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Universal entire functions with gap power series

by Wolfgang Luh[1], Valeri A. Martirosian[2,*] and Jürgen Müller[1]

[1]*Fachbereich IV, Universität Trier, D-54286 Trier, Germany*

[2]*Department of Mathematics, Yerevan State University, Alex Manoogian St. 1, Yerevan 375049, Armenia*

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ABSTRACT

Let $\mathcal{M}$ be the family of all compact sets in $\mathbb{C}$ which have connected complement. For $K \in \mathcal{M}$ we denote by $A(K)$ the set of all functions which are continuous on K and holomorphic in its interior.

Suppose that $\{z_n\}$ is any unbounded sequence of complex numbers and let Q be a given subsequence of $\mathbb{N}_0$.

If Q has density $\Delta(Q) = 1$ then there exists a universal entire function φ with lacunary power series

(1) $\varphi(z) = \sum_{\nu=0}^{\infty} \varphi_\nu z^\nu, \quad \varphi_\nu = 0$ for $\nu \notin Q$,

which has for all $K \in \mathcal{M}$ the following properties simultaneously:

(2) the sequence $\{\varphi(z + z_n)\}$ is dense in $A(K)$;

(3) the sequence $\{\varphi(zz_n)\}$ is dense in $A(K)$ if $0 \notin K$.

Also a converse result is proved: If φ is an entire function of the form (1) which satisfies (3), then Q must have maximal density $\Delta_{\max}(Q) = 1$.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

For a compact set K in the complex plane $\mathbb{C}$ we denote by $A(K)$ the set of all complex valued functions, which are continuous on K and holomorphic in its interior K° (where possibly $K^\circ = \emptyset$). Introducing the norm

$$\|f\| := \max_K |f(z)|$$

$A(K)$ becomes a Banach space.

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By $\mathcal{M}$ we denote the family of all compact sets which have connected complement and $\mathcal{M}_0$ stands for the subfamily of all $K \in \mathcal{M}$ with $0 \notin K$.

The problems of the existence of so called 'universal functions' and the 'universal approximation' of functions are classical, and there is an extensive literature on the theory of functions, which are universal in different respects. The first example is due to G.D. Birkhoff [3], who proved the existence of an entire function ϕ with the property that for an arbitrary entire function f there exists a subsequence $\{n_k\}_{k\in\mathbb{N}}$ of the natural numbers $\mathbb{N}$, such that $\{\phi(z+n_k)\}_{k\in\mathbb{N}}$ converges to $f(z)$, compactly on $\mathbb{C}$. Hence the sequence of 'additive translates' $\{\phi(z+n)\}_{n\in\mathbb{N}}$ is dense in the space of all entire functions endowed with the topology of compact convergence.

P. Zappa [16] has established a theorem analogous to that of Birkhoff for the punctured plane $\mathbb{C}^* := \mathbb{C}\backslash\{0\}$. He proved the existence of a holomorphic function ϕ on $\mathbb{C}^*$ with the property that for any compact set $K \subset \mathbb{C}^*$, whose complement is connected in $\mathbb{C}^*$, the sequence $\{\phi(nz)\}_{n\in\mathbb{N}}$ is dense in $A(K)$.

In a recent paper L. Bernal-González and A. Montes-Rodriguez [2] characterized the sequences $\{\varphi_n\}_{n\in\mathbb{N}}$ of automorphisms of a domain $G \subset \mathbb{C}$ with the property, that there exists a universal holomorphic function ϕ on G such that the sequence $\{\phi\circ\varphi_n\}$ is dense in the space of all holomorphic functions on G.

For a brief resumé of the history of universal functions we refer to the articles [3] and [7], where further bibliographical remarks are given.

In the classical theory of universal functions the approximation theorems of C. Runge and S.N. Mergelian (see [15], [10] and [5]) are fundamental for the construction of those functions.

In this note we deal with the question whether refinements of results concerning universal functions are obtainable, if approximation by lacunary polynomials is taken as a basic tool for the construction of those functions. Other results involving lacunary approximation were obtained by M. Dixon and J. Korevaar [4], N.U. Arakelian and V.A. Martirosian [1], V.A. Martirosian [9] and J. Müller [11], [12].

If Q is a subsequence of $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$, we will denote by $\Delta(Q)$ its density (if it exists) and by $\Delta_{\max}(Q)$ its maximal density in the sense of G. Pólya [13], that is

$$\Delta_{\max}(Q) := \lim_{\mu\to 1^-}\left(\limsup_{r\to\infty}\frac{n_Q(r)-n_Q(\mu r)}{(1-\mu)r}\right),$$

where $n_Q(r)$ is the number of elements in $Q\cap[0,r]$.

In our main result we prove the existence of an entire function with a lacunary power series which has two universal properties simultaneously.

Theorem 1. *Let Q be any subsequence of $\mathbb{N}_0$ with density $\Delta(Q) = 1$ and let $S = \{z_n\}_{n\in\mathbb{N}}$ be any unbounded sequence of complex numbers. Then there exists an entire function ϕ with lacunary power series*

$$(1) \qquad \phi(z) = \sum_{\nu=0}^{\infty} \phi_\nu z^\nu, \quad \phi_\nu = 0 \text{ for } \nu \notin Q$$

which is universal in the double sense that

(A) The sequence of 'additive translates' $\{\phi(z + z_n)\}_{n \in \mathbb{N}}$ *is dense in* $A(K)$ *for all* $K \in \mathcal{M}$.

(B) The sequence of 'multiplicative translates' $\{\phi(zz_n)\}_{n \in \mathbb{N}}$ *is dense in* $A(K)$ *for all* $K \in \mathcal{M}_0$.

In the following result we investigate the sharpness of the density property of Q. We shall prove the following partial converse of Theorem 1.

Theorem 2. *Let* Q *be a given subsequence of* $\mathbb{N}_0$ *and let* $S = \{z_n\}_{n \in \mathbb{N}}$ *be any unbounded sequence of complex numbers. Suppose that* ϕ *is an entire function of the form* (1) *which is universal in the sense that it has property (B) of Theorem* 1. *Then* $\Delta_{\max}(Q) = 1$.

2. AUXILIARY RESULTS

For the proofs of our results some Lemmas are needed. We start with the following result.

Lemma 1. *There exists a sequence* $\{K_n\}_{n \in \mathbb{N}}$ *of sets* $K_n \in \mathcal{M}_0$ *with the property that for any* $K \in \mathcal{M}_0$ *there exists an* $n_0 \in \mathbb{N}$ *with* $K \subset K_{n_0}$.

A short proof can be found in [8].

The following result (compare with [4]) is an essential tool for our construction of universal functions.

Lemma 2. *Let* Q *be a subsequence of* $\mathbb{N}_0$ *with density* $\Delta(Q) = 1$ *and let* K *be a given set in* $\mathcal{M}$ *with* $K^\circ \neq \emptyset$ *and* $0 \in K^\circ$. *Suppose that the function* f *is holomorphic on* K *and that in a neighborhood of the origin it has a power series representation of the form*

$$f(z) = \sum_{\nu=0}^{\infty} f_\nu z^\nu, \quad \text{where} \quad f_\nu = 0 \text{ for } \nu \notin Q.$$

Then for every $\varepsilon > 0$ *there exists a 'lacunary' polynomial*

$$(2) \qquad P(z) = \sum_{\nu=0}^{N} p_\nu z^\nu, \quad \text{where} \quad p_\nu = 0 \text{ for } \nu \notin Q$$

such that

$$\max_K |f(z) - P(z)| < \varepsilon.$$

Proof. According to the Riesz representation theorem and the Hahn-Banach theorem, it is sufficient to show that for every Borel measure μ on K with

$$\int_K \zeta^n d\mu(\zeta) = 0 \quad \text{for } n \in Q \tag{3}$$

we have

$$\int_K f(\zeta) d\mu(\zeta) = 0.$$

Let a Borel measure μ on K with (3) be given, and let

$$h(z) := \int_K \frac{d\mu(\zeta)}{\zeta - z} \quad \text{for } z \in \hat{\mathbb{C}} \backslash K$$

be the Cauchy transform of μ. Then h is holomorphic in $\hat{\mathbb{C}} \backslash K$, and for $|z| > \max_{\zeta \in K} |\zeta|$ we have

$$\begin{aligned} h(z) &= \int_K \left(-\sum_{n=0}^{\infty} \frac{\zeta^n}{z^{n+1}} \right) d\mu(\zeta) = \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} \left(-\int_K \zeta^n d\mu(\zeta) \right) \\ &= -\sum_{n \notin Q} \frac{\alpha_n}{z^{n+1}} \end{aligned} \tag{4}$$

with $\alpha_n := \int_K \zeta^n d\mu(\zeta)$ for $n \notin Q$. Since $\Delta(Q) = 1$ and thus $\Delta(\mathbb{N}_0 \backslash Q) = 0$, the Fabry gap theorem [6] shows that h has a holomorphic extension to $|z| > \delta$, where $\delta := \operatorname{dist}(0, \partial K)$. Therefore the expansion (4) holds compactly in $|z| > \delta$.

Let $\Omega \subset \mathbb{C}$ be an open set containing K and such that f is holomorphic in Ω. Then there exists a contour Γ in $\Omega \backslash K$ such that

$$\operatorname{ind}_\Gamma(\alpha) = \begin{cases} 1, & \alpha \in K \\ 0, & \alpha \notin \Omega \end{cases}$$

(see for example [14], Theorem 13.5). From Cauchy's theorem we obtain

$$\frac{1}{2\pi i} \int_\Gamma \frac{f(w)}{w^{n+1}} dw = f_n = 0 \quad \text{for} \quad n \notin Q.$$

Since $\Gamma \subset \Omega \backslash K$, we have uniform convergence of (4) on Γ and therefore, according to Fubini's theorem, we find

$$\begin{aligned} \int_K f(\zeta) d\mu(\zeta) &= \int_K \frac{1}{2\pi i} \left[\int_\Gamma \frac{f(w)}{w - \zeta} dw \right] d\mu(\zeta) = \\ &= \frac{1}{2\pi i} \int_\Gamma f(w) \left[\int_K \frac{d\mu(\zeta)}{w - \zeta} \right] dw = \\ &= \sum_{n \notin Q} \alpha_n \frac{1}{2\pi i} \int_\Gamma \frac{f(w)}{w^{n+1}} dw = 0. \end{aligned}$$

□

In order to state the next result, we use the following notation. If a compact set K and a number $\delta \in [0, \pi)$ are given, we define

$$K_\delta := \bigcup_{|\varphi| \le \delta} \{ w : w = z e^{i\varphi}, z \in K \}.$$

Lemma 3. *Let Q be a subsequence of $\mathbb{N}_0$ with the density $\Delta(Q) = \Delta \in [0, 1)$ and let f be an entire function with a power series representation*

$$f(z) = \sum_{\nu=0}^{\infty} f_\nu z^\nu, \qquad \textit{where } f_\nu = 0 \textit{ for } \nu \notin Q.$$

Suppose that there exists a sequence of closed disks

$$C_m := \{z : |z - \zeta_m| \le r_m\}$$

with the properties

$$\lim_{m \to \infty} \zeta_m = \infty, \qquad 1 < \liminf_{m \to \infty} \frac{|\zeta_m|}{r_m} < \infty$$

and a constant C (independent of m) such that

$$|f(z)| \le C \qquad \textit{for all } z \in (C_m)_{\Delta\pi}.$$

Then f is a constant.

For a proof see [9, Lemma 8].

3. PROOF OF THEOREM 1

1. We assume that $\{K_n\}_{n \in \mathbb{N}}$ is a sequence of compacta as in Lemma 1 and that $\{\Pi_n\}_{n \in \mathbb{N}}$ is an enumeration of all the polynomials $\Pi_n(z) \not\equiv 0$ whose coefficients have rational real and imaginary parts. Let $\mathcal{L} = \{(K_n^*, \Pi_n^*)\}_{n \in \mathbb{N}}$ be a countable listing of all the pairs (K_ν, Π_μ) with the property that every such pair occurs infinitely often on $\mathcal{L}$.

2. By an inductive procedure we may choose a sequence $\rho = \{\rho_n\}_{n \in \mathbb{N}_0}$ of natural numbers with $\rho_0 = 1$ and $\rho_n \ge n + 1$ and subsequences $a = \{a_n\}_{n \in \mathbb{N}}$ and $b = \{b_n\}_{n \in \mathbb{N}}$ of $\{z_n\}_{n \in \mathbb{N}}$ with $|a_n| \ge n$ such that the following properties hold:

If $A_n := a_n K_n^*$, if $B_n := \{z : |z - b_n| \le n\}$ and $C_n := \{z : |z| \le \rho_n\}$, then we have

$$A_n \cap B_n = \emptyset, \quad (A_n \cup B_n) \cap C_{n-1} = \emptyset, \quad A_n \cup B_n \subset C_n.$$

Obviously the sets A_n, B_n, C_{n-1} are pairwise disjoint.

3. We now construct a sequence $\{P_n\}_{n \in \mathbb{N}_0}$ of polynomials of the form (2). Let us assume, that the polynomials $P_0 \equiv 0, P_1, \dots, P_{n-1}$ have already been determined. Then according to Lemma 2 there exists a polynomial P_n of the form (2) such that the following three conditions hold simultaneously:

$$\max_{C_{n-1}} |P_n(z) - P_{n-1}(z)| < \frac{1}{n^2}, \tag{5}$$

$$\max_{B_n} |P_n(z) - \Pi_n(z - b_n)| < \frac{1}{n}, \tag{6}$$

$$\max_{A_n} \left| P_n(z) - \Pi_n^*\left(\frac{z}{a_n}\right) \right| < \frac{1}{n}. \tag{7}$$

By induction we obtain the sequence $\{P_n\}_{n \in \mathbb{N}_0}$.

4. Let the function ϕ be defined by

$$\phi(z) := \sum_{\mu=1}^{\infty} \{P_\mu(z) - P_{\mu-1}(z)\}.$$

Then (5) implies that ϕ is an entire function and it obviously has the desired representation (1).

For all $\mu \geq n+1$ we have $B_n \subset C_n \subset C_{\mu-1}$ and we obtain from (6) and (5)

$$\max_{|z|\leq n} |\phi(z+b_n) - \Pi_n(z)| = \max_{B_n} |\phi(z) - \Pi_n(z-b_n)| \leq$$

$$(8) \qquad \leq \max_{B_n} |P_n(z) - \Pi_n(z-b_n)| + \sum_{\mu=n+1}^{\infty} \max_{C_{\mu-1}} |P_\mu(z) - P_{\mu-1}(z)| <$$

$$< \frac{2}{n}.$$

Analogously we have $A_n \subset C_n \subset C_{\mu-1}$ for all $\mu \geq n+1$ and (7) and (5) imply

$$\max_{K_n^*} |\phi(a_n z) - \Pi_n^*(z)| = \max_{A_n} \left|\phi(z) - \Pi_n^*\left(\frac{z}{a_n}\right)\right| \leq$$

$$(9) \qquad \leq \max_{A_n} \left|P_n(z) - \Pi_n^*\left(\frac{z}{a_n}\right)\right| + \sum_{\mu=n+1}^{\infty} \max_{C_{\mu-1}} |P_\mu(z) - P_{\mu-1}(z)| <$$

$$< \frac{2}{n}.$$

5. Let K be given set in $\mathcal{M}$ and let f be a function in $A(K)$. By the theorem of Mergelian there exists a subsequence $\{k_s\}_{s\in\mathbb{N}}$ of $\mathbb{N}$ such that

$$(10) \qquad \max_K |f(z) - \Pi_{k_s}(z)| < \frac{1}{s} \quad \text{for all } s \in \mathbb{N}.$$

a) Since $K \subset \{z : |z| \leq k_s\}$ for all sufficiently large s, it follows from (8) and (10) for those s that

$$\max_K |\phi(z+b_{k_s}) - f(z)| \leq$$

$$\leq \max_{|z|\leq k_s} |\phi(z+b_{k_s}) - \Pi_{k_s}(z)| + \max_K |\Pi_{k_s}(z) - f(z)| <$$

$$< \frac{2}{k_s} + \frac{1}{s}.$$

Therefore the sequence $\{\phi(z+z_n)\}_{n\in\mathbb{N}}$ is dense in $A(K)$.

b) Let us now suppose that $K \in \mathcal{M}_0$. Then by Lemma 1 there exists an n_0 with $K \subset K_{n_0}$. By our definition of $\mathcal{L}$ there exists a strictly increasing sequence $\{n_\ell^{(s)}\}_{\ell\in\mathbb{N}}$ of natural numbers such that $(K_{n_0}, \Pi_{k_s}) = \left(K^*_{n_\ell^{(s)}}, \Pi^*_{n_\ell^{(s)}}\right)$ for all $\ell \in \mathbb{N}$.

The estimate (9) therefore guarantees, that for any $s \in \mathbb{N}$ there exists a $p_s := n_{\ell_s}^{(s)}$ such that

$$\max_{K_{n_0}} |\phi(a_{p_s} z) - \Pi_{k_s}(z)| < \frac{2}{s},$$

which together with (10) implies that

$$\max_K |\phi(a_{p_s}z) - f(z)| < \frac{3}{s}.$$

Therefore the sequence $\{\phi(zz_n)\}_{n\in\mathbb{N}}$ is dence in $A(K)$. This completes the proof of Theorem 1. $\square$

4. PROOF OF THEOREM 2

Suppose that the sequence Q satisfies $\Delta_{\max}(Q) < 1$ and that there exists a universal entire function ϕ of the form (1) for which the property (B) of Theorem 1 holds.

From the definition of maximal density it follows that we may assume without loss of generality that Q has ordinary density $\Delta(Q) = \Delta \in [0, 1)$.

We consider a disk $C := \{z : |z - \zeta| \le r\}$ with $|\zeta| > r$ and $C_{\pi\Delta} \in \mathcal{M}_0$. Since it is assumed that ϕ satisfies condition (B), there exists a subsequence $\{n_k\}_{k\in\mathbb{N}}$ of $\mathbb{N}$ such that $z_{n_k} \to \infty$ and

$$|\phi(zz_{n_k})| \le 1 \quad \text{for all} \quad z \in C_{\pi\Delta}$$

or equivalently

$$|\phi(z)| \le 1 \quad \text{for all} \quad z \in z_{n_k} \cdot C_{\pi\Delta}.$$

If we now apply Lemma 3 to the function ϕ, we derive that ϕ must be a constant, which is impossible. This completes the proof of Theorem 2. $\square$

Remark. The authors also investigated the existence of universal functions with lacunary power series in general open sets. In this situation more subtle density properties of the sequence Q are essential. The results will soon be published elsewhere.

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