactly two primary components, namely $\mathfrak{a}_\lambda$ and $\mathfrak{m}$.

Example 7.5. As a generalization of Example 7.4, for $n\geqslant 3$, let

$$I=\big(x_1^{n-1}-x_2\cdots x_n,\ldots,x_n^{n-1}-x_1\cdots x_{n-1}\big)$$

be the PCB ideal associated to the $n\times n$ PCB matrix L with diagonal entries $n-1$ and off-diagonal entries -1. One can check that the invariant factors of L are $d_1=1,d_2=n,\ldots,d_{n-1}=n$ and that a normal decomposition of $PLQ=D$ is given by

$$P=\begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 & 0\\ -1 & 1 & 0 & \dots & 0 & 0 & 0\\ 0 & -1 & 1 & \dots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \dots & 1 & 0 & 0\\ 0 & 0 & 0 & \dots & -1 & 1 & 0\\ 1 & 1 & 1 & \dots & 1 & 1 & 1 \end{pmatrix} \quad\text{and}\quad Q=\begin{pmatrix} 1 & n-2 & n-3 & \dots & 2 & 1 & 1\\ 1 & n-1 & n-3 & \dots & 2 & 1 & 1\\ 1 & n-1 & n-2 & \dots & 2 & 1 & 1\\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots\\ 1 & n-1 & n-2 & \dots & 3 & 1 & 1\\ 1 & n-1 & n-2 & \dots & 3 & 2 & 1\\ 0 & 0 & 0 & \dots & 0 & 0 & 1 \end{pmatrix}.$$

In particular, $d=d_1\dots d_{n-1}=n^{n-2}$ and $\nu(I)=(1,\dots,1)$. Therefore, by Theorem 7.1, I has at most $d+1=n^{n-2}+1$ prime components. Suppose that $k=\mathbb{C}$. For $\lambda\in\Lambda(D)$, let $\mathfrak{a}_\lambda=\ker(\varphi_\lambda)$ where $\varphi_\lambda: A\to k[t]$ is the natural map defined by the rule $\varphi_\lambda(x_i)=\lambda_i\lambda_{i+1}^{-1}t$, for $i=1,\dots,n-2$, $\varphi_\lambda(x_{n-1})=\lambda_{n-1}t$ and $\varphi_\lambda(x_n)=t$. Then each $\mathfrak{a}_\lambda$ is a prime ideal. If $n=3$, $I=\bigcap_{\lambda\in\Lambda(D)}\mathfrak{a}_\lambda$, whereas if $n\geqslant 4$, $I=\bigcap_{\lambda\in\Lambda(D)}\mathfrak{a}_\lambda\cap\mathfrak{Q}$ with $\mathfrak{Q}=I+(x^{b(n)})$ an $\mathfrak{m}$-primary ideal; in each case, these decompositions are irredundant.

Acknowledgments

We gratefully acknowledge Josep Àlvarez Montaner's interest and our useful discussions with him. We are grateful to Rafael H. Villarreal for his stimulating and generous comments. The authors gratefully acknowledge financial support from the research grant MTM2010-20279-C02-01 and the Universitat Politècnica de Catalunya during the development of this research. The first author benefited from the hospitality offered by the Departament de Matemàtica Aplicada 1 at UPC during his research visit there.

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