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Advances in Mathematics 229 (2012) 731–769

ADVANCES IN
Mathematics

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Compactness results for immersions of prescribed Gaussian curvature I – analytic aspects

Graham Smith

Departament de Matemàtiques, Facultat de Ciències, Edifici C, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Spain

Received 10 December 2009; accepted 17 October 2011

Available online 8 November 2011

Communicated by Luis Caffarelli

Dedicated to Hennie Smith
22/12/1914–08/12/2009
With affection

Abstract

We extend recent results of Guan and Spruck, proving existence results for constant Gaussian curvature hypersurfaces in Hadamard manifolds.

MSC: primary 58E12; secondary 35J25, 35J60, 53A10, 53C21, 53C42

Keywords: Gaussian curvature; Plateau problem; Monge–Ampère equation; Non-linear elliptic PDEs

1. Introduction

Let $M := M^{n+1}$ be an $(n+1)$-dimensional Riemannian manifold. An immersed hypersurface in M is a pair $(\Sigma, \partial\Sigma) := ((S, \partial S), i)$ where $(S, \partial S)$ is a compact, n-dimensional manifold with boundary and $i : S \to M$ is an immersion (that is, a smooth mapping whose derivative is everywhere injective). Throughout the sequel we abuse notation and denote $(S, \partial S)$ also by $(\Sigma, \partial\Sigma)$. We recall that the shape operator of the immersion is defined at each point by taking the covariant derivative in M of the unit, normal vector field over Σ at that point, and that the

E-mail address: grahamandrewsmith@gmail.com.

0001-8708/$ – see front matter
doi:10.1016/j.aim.2011.10.008

Gaussian curvature (also called the extrinsic curvature) is then defined as a function over S to be equal to the determinant of the shape operator at each point.

Geometers have studied the concept of Gaussian curvature ever since Gauss first proved in [6] his famous *Teorema Egregium* which states that the Gaussian curvature of a surface immersed in $\mathbb{R}^3$ only depends on its intrinsic geometry and not on the immersion, which explains, for example, why a flat sheet of paper cannot be smoothly wrapped round a portion of the sphere. In more recent times, the Gaussian curvature of a hypersurface has revealed itself as an interesting object of study also from the perspective of geometric analysis as a straightforward and archetypal case of a much larger class of problems, including those of affine geometry, mass transport, Calabi–Yau geometry and so on, all of whose underlying equations are of so-called Monge–Ampère type.

In studying hypersurfaces of constant curvature of any sort, the most natural problems to study are those of Plateau and Minkowski, which ask respectively for the existence of hypersurfaces of constant curvature with prescribed boundary, or without boundary but instead satisfying certain topological conditions. The study of these problems has enjoyed a rich development over the last century, with the application of a wide variety of different techniques, including, for example, polyhedral approximation, used by Pogorelov to solve the Minkowski problem for convex, immersed spheres of prescribed Gaussian curvature in Euclidean space (cf. [16]), and, more recently, the continuity method, as used by Caffarelli, Nirenberg and Spruck (cf. [3]) to solve the Plateau problem for locally strictly convex (LSC) hypersurfaces which are graphs over a given hyperplane in Euclidean space. The ideas of Caffarelli, Nirenberg and Spruck were further developed in one direction by Rosenberg and Spruck (cf. [17]) to prove the existence of LSC hypersurfaces of constant extrinsic curvature in hyperbolic space with prescribed asymptotic boundary in the sphere at infinity (which was in turn generalized by Guan and Spruck in [11] and [12] to treat more general notions of curvature). Likewise they were developed in another direction by Guan and Spruck in [9] to prove existence of LSC hypersurfaces of constant extrinsic curvature in Euclidean space with prescribed boundary in the unit sphere. This led Spruck to conjecture in [23] that any compact, codimension 2, immersed submanifold in Euclidean space which is the boundary of an LSC, immersed hypersurface is also the boundary of an LSC, immersed hypersurface of constant Gaussian curvature, a conjecture which was confirmed simultaneously by Guan and Spruck in [10] and Trudinger and Wang in [24] using in both cases a combination of Caffarelli, Nirenberg and Spruck's continuity method alongside an elegant application of the Perron method.

With the exception of [17], the above results essentially concern submanifolds of $\mathbb{R}^{n+1}$ and mild generalizations of this setup, and since most of the techniques used above rely in some way or another on the geometry of Euclidean space, the problem in general ambient manifolds has remained largely open. Nonetheless, in [14], Labourie showed how pseudo-holomorphic geometry may be applied in conjunction with a parametric version of the continuity method to solve the Plateau problem in the case where M is a 3-dimensional Hadamard manifold. However, since this approach relies on techniques of holomorphic function theory, it does not easily generalize to the higher dimensional case, which has therefore hitherto remained unsolved. It is to fill this gap that we present in this and our forthcoming work [18] an approach which allows us to solve the Plateau problem for hypersurfaces of constant (or prescribed) Gaussian curvature in general manifolds, thus generalizing the results [10] and [24] of Guan and Spruck and Trudinger and Wang on the one hand and the result [14] of Labourie on the other. In the interest of simplicity, we henceforth restrict attention to **Hadamard manifolds**, which, we recall, are, by definition, complete, simply connected manifolds of non-positive sectional curvature. We leave the enthusiastic reader to investigate the few extra technical conditions required to state and prove the results in general manifolds.

In the current paper, we will essentially be concerned with the local problem of finding solutions under conditions that are typically only valid over small regions. We will be mostly interested in the analysis required to obtain a priori estimates and compactness results. In the forthcoming work, geometric results will be developed which will allow us to apply the estimates obtained here also to the global problem, as we will briefly discuss towards the end of this introduction.

Thus let M be a Hadamard manifold. Let $(\Sigma_0, \partial\Sigma_0)$ be a smooth, convex, immersed hypersurface in M with smooth boundary. Let N be the exterior, unit, normal vector field over Σ_0 and define $\mathcal{E} : \Sigma_0 \times]-\infty, 0] \to M$ by

$$\mathcal{E}(x,t) = \mathrm{Exp}(-t\mathsf{N}).$$

We have chosen here an unusual sign convention which we prefer for technical reasons. We say that a $C^{0,1}$ hypersurface $(\Sigma, \partial\Sigma)$ is a **graph below** Σ_0 if and only if there exists a $C^{0,1}$ function $f : \Sigma_0 \to]-\infty, 0]$ and a homeomorphism $\varphi : \Sigma_0 \to \Sigma$ such that:

(i) f vanishes along $\partial\Sigma_0$ (i.e. $\partial\Sigma = \partial\Sigma_0$); and
(ii) for all $p \in \Sigma_0$:

$$\varphi(p) = \mathcal{E}\big(p, f(p)\big) = \mathrm{Exp}_p\big(-f(p)\mathsf{N}(p)\big).$$

Let $(\hat{\Sigma}, \partial\hat{\Sigma})$ be a $C^{0,1}$, convex, immersed hypersurface in M which is a graph below Σ_0.

We denote by $C_0^\infty(\Sigma_0)$ the space of smooth functions over Σ_0 which vanish along the boundary, and we identify surfaces which are graphs below Σ_0 with functions in $C_0^\infty(\Sigma_0)$. Gaussian curvature defines an operator $K : C_0^\infty(\Sigma_0) \to C^\infty(\Sigma_0)$ such that, for all $f \in C_0^\infty(\Sigma_0)$ and for all $p \in \Sigma$, $K(f)(p)$ is the Gaussian curvature of the graph of f at the point below p. When the graph of f is convex, the linearization DK_f of K at f is a second order, elliptic, partial differential operator. In particular, this is the case for DK_0, the linearization of K at the zero function, and we say that $(\Sigma_0, \hat{\Sigma})$ is **stable** if and only if, for all $\psi \in C_0^\infty(\Sigma_0)$, if $DK_0 \cdot \psi \geqslant 0$, then $\psi < 0$ over the interior of Σ_0.

We say that $(\Sigma_0, \hat{\Sigma})$ is **rigid** if and only if there exists no other smooth hypersurface Σ lying between Σ_0 and $\hat{\Sigma}$ such that $K(\Sigma) = K(\Sigma_0)$.

In general, stable and rigid pairs of surfaces are relatively easy to construct inside small regions. For example, if Σ_0 is a bounded portion of a hypersurface in hyperbolic space which lies at constant distance from a totally geodesic hypersurface, then $(\Sigma_0, \hat{\Sigma})$ is both stable and rigid for any choice of $\hat{\Sigma}$. We refer the reader to Section 9 for more details.

We prove the following local result:

Theorem 1.1. *Choose $k > 0$ and suppose that the Gaussian curvature of Σ_0 is less than k. Suppose, moreover, that for some $\epsilon > 0$ the Gaussian curvature of $\hat{\Sigma}$ is no less than $k + \epsilon$ in the weak (Alexandrov) sense and that the second fundamental form of $\hat{\Sigma}$ is also no less than ϵ in the weak (Alexandrov) sense. If $(\Sigma_0, \hat{\Sigma})$ is stable and rigid, then there exists a smooth, convex, immersed hypersurface Σ_k such that:*

(i) *Σ_k is a graph below Σ_0;*
(ii) *Σ_k lies between Σ_0 and $\hat{\Sigma}$ as a graph below Σ_0; and*
(iii) *the Gaussian curvature of Σ_k is constant and equal to k.*

Remark. This follows immediately from Lemma 10.2.

Remark. The weak (Alexandrov) notion of lower (and upper) bounds for curvature is defined in Section 4.

Remark. The hypothesis that M be a Hadamard manifold is only made for simplicity of presentation. The same result, with appropriate modifications, continues to hold in more general manifolds.

When M is in addition a space form (and thus, up to rescaling, isometric to $\mathbb{R}^{n+1}$ or $\mathbb{H}^{n+1}$), the Perron method may be applied to solve the following more general boundary value problem: let $\Gamma = (\Gamma_1, \ldots, \Gamma_n)$ be a disjoint collection of closed, smooth, embedded, $(n-1)$-dimensional submanifolds of M. Applying the machinery developed by Guan and Spruck in [10] along with Lemma 11.3 (which constitutes the more precise version of Theorem 1.1 when $M = \mathbb{H}^{n+1}$) in place of Theorem 1.1 of [8], we immediately obtain:

Theorem 1.2. *Let M be a space form of non-positive curvature. Choose $k > 0$. Suppose that there exists a C^2, LSC, immersed hypersurface $\hat{\Sigma} \subseteq M$ of Gaussian curvature no less than k such that $\partial\hat{\Sigma} = \Gamma$. Then there exists a smooth (up to the boundary), locally strictly convex, immersed hypersurface $\Sigma \subseteq M$ with $\partial\Sigma = \Gamma$ of constant Gaussian curvature equal to k. Moreover, Σ is homeomorphic to $\hat{\Sigma}$.*

The proof of Theorem 1.1 follows the analysis of Caffarelli, Nirenberg and Spruck first laid out in [2] and first applied to constant curvature hypersurfaces by the same authors in [3]. Our current work uses two key developments which simplify the analysis. The first, which is merely a question of perspective, is to analyze the Gauss Curvature Equation intrinsically along the hypersurface as in Section 6, and the second is the use of Sard's Lemma in Section 8 to generate smooth families of hypersurfaces interpolating between the data and the desired solution, which simplifies the topological approach already suggested by the work [8] of Guan. In Section 12 we show how our techniques can be easily adapted to recover both the results [8] of Guan and [17] of Rosenberg and Spruck.

As discussed previously, our main aim is to obtain a global existence result which confirms the natural extension of Spruck's conjecture (cf. [23]) to more general manifolds. As we shall see in our forthcoming work [18], the most significant new obstacle is the geometric problem of developing compactness results in general manifolds for LSC immersions with prescribed boundary. Having solved this problem, we return to Theorem 1.1, this time removing Σ_0 whilst also allowing $\hat{\Sigma}$ to vary between the data of the problem and a setup which may readily be shown to be stable and rigid. Then, proceeding as before, we obtain the following result, which is a mild simplification of the main result of [18]:

Theorem 1.3. *Let M be a Hadamard manifold. Let $(\hat{\Sigma}, \partial\hat{\Sigma})$ be a locally strictly convex, immersed hypersurface in M whose boundary intersects itself transversally. Let $k > 0$ be such that the Gaussian curvature of $\hat{\Sigma}$ is everywhere strictly greater than k.*

Suppose that there exists a convex subset $K \subseteq M$ with smooth boundary and an open subset $\Omega \subseteq \partial K$ such that:

(i) *$\partial\Omega$ is smooth; and*
(ii) *$(\hat{\Sigma}, \partial\hat{\Sigma})$ is isotopic through locally strictly convex immersions to a finite covering of $(\Omega, \partial\Omega)$,*

then there exists a locally strictly convex, immersed hypersurface $(\Sigma, \partial\Sigma)$ *in* M *such that:*

(i) $\partial\Sigma = \partial\hat{\Sigma}$*; and*
(ii) Σ *has constant Gaussian curvature equal to* k.

Remark. Since the submission of these papers, we have shown (cf. [22]) that any locally strictly convex immersion is isotopic through locally strictly convex immersions to such a covering of an open subset of the boundary of a convex set, and so this condition is in fact redundant. We have chosen nonetheless to retain it here in order to keep this and the forthcoming paper as self-contained as possible.

This paper is structured as follows:

(a) in Section 2, we show how first order bounds arise as a consequence of convexity;
(b) in Section 3, we derive the Gauss Curvature Equation for a graph in a general Riemannian manifold;
(c) in Section 4, we introduce the concept of weak (Alexandrov) lower and upper bounds for curvature;
(d) in Sections 5 and 6 we obtain a priori second order bounds over the boundary and then over the whole hypersurface respectively. These bounds are then applied in Section 7 to obtain the compactness result, Lemma 7.1;
(e) in Section 8, we use Sard's Lemma to obtain smooth (albeit possibly empty) 1-dimensional families of hypersurfaces interpolating between the data and the solutions. These are used in conjunction with the concepts of stability, rigidity and local rigidity developed in Sections 9 and 10 to prove in Section 10 the existence result, Lemma 10.2, which immediately yields Theorem 1.1;
(f) in Section 11, we restrict attention to space forms, proving Lemma 11.3, which, in conjunction with the machinery developed by Guan and Spruck in [10] immediately yields Theorem 1.2;
(g) in Section 12, we show how minor adaptations of these techniques allow us to obtain both the results [8] of Guan (Theorem 12.1) and [17] of Rosenberg and Spruck (Theorem 12.2); and
(h) in Appendix A, we prove the regularity of limiting hypersurfaces which are themselves strictly convex. This result may be found in the notes of Caffarelli [1], but given their general public unavailability, we consider it preferable to provide our own proof here.

This paper was written whilst the author was staying at the Mathematics Department of the University Autonoma de Barcelona, Bellaterra, Spain.

2. First order control

Let $M := M^{n+1}$ be an $(n+1)$-dimensional Riemannian manifold. Let $(\Sigma_0, \partial\Sigma_0)$ be a convex, immersed hypersurface with boundary. Let N_0 and A_0 denote the outward pointing unit normal and the second fundamental form respectively of Σ_0. We define $\mathcal{E} : \Sigma_0 \times]-\infty, 0] \to M$ by

$$\mathcal{E}(x, t) = \mathrm{Exp}\big(-t\mathsf{N}_0(x)\big).$$

Remark. The change of sign ensures that convex hypersurfaces correspond to graphs of convex functions.

We will say that a $C^{0,1}$ hypersurface, Σ, is a **graph below** Ω if and only if there exists a $C^{0,1}$ function $f:\overline{\Omega}\to\,]-\infty,0]$ and a homeomorphism $\varphi:\overline{\Omega}\to\Sigma$ such that:

(i) f vanishes along $\partial\Omega$ (i.e. $\partial\Sigma=\partial\Omega$); and
(ii) for all $p\in\Omega$:

$$\varphi(p)=\mathcal{E}\big(p,f(p)\big)=\mathrm{Exp}_p\big(-f(p)\mathsf{N}_0(p)\big).$$

We refer to f as the graph function of Σ. In particular, since f is Lipschitz, its graph is never vertical, even along the boundary. Consider the family of graphs over Ω. We define the partial order "$<$" on this family such that if Σ and Σ' are two graphs over Ω and f and f' are their respective graph functions, then

$$\Sigma<\Sigma' \quad\Leftrightarrow\quad f(p)<f'(p)\quad\text{for all } p\in\Omega.$$

Since $\partial\Omega$ is smooth, for all $p\in\partial\Omega$, the set of supporting hyperplanes in TM to $\partial\Omega$ at p is parametrized by $\mathbb{R}$. Supporting hyperplanes may be locally considered as graphs over Ω, and we obtain an analogous partial order on this set which we also denote by $<$.

Let $\hat{\Sigma}$ be a $C^{0,1}$ convex hypersurface which is a graph over Ω. Let $(\Sigma_n)_{n\in\mathbb{N}}$ be a sequence of convex graphs over Ω such that for all $n\in\mathbb{N}$, $\Sigma_n>\hat{\Sigma}$. For all n, let f_n be the graph function of Σ_n.

Lemma 2.1. *$(f_n)_{n\in\mathbb{N}}$ is uniformly bounded in the $C^{0,1}$ sense.*

Proof. For all $n\in\mathbb{N}\cup\{\infty\}$, define U_n by

$$U_n=\big\{\mathrm{Exp}_p\big(-t\mathsf{N}_0(p)\big)\text{ s.t. } p\in\overline{\Omega}\text{ and } 0\leqslant t\leqslant f_n(p)\big\}.$$

By compactness of the family of convex sets, after extraction of a subsequence, there exists U_0 towards which $(U_n)_{n\in\mathbb{N}}$ converges in the Hausdorff sense. Moreover, the supporting hyperplanes of U_0 are transverse to the normal geodesics leaving H. Indeed, suppose the contrary and let $p_0\in\partial U_0$ be a point where the supporting hyperplane is not transverse to the normal geodesic leaving Σ_0. Taking limits $\partial U_0\geqslant\hat{\Sigma}$. Since the tangent to $\hat{\Sigma}$ along $\partial\hat{\Sigma}$ is not vertical, it follows that p_0 lies over an interior point of Σ_0. Let $(p_n)_{n\in\mathbb{N}}\in(\partial U_n)_{n\in\mathbb{N}}$ be a sequence converging to p_0. For all $n\in\mathbb{N}\cup\{0\}$, let $q_n\in\Sigma_0$ be the orthogonal projection of p_n onto Σ_0 and let γ_n be the geodesic segment joining q_n to p_n. For all $n\in\mathbb{N}$, $\gamma_n\subseteq U_n$. Taking limits, $\gamma_0\subseteq U_0$. It follows that γ_0 is an interior tangent to ∂U_0 at p_0. Therefore, by convexity, $\gamma_0\subseteq\partial U_0$. In particular, U_0 has a vertical supporting tangent at q_0, which is absurd. By compactness, we deduce that the supporting tangent hyperplanes of $(\partial U_n)_{n\in\mathbb{N}}$ are uniformly transverse to the foliation of normal geodesics leaving Σ_0, and the result follows. □

3. The Gauss Curvature Equation

Let $M:=M^{n+1}$ be an $(n+1)$-dimensional Riemannian manifold. Let $(\Sigma_0,\partial\Sigma_0)\subseteq M$ be a convex, immersed hypersurface with boundary. Let N_0 and A_0 denote the outward pointing, unit normal and the second fundamental form respectively of Σ_0. Using the exponential map, we identify an open subset of M with $\Sigma_0 \times]-\infty, 0]$.

We will prove:

Proposition 3.1. *Let $f : \Sigma_0 \to]-\infty, 0]$ be a smooth function. The Gaussian curvature of the graph of f is given by*

$$K = \psi(x, f, \nabla f)^{-1} \operatorname{Det}\big(\operatorname{Hess}(f) + \Psi(x, f, \nabla f)\big)^{1/n},$$

where:

(i) *$\psi = \psi(x, t, p)$ is a smooth, strictly positive function and, for all $R > 0$ there exists $\epsilon > 0$ such that if $|t| < \epsilon$ then $\psi(x, t, p)$ is convex in p for $\|p\| \leqslant R$; and*
(ii) *there exists a smooth function Ψ_0 such that:*

$$\Psi(x, f, \nabla f)_{ij} = A_{0,ij} + f_{;i} f_{;k} A^k_{0\,j} + f_{;j} f_{;k} A^k_{0\,i} + f \Psi_0(x, f, \nabla f).$$

Moreover, the graph of f is convex if and only if $\operatorname{Hess}(f) + \Psi(x, f, \nabla f)$ is positive definite.

Example. We view $\mathbb{H}^n$ as a totally geodesic, embedded hypersurface in $\mathbb{H}^{n+1}$. Let g_0 and g be the metrics of $\mathbb{H}^n$ and $\mathbb{H}^{n+1}$ respectively. We consider the foliation of $\mathbb{H}^{n+1}$ by geodesics normal to $\mathbb{H}^n$. Exceptionally, we reparametrize geodesics in a non-uniform manner in order to make this parametrization conformal which simplifies the calculation of the connexion 2-form. Let $\alpha :]-\pi/2, \pi/2[\to \mathbb{R}$ be such that, for all θ:

$$\cos(\theta)\cosh\big(\alpha(\theta)\big) = 1.$$

Let N be the unit, normal vector field over $\mathbb{H}^n$ in $\mathbb{H}^{n+1}$. We define $\Phi : \mathbb{H}^n \times]-\pi/2, \pi/2[\to \mathbb{H}^{n+1}$ by

$$\Phi(x, \theta) = \operatorname{Exp}\big(-\alpha(t)\mathsf{N}(x)\big).$$

We easily obtain:

$$\Phi^* g = \frac{1}{\cos^2(\theta)}\big(g_0 \oplus d\theta^2\big).$$

If Ω denotes the connexion 2-form of the Levi-Civita covariant derivative of $\Phi^* g$ with respect to that of the product metric, then, for all X, Y tangent to $\mathbb{H}^n$:

$$\Omega(X, Y) = -\langle X, Y\rangle \tan(\theta)\partial_\theta,$$

$$\Omega(X, \partial_\theta) = \tan(\theta) X,$$

$$\Omega(\partial_\theta, \partial_\theta) = \tan(\theta)\partial_\theta.$$

Thus, if $\Omega \subseteq \mathbb{H}^n$ is an open set, and if $f : \Omega \to]-\pi/2, \pi/2[$ is a smooth function, then the Gaussian curvature of the graph of f is given by

$$K = \cos(f)^3 \big(1 + \|\nabla f\|^2\big)^{-(n+2)/2n} \operatorname{Det}\big(f_{;ij} - \tan(f)(f_{;j} f_{;j} + \delta_{ij})\big)^{1/n}.$$

We will return to this formula in later examples.

Let ∇^0 denote the Levi-Civita covariant derivative of the product metric on $\Sigma_0 \times]-\infty, 0]$. Let g denote the pull back of the metric over M through the exponential map. Let Vol denote the volume form of g and let ∇ denote the Levi-Civita covariant derivative of g. Trivially, ∇ coincides with the pull back through the exponential map of the Levi-Civita covariant derivative of M.

Proposition 3.2. *Let $\Omega := \nabla - \nabla^0$ be the connection 2-form of ∇ with respect to ∇^0. There exists a smooth 2-form Ω_0 such that, if X and Y are tangent to Σ, then:*

$$\begin{aligned}
\Omega_{(x,t)}(X, Y) &= A_0(X, Y)\partial_t + t\Omega_{0,(x,t)}(X, Y), \\
\Omega_{(x,t)}(X, \partial_t) &= -A_0 X + t\Omega_{0,(x,t)}(X, \partial_t), \\
\Omega_{(x,t)}(\partial_t, \partial_t) &= t\Omega_{0,(x,t)}(\partial_t, \partial_t).
\end{aligned}$$

Proof. When $t = 0$, by definition of A_0:

$$\begin{aligned}
\nabla_X Y &= \nabla^0_X Y + \langle \nabla_X Y, \mathsf{N}_0\rangle \mathsf{N}_0 \\
&= \nabla^0_X Y - A_0(X, Y)\mathsf{N}_0.
\end{aligned}$$

Thus, since $\mathsf{N}_0 = -\partial_t$, at t_0:

$$\nabla_X Y = \nabla^0_X Y + A_0(X, Y)\partial_t.$$

Likewise

$$\nabla_X \partial_t = -\nabla_X \mathsf{N}_0 = -A_0 X.$$

Finally, since the vertical lines are geodesics:

$$\nabla_{\partial_t} \partial_t = 0.$$

The result follows. □

Define $\hat{f} : \Sigma_0 \times]-\infty, 0] \to \mathbb{R}$ by

$$\hat{f}(x, t) = f(x) - t.$$

The graph of f is the level set $\hat{f}^{-1}(\{0\})$. Observe that $\nabla \hat{f}$ is parallel to the downwards pointing unit normal over the graph of f. Let A_f denote the second fundamental form of this graph. For all i, we define the vector field $\hat{\partial}_i = (\partial_i, f_{;i})_{(x,f(x))}$. $(\hat{\partial}_1, \ldots, \hat{\partial}_n)$ forms a basis of the tangent space of the graph of f.

Proof of Proposition 3.1. By definition:

$$K^n = \mathrm{Det}\big(A_f(\hat{\partial}_i, \hat{\partial}_j)\big) / \mathrm{Det}\big(g(\hat{\partial}_i, \hat{\partial}_j)\big).$$

However, since the graph of f is the level set $\hat{f}^{-1}(0)$:

$$A_f = \frac{1}{\|\nabla \hat{f}\|_g}\big(\mathrm{Hess}(\hat{f})\big).$$

Moreover:

$$\mathrm{Hess}(\hat{f}) = \mathrm{Hess}^0(\hat{f}) - d\hat{f}(\Omega) = \mathrm{Hess}(f) - d\hat{f}(\Omega).$$

It follows that K has the specified form with:

$$\psi(x, f, \nabla f) = \|\nabla \hat{f}\|_g \,\mathrm{Det}\big(g(\hat{\partial}_i, \hat{\partial}_j)\big)^{1/n},$$

and

$$\Psi(x, f, \nabla f) = -d\hat{f}(\Omega).$$

When $t = 0$:

$$\psi(x, 0, p) = \big(1 + \|p\|^2\big)^{(n+2)/2n}.$$

Thus, since the function $p \mapsto (1 + \|p\|^2)^\alpha$ is locally uniformly strictly convex for $\alpha > 1/2$, (i) follows.

Likewise, by Proposition 3.2:

$$\begin{aligned}\Psi(x, 0, p)(\hat{\partial}_i, \hat{\partial}_j) &= d\hat{f}(\mathsf{N})A_0(\partial_i, \partial_j) + f^{;j} d\hat{f}(A_0\partial_i) + f^{;i} d\hat{f}(A_0\partial_j)\\ &= A_{0,ij} + f_{;i} f_{;k} A^k_{0\,j} + f_{;j} f_{;k} A^k_{0\,i}.\end{aligned}$$

(ii) follows.

Finally, the graph of f is convex if and only if A_f is positive definite, and this completes the proof. □

4. Interlude – Maximum Principles

Let $M := M^{n+1}$ be an $(n+1)$-dimensional Riemannian manifold.

Definition 4.1. Let Σ be a $C^{0,1}$ convex, immersed hypersurface in M. Choose $k > 0$. For $P \in \Sigma$, we say that the Gaussian curvature of Σ is at least (resp. at most) k in the weak (Alexandrov) sense at P if and only if there exists a smooth, convex, immersed hypersurface Σ' such that:

(i) Σ' is an exterior (resp. interior) tangent to Σ at P; and
(ii) the Gaussian curvature of Σ' at P is equal to k.

This notion is well adapted to the weak Geometric Maximum Principle:

Lemma 4.2 *(Weak Geometric Maximum Principle). Let Σ_1, Σ_2 be two $C^{0,1}$, convex, immersed hypersurfaces in M. Choose $P \in \Sigma_1$. If Σ_2 is an interior tangent to Σ_1 at P, then the Gaussian curvature of Σ_2 at P is no less than the Gaussian curvature of Σ_1 at P in the weak (Alexandrov) sense.*

Proof. Let Σ_1' be a smooth, convex hypersurface which is an exterior tangent to Σ_1 at P. Likewise, let Σ_2' be a smooth convex hypersurface which is an interior tangent to Σ_2 at P. Let A_1 and A_2 be the respective second fundamental forms of Σ_1' and Σ_2' respectively. Since Σ_2' is an interior tangent to Σ_1' at P:

$$A_2 \geqslant A_1.$$

The result follows. □

Remark. This result is often used in conjunction with foliations by constant curvature hypersurfaces which then act as barriers. In the case where $M = \mathbb{H}^{n+1}$, if we identify $\mathbb{H}^{n+1}$ with the upper half space in $\mathbb{R}^{n+1}$, then we obtain families of constant curvature hypersurfaces by considering intersections of spheres in $\mathbb{R}^{n+1}$ with $\mathbb{H}^{n+1}$. If the centre of such a sphere lies on $\mathbb{R}^n$, then its intersection with $\mathbb{H}^{n+1}$ has zero curvature. Otherwise, if the sphere is not entirely contained in $\mathbb{H}^{n+1}$, then the intersection has curvature less than 1, and if it is contained in $\mathbb{H}^{n+1}$, then the intersection has curvature greater than 1.

We also have the strong Geometric Maximum Principle:

Lemma 4.3 *(Strong Geometric Maximum Principle). Let $(\Sigma_1, \partial\Sigma_1)$ and $(\Sigma_2, \partial\Sigma_2)$ be smooth, convex, immersed hypersurfaces in M of constant Gaussian curvature equal to k.*

(i) *If P is an interior point of Σ_1, and if Σ_2 is an exterior tangent to Σ_1 at P, then $\Sigma_1 = \Sigma_2$.*
(ii) *Suppose in addition that $\partial\Sigma_1 = \partial\Sigma_2$. If P is a boundary point of Σ_1 and if Σ_2 is an exterior tangent to Σ_1 at P, then $\Sigma_1 = \Sigma_2$.*

Proof. Σ_2 is a graph below Σ_1 near P. Let U be a neighborhood of P in Σ_1 over which Σ_2 is a graph. Let A be the shape operator of Σ_1 and let f be the graph function of Σ_2. By Proposition 3.1:

$$\mathrm{Det}\big(\mathrm{Hess}(f) + \Psi(x, f, \nabla f)\big)^{1/n} = k\psi(x, f, \nabla f),$$

for some Ψ and ψ. However

$$\mathrm{Det}(A) = k.$$

Thus, by concavity of $\mathrm{Det}^{1/n}$:

$$\frac{k}{n}\,\mathrm{Tr}\big(A^{-1}\big(\mathrm{Hess}(f)+\Psi(x,f,\nabla f)-A\big)\big)\geqslant k\big(\psi(x,f,\nabla f)-1\big).$$

Moreover, by the proof of Proposition 3.1:

$$\psi(x,f,\nabla f)=\big(1+\|\nabla f\|^2\big)^{n+2/2n}+f\psi_0(x,f,\nabla f).$$

For some smooth function ψ_0. Thus

$$k\big(\psi(x,f,\nabla f)-1\big)=c_1 f+\langle b_1,\nabla f\rangle,$$

for some smooth function c_1 and vector field b_1. Likewise, by Proposition 3.1:

$$\mathrm{Tr}\big(A^{-1}\big(\Psi(x,f,\nabla f)-A\big)\big)=c_2 f+\langle b_2,\nabla f\rangle,$$

for some smooth function c_2 and vector field b_2. Thus

$$\mathrm{Tr}\big(A^{-1}\,\mathrm{Hess}(f)\big)+\langle b,\nabla f\rangle+cf\geqslant 0,$$

for some smooth function c and vector field b. Since $f\leqslant 0$ and $f(P)=0$, in both cases (i) and (ii), it follows by the Strong Maximum Principle (Theorems 3.5 and 3.6 of [7]) that $f=0$ over a neighborhood of P. The result now follows by unique continuation of constant Gaussian curvature hypersurfaces. □

5. Second order bounds along the boundary

Let $M:=M^{n+1}$ be an $(n+1)$-dimensional Riemannian manifold. Let $(\Sigma_0,\partial\Sigma_0)\subseteq M$ be a smooth, strictly convex, immersed hypersurface. Using the exponential map, we identify a subset of M with $\Sigma_0\times\,]-\infty,0]$. Let $\phi:M\to\,]0,\infty[$ be a smooth, positive function. Let $(\hat{\Sigma},\partial\hat{\Sigma})$ be a $C^{0,1}$, convex, immersed hypersurface such that:

(i) $\hat{\Sigma}$ is a graph below Σ_0;
(ii) $\partial\hat{\Sigma}=\partial\Sigma_0$; and
(iii) for all $x\in\hat{\Sigma}$, the Gaussian curvature of $\hat{\Sigma}$ is greater than $\phi(x)+\epsilon$ in the weak (Alexandrov) sense, for some $\epsilon>0$.

$\hat{\Sigma}$ serves as a lower barrier for our problem. Let $(\Sigma,\partial\Sigma)\subseteq M$ be a smooth, convex, immersed hypersurface such that:

(i) $\hat{\Sigma}\leqslant\Sigma\leqslant\Sigma_0$;
(ii) $\partial\Sigma=\partial\Sigma_0$; and
(iii) for all $x\in\Sigma$, the Gaussian curvature of Σ at x is equal to $\phi(x)$.

We aim to obtain bounds for the norm of the second fundamental form of Σ along the boundary which only depend on the data. To this end, we denote by $\mathcal{B}$ the family of constants which depend continuously on the data: M, Σ_0, $\hat{\Sigma}$, ϵ, ϕ and the C^1 jet of Σ (formally, $\mathcal{B}$ is the set of continuous

– or even locally bounded – functions over the space of data). When supplementary data, D (such as, for example, a vector field) is added, we denote by $\mathcal{B}(D)$ the family of constants which, in addition, also depend on D.

We will prove:

Proposition 5.1. *Let Σ, $\mathcal{B}$ be as described above. If Σ_0 is strictly convex, then there exists K in $\mathcal{B}$ such that, if A is the second fundamental form of Σ, then, for all $P \in \partial\Sigma$:*

$$\|A(P)\| \leqslant K.$$

Remark. The strict convexity of Σ_0 is only required in the last step of the proof, where it is used to obtain uniform, strict lower bounds for the restriction of the second fundamental form to the tangent space of $\partial\Sigma_0$. In other cases, such as where Σ_0 is totally geodesic, for example, this may shown using other means (cf. Section 11).

Let $P \in \partial\Sigma_0$ be a point on the boundary. For the sake of later applications (cf. [18]), we underline that $\hat{\Sigma}$ need only exist locally. We thus let $\hat{\Sigma}_P \subseteq M$ be a smooth, convex, immersed hypersurface such that:

(i) $\hat{\Sigma}_P$ is a graph below Σ_0;
(ii) $P \in \hat{\Sigma}_P$; and
(iii) for all $x \in \hat{\Sigma}_P$, the Gaussian curvature of $\hat{\Sigma}_P$ at x is greater than $\phi(x) + \epsilon$.

Bearing in mind the results of Section 3, we will consider Σ and $\hat{\Sigma}_P$ as graphs near P over a hypersurface whose second fundamental form vanishes at P. Thus, let $\Sigma_1 \subseteq M$ be an immersed hypersurface in M which is tangent to Σ_0 at P and which is totally geodesic at P.

Let $\Omega \subseteq \Sigma_1$ be an open set with $P \in \partial\Omega$ and $f_0 : \Omega \to \mathbb{R}$ a function such that:

(i) Σ_0 is the graph of f_0 over Ω; and
(ii) $f_0(\partial\Omega) = \partial\Sigma_0$.

Remark. Observe that both $\hat{\Sigma}_P$ and Σ_1 are local objects, only defined near P, as opposed to $\hat{\Sigma}$, for example, which is a global object, sharing the same boundary as Σ_0.

We observe in passing that, by convexity, after reducing Σ_1 if necessary, f_0 may be made to be positive. $\partial\Omega$ consists of two components: we denote by $\partial_b\Omega$ the subset of $\partial\Omega$ which lies below the boundary of Σ_0 and we denote by $\partial_i\Omega$ the subset of $\partial\Omega$ which lies below the interior of Σ_0.

Proposition 5.2. *Let Σ, $\mathcal{B}$ be as described at the beginning of this section. Let Ω be as described above. For all $P \in \partial\Sigma$, there exist $\delta > 0$ in $\mathcal{B}(P)$ and a neighborhood U of P in Σ which is a graph over $B_\delta(P) \cap \Omega$.*

Proof. The radius over which Σ is a graph over Σ_1 is determined by the C^1 jet of Σ, which is among the data defining $\mathcal{B}$. The result follows. □

We thus replace Ω with $\Omega \cap B_\delta(P)$ and let $f, \hat{f} : \Omega \to \mathbb{R}$ be the functions of which Σ and $\hat{\Sigma}_P$ respectively are the graphs below Σ_1.

By Proposition 3.1, there exist functions ψ and Ψ and a positive number $R > 0$, which only depends on M, ϕ and Σ_1 such that:

$$\mathrm{Det}\big(\mathrm{Hess}(f) + \Psi(x, f, \nabla f)\big)^{1/n} = \psi(x, f, \nabla f).$$

Moreover:

(i) $\mathrm{Hess}(f) + \Psi(x, f, \nabla f)$ is positive definite;
(ii) $\Psi(x, t, p), (\partial_{p_k}\Psi)(x, t, p) = O(d(x, P)) + O(t)$ where $d(\cdot, P)$ is the distance in M to P; and
(iii) for t sufficiently small, $p \mapsto \psi(x, t, p)$ is a convex function in p for $\|p\| \leqslant R$.

We define the matrix B by

$$B = \frac{1}{n}\psi(x, f, \nabla f)\big(\mathrm{Hess}(f) + \Psi(x, f, \nabla f)\big)^{-1}.$$

We define the operator $\mathcal{L}$ by

$$\mathcal{L}g = B^{ij}g_{;ij} + B^{ij}(\partial_{p_k}\Psi)_{ij}g_{;k} - (\partial_{p_k}\psi)g_{;k}.$$

Proposition 5.3. *Let Σ, $\mathcal{B}$ be as described at the beginning of this section. For all $P \in \partial\Sigma$, there exists $\delta_1 > 0$ and $\epsilon_1 > 0$ in $\mathcal{B}(P)$ such that for $d(x, P) < \delta_1$:*

$$\mathcal{L}(f - \hat{f}) \leqslant -\epsilon_1\left(1 + \sum_{i=1}^{n} B^{ii}\right).$$

Remark. This inequality lies at the heart of the Caffarelli–Nirenberg–Spruck technique. The aim is to build functions which are subharmonic with respect to $\mathcal{L}$, the key observation being that the appropriate term with respect to which bounds should be obtained is the trace of the matrix defining the generalized Laplacian $\mathcal{L}$, in this case $\sum_{i=1}^{n} B^{ii}$. We encourage the interested reader to compare this proposition with the relation shown on Line 13 of p. 376 of [2], where the function $f - \hat{f}$ here plays the role of the function x_n in the construction of their barrier function w. In addition, a clearer view of the main elements of the proof may be obtained by observing the effect of setting the constant η_2 to be equal to 0, amounting to not perturbing $\hat{f}$. Finally, observe how the proof depends on the concavity of the determinant function as well as the convexity of $\psi(x, t, p)$ with respect to p, which is a recurring theme whenever this technique is applied.

Proof of Proposition 5.3. There exists $\eta_1 > 0$ in $\mathcal{B}$ such that, near p:

$$\mathrm{Det}\big(\mathrm{Hess}(\hat{f}) + \Psi(x, \hat{f}, \nabla\hat{f})\big)^{1/n} \geqslant \psi(x, \hat{f}, \nabla\hat{f}) + 2\eta_1.$$

Define $\delta : \Sigma_1 \to \mathbb{R}$ by

$$\delta(x) = d_1(x, P)^2,$$

where d_1 denotes the intrinsic distance in Σ_1. Near P:

$$\mathrm{Hess}(\delta) \geqslant \mathrm{Id}.$$

There exists $\eta_2 > 0$ in $\mathcal{B}$ such that, if we define $\hat{g}$ by

$$\hat{g} = \hat{f} - \eta_2\delta,$$

then, near P:

$$\mathrm{Det}\big(\mathrm{Hess}(\hat{g}) + \Psi(x, \hat{g}, \nabla\hat{g})\big)^{1/n} \geqslant \psi(x, \hat{g}, \nabla\hat{g}) + \eta_1.$$

Since $\mathrm{Det}^{1/n}$ is a concave function:

$$\begin{aligned}
&\mathrm{Det}\big(\mathrm{Hess}(\hat{g}) + \Psi(x, \hat{g}, \nabla\hat{g})\big)^{1/n} - \mathrm{Det}\big(\mathrm{Hess}(f) + \Psi(x, f, \nabla f)\big)^{1/n} \\
&\quad \leqslant B^{ij}\big(\hat{g}_{;ij} + \Psi_{ij}(x, \hat{g}, \nabla\hat{g}) - f_{;ij} - \Psi_{ij}(x, f, \nabla f)\big) \\
&\quad \leqslant B^{ij}(\hat{f} - f)_{;ij} - \eta_2 \sum_{i=1}^{n} B^{ii} + B^{ij}\big(\Psi_{ij}(x, \hat{g}, \nabla\hat{g}) - \Psi_{ij}(x, f, \nabla f)\big).
\end{aligned}$$

Bearing in mind that $\Psi(x, t, p) = O(d(x, P)) + O(t)$, near P:

$$B^{ij}(f - \hat{f})_{;ij} \leqslant -\eta_1 - \frac{\eta_2}{2}\sum_{i=1}^{n} B^{ii} + \psi(x, f, \nabla f) - \psi(x, \hat{g}, \nabla\hat{g}).$$

However, sufficiently close to p:

$$\psi(x, f, \nabla\hat{g}) - \psi(x, \hat{g}, \nabla\hat{g}) \leqslant \eta_1/3.$$

Moreover, by convexity of ψ:

$$\psi(x, f, \nabla f) - \psi(x, f, \nabla\hat{g}) \leqslant (\partial_{p_k}\psi)(f_{;k} - \hat{f}_{;k} + \eta_2\delta_{;k}).$$

Since $\delta_{;k}$ is continuous and vanishes at P, we conclude that, near P:

$$B^{ij}(f - \hat{f})_{;ij} - (\partial_{p_k}\psi)(f_{;k} - \hat{f}_{;k}) \leqslant -\eta_1/3 - \eta_2/2\sum_{i=1}^{n} B^{11}.$$

Bearing in mind that, for all k, $(\partial_{p_k}\Psi)(x, t, \xi) = O(d(x, P)) + O(t)$, the result follows. □

Let X be a vector field over Σ_1.

Proposition 5.4. *Let Σ, $\mathcal{B}$ be as described at the beginning of this section. For all $P \in \partial\Sigma$, there exists K in $\mathcal{B}(P, X)$ such that near P:*

$$\big|\mathcal{L}(Xf)\big| \leqslant K\left(1 + \sum_{i=1}^{n} B^{ii}\right).$$

Remark. We encourage the interested reader to compare this relation to 2.12 of [2].

Proof of Proposition 5.4. Differentiating the Gaussian Curvature Equation yields, for all k:

$$B^{ij}\big(f_{;ijk} + (\partial_{x_k}\Psi)_{ij} + (\partial_t\Psi)_{ij} f_{;k} + (\partial_{p_l}\Psi)_{ij} f_{;lk}\big) = (\partial_{x_k}\psi) + (\partial_t\psi) f_{;k} + (\partial_{p_l}\psi) f_{;lk}.$$

However

$$f_{;lk} = f_{;kl}.$$

Moreover

$$f_{;ijk} = f_{;kij} + R^{\Sigma_1\,p}_{jki} f_{;p},$$

where R^{Σ_1} is the Riemann curvature tensor of Σ_1. There therefore exists K_1 in $\mathcal{B}(P, \Sigma_1)$ such that:

$$\big|B^{ij}\big(f_{;kij} + (\partial_{p_l}\Psi)_{ij} f_{;kl}\big) - (\partial_{p_l}\psi) f_{;kl}\big| \leqslant K_1\left(1 + \sum_{i=1}^{n} B^{ii}\right).$$

Moreover, bearing in mind the definition of B, we obtain:

$$\begin{aligned} B^{ij} f_{;ki} &= B^{ij}\big((f_{;ki} + \Psi_{ki}) - \Psi_{ki}\big) \\ &= \psi(x, f, \nabla f) - B^{ij}\Psi_{ki}. \end{aligned}$$

However

$$\begin{aligned} \mathcal{L}(Xf) = {} & X^k\big(B^{ij}\big(f_{;kij} + (\partial_{p_l}\Psi)_{ij} f_{;kl}\big) - (\partial_{p_l}\psi) f_{;kl}\big) \\ & + f_{;k}\big(B^{ij}\big(X^k_{;ij} + (\partial_{p_l}\Psi)_{ij} X^k_{;l}\big) - (\partial_{p_l}\psi) X^k_{;l}\big) + 2B^{ij}\big(f_{;ki} X^k_{;j}\big). \end{aligned}$$

The result follows by combining the above relations. □

Corollary 5.5. *Let Σ, $\mathcal{B}$ be as described at the beginning of this section. For all $P \in \partial\Sigma$, there exists K in $\mathcal{B}(P, X)$ such that near P:*

$$\big|\mathcal{L}X(f - f_0)\big| \leqslant K\left(1 + \sum_{i=1}^{n} B^{ii}\right).$$

We define $\delta : \Sigma_1 \to\ ]0, \infty[$ by

$$\delta(x) = d_1(x, P)^2,$$

where $d_1(\cdot, P)$ denotes the distance in Σ_1 to P.

Proposition 5.6. *Let Σ, $\mathcal{B}$ be as described at the beginning of this section. For all $P \in \partial\Sigma$, there exists K in $\mathcal{B}(P, X)$ such that near p:*

$$|\mathcal{L}\delta| \leqslant K\left(1 + \sum_{i=1}^{n} B^{ii}\right).$$

Proof. Trivial. □

Proof of Proposition 5.1. Let P, Σ_1 and Ω be as before. Let X be a vector field over Ω which is tangent to $\partial_b\Omega$. By Propositions 5.3 and 5.6 and Corollary 5.5, there exists $\eta, K > 0$ in $\mathcal{B}(P, X)$ such that:

$$\left|\mathcal{L}X(f - f_0)\right| \leqslant K\left(1 + \sum_{i=1}^{n} B^{ii}\right),$$

$$|\mathcal{L}\delta| \leqslant K\left(1 + \sum_{i=1}^{n} B^{ii}\right),$$

$$\mathcal{L}(f - \hat{f}) \leqslant -\eta\left(1 + \sum_{i=1}^{n} B^{ii}\right).$$

Moreover, we may assume that, throughout Ω:

$$\left|X(f - f_0)\right| \leqslant K.$$

By definition of X, $X(f - f_0)$ vanishes along $\partial_b\Omega$. Since $\partial_i\Omega$ is bounded away from P, there therefore exists $A_+ > 0$ in $\mathcal{B}(P, X)$ such that, over $\partial\Omega$:

$$X(f - f_0) - A_+\delta \leqslant 0.$$

There exists $B_+ > 0$ in $\mathcal{B}(P, X)$ such that, throughout Ω:

$$\mathcal{L}\big(X(f - f_0) - A_+\delta - B_+(f - \hat{f})\big) \geqslant 0.$$

Moreover, since $f - \hat{f} \geqslant 0$, this function is also negative along $\partial\Omega$. Thus, by the Maximum Principle, throughout Ω:

$$X(f - f_0) \leqslant A_+\delta + B_+(f - \hat{f}).$$

Likewise, there exist $A_-, B_- > 0$ in $\mathcal{B}(P, X)$ such that:

$$X(f - f_0) \geqslant -A_-\delta - B_-(f - \hat{f}).$$

There therefore exists $K_1 > 0$ in $\mathcal{B}(P, X)$ such that:

$$\left|d\big(X(f - f_0)\big)(P)\right| \leqslant K_1.$$

Thus, increasing K_1 if necessary:

$$\left|d(Xf)(P)\right| \leqslant K_1.$$

Let N be the unit normal vector field along $\partial\Sigma$ pointing into Σ. We have shown that there exists $K_2 > 0$ in $\mathcal{B}$ such that, for any vector field, X, tangent to $\partial\Sigma$:

$$\left\|A(X, \mathsf{N})\right\| \leqslant K_2\|X\|.$$

The restriction of A to $\partial\Sigma$ is determined by the norm of the second fundamental form of $\partial\Sigma = \partial\Sigma_0$. There therefore exists $K_3 > 0$ in $\mathcal{B}$ such that, if X and Y are vector fields tangent to $\partial\Sigma$, then

$$\left\|A(X, Y)\right\| \leqslant K_3\|X\|\|Y\|.$$

Finally, since Σ lies between Σ_0 and $\hat{\Sigma}$, both of which are strictly convex, there exists $\epsilon_1 > 0$ in $\mathcal{B}$ such that, throughout $\partial\Sigma$:

$$A|_{T\partial\Sigma} \geqslant \epsilon_1 \,\mathrm{Id}.$$

Since $\mathrm{Det}(A) = \phi$, $A(\mathsf{N}, \mathsf{N})$ may be estimated in terms of the other components of A, and there therefore exists $K_4 > 0$ in $\mathcal{B}$ such that, throughout $\partial\Sigma$:

$$\left\|A(\mathsf{N}, \mathsf{N})\right\| \leqslant K_4.$$

The result now follows. □

6. Second order bounds over the interior

Let $M := M^{n+1}$ be a Hadamard manifold. Let $\Omega \subseteq M$ be a relatively compact open subset. Let $(\Sigma, \partial\Sigma) \subseteq M^{n+1}$ be a smooth, convex hypersurface and suppose that $\Sigma \subseteq \Omega$. Let N and A be the unit, exterior, normal vector and the shape operator of Σ respectively. In this section, it will be convenient to use the logarithm of the extrinsic curvature. Let $\phi : M \to \mathbb{R}$ be a strictly positive smooth function. We prove global second order estimates given second order estimates along the boundary for the problem:

$$\mathrm{Log}\big(\mathrm{Det}(A)\big) = \phi(x).$$

We denote by $\|A|_{\partial\Sigma_0}\|$ the supremum over $\partial\Sigma_0$ of the norm of A. We will prove:

Proposition 6.1. *There exists $K > 0$ in $\mathcal{B}(\|A|_{\partial\Sigma_0}\|)$ such that:*

$$\|A\| \leqslant K.$$

In the sequel, we raise and lower indices with respect to A. Thus

$$A^{ij}A_{jk} = \delta^i_k,$$

where δ is the Krönecker delta function.

Proposition 6.2.

(i) *For all* p*:*

$$A^{ij}A_{ij;p} = \phi_{;p}.$$

(ii) *For all* p, q*:*

$$A^{ij}A_{ij;pq} = A^{im}A^{jn}A_{ij;p}A_{mn;q} + \phi_{;pq}.$$

Proof. This follows by differentiating the equation $\mathrm{Log}(\mathrm{Det}(A)) = \phi$. □

We recall the commutation rules of covariant differentiation in a Riemannian manifold:

Lemma 6.3. *Let* R^{Σ} *and* R^{M} *be the Riemann curvature tensors of* Σ *and* M *respectively. Then:*

(i) *for all* i, j, k*:*

$$A_{ij;k} = A_{kj;i} + R^{M}_{ki\nu j},$$

where ν *represents the direction normal to* Σ*; and*

(ii) *for all* i, j, k, l*:*

$$A_{ij;kl} = A_{ij;lk} + R^{\Sigma\,p}_{kli}A_{pj} + R^{\Sigma\,p}_{klj}A_{pi}.$$

Corollary 6.4. *For all* i*,* j*,* k *and* l*:*

$$A_{ij;kl} = A_{kl;ij} + R^{M}_{kj\nu i;l} + R^{M}_{li\nu k;j} + R^{\Sigma\,p}_{jlk}A_{pi} + R^{\Sigma\,p}_{jli}A_{pk}.$$

Proof.

$$\begin{aligned}
A_{ji;kl} &= A_{ki;jl} + R^{M}_{kj\nu i;l} \\
&= A_{ik;lj} + R^{M}_{kj\nu i;l} + R^{\Sigma\,p}_{jlk}A_{pi} + R^{\Sigma\,p}_{jli}A_{pk} \\
&= A_{lk;ij} + R^{M}_{kj\nu i;l} + R^{M}_{li\nu k;j} + R^{\Sigma\,p}_{jlk}A_{pi} + R^{\Sigma\,p}_{jli}A_{pk}.
\end{aligned}$$

The result follows. □

Choose $P \in \Sigma$. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A at P. Choose an orthonormal basis, $(e_1, \ldots, e_n)$ of $T_P\Sigma$ with respect to which A is diagonal such that $a := \lambda_1 = A_{11}$ is the highest eigenvalue of A at P. We extend this to a frame in a neighborhood of P by parallel transport along geodesics. We likewise extend a to a function defined in a neighborhood of P by

$$a = A(e_1, e_1).$$

Viewing λ_1 also as a function defined near P, $\lambda_1 \geqslant a$ and $\lambda_1 = a$ at P.

Proposition 6.5. *For all i, at P:*

$$a_{;ii} = A_{11;ii}.$$

Proof. Bearing in mind that $\nabla_{e_i} e_i = 0$ at P:

$$\begin{aligned} a_{;ii} &= D_{e_i} D_{e_i} a \\ &= D_{e_i} D_{e_i} A(e_1, e_1) \\ &= D_{e_i} (\nabla A)(e_1, e_1; e_i) - 2D_{e_i} A(\nabla_{e_i} e_1, e_1) \\ &= \left(\nabla^2 A\right)(e_1, e_1; e_i, e_i) - 2A(\nabla_{e_i} \nabla_{e_i} e_1, e_1). \end{aligned}$$

Since e_1 is defined by parallel transport along geodesics emanating from P, for all i, $\nabla_{e_i} \nabla_{e_i} e_1 = 0$ at P, and the result follows. □

We define the Laplacian Δ such that, for all functions f:

$$\Delta f = A^{ij} f_{;ij}.$$

Proposition 6.6. *There exists $K > 0$, which only depends on M and ϕ such that, if $a > 1$, then*

$$\Delta \operatorname{Log}(a)(P) \geqslant -K\left(1 + \sum_{i=1}^{n} \frac{1}{\lambda_i}\right).$$

Proof. By Corollary 6.4:

$$\begin{aligned} a_{;ii} &= A_{11;ii} \\ &= A_{ii;11} + R^{M}_{i1\nu1;i} + R^{M}_{i1\nu i;1} + R^{\Sigma\, p}_{1ii} A_{p1} + R^{\Sigma\, p}_{1i1} A_{pi}. \end{aligned}$$

However, at P:

$$\sum_{i=1}^{n} \frac{1}{\lambda_1 \lambda_i} A_{ii;11} = \sum_{i,j=1}^{n} \frac{1}{\lambda_i \lambda_j \lambda_1} A_{ij;1} A_{ij;1} + \frac{1}{\lambda_1} \phi_{;11}.$$

Thus, at P:

$$\begin{aligned} \Delta \operatorname{Log}(a) \geqslant{}& \frac{1}{\lambda_1} \phi_{;11} + \sum_{i,j=1}^{n} \frac{1}{\lambda_i \lambda_j \lambda_1} A_{ij;1} A_{ij;1} - \sum_{i=1}^{n} \frac{1}{\lambda_1^n \lambda_i} A_{11;i} A_{11;i} \\ &+ \sum_{i=1}^{n} \frac{1}{\lambda_1 \lambda_i} \left(R^{M}_{i1\nu1;i} + R^{M}_{i1\nu i;1}\right) + \sum_{i,j=1}^{n} \frac{1}{\lambda_1 \lambda_i} \left(R^{\Sigma\, p}_{1ii} A_{p1} + R^{\Sigma\, p}_{1i1} A_{pi}\right). \end{aligned}$$

We consider each contribution separately. Since, for all $a, b \in \mathbb{R}$, $(a+b)^2 \leqslant 2a^2 + 2b^2$, by Lemma 6.3, for all $i \geqslant 2$:

$$A_{11;i}^2 = \left(A_{i1;1} + R^M_{i1\nu 1}\right)^2 \leqslant 2A_{i1;1}^2 + 2\left(R^M_{i1\nu 1}\right)^2.$$

Thus, bearing in mind that $\lambda_1 \geqslant 1$, there exists K_1, which only depends on M such that:

$$\sum_{i,j=1}^n \frac{1}{\lambda_i\lambda_j\lambda_1} A_{ij;1}A_{ij;1} - \sum_{i=1}^n \frac{1}{\lambda_1^2\lambda_i} A_{11;i}A_{11;i} \geqslant -K_1 \sum_{i=1}^n \frac{1}{\lambda_i}.$$

For all ξ, X and Y:

$$\nabla^\Sigma \xi(Y;X) = \nabla^M \xi(Y;X) - A(X,Y)\xi(N);$$

and

$$X\xi(\mathsf{N}) = \nabla^M \xi(\mathsf{N};X) + \xi(AX).$$

Thus

$$R^M_{i1\nu 1;i} = \left(\nabla^M R^M\right)_{i1\nu 1;i} + \lambda_i(1-\delta_{i1})R^M_{1\nu\nu 1} + \lambda_i R^M_{i1i1},$$
$$R^M_{i1\nu i;1} = \left(\nabla^M R^M\right)_{i1\nu i;1} - \lambda_1(1-\delta_{i1})R^M_{i\nu\nu i} - \lambda_1 R^M_{i1i1}.$$

Bearing in mind that $\lambda_1 \geqslant 1$, there exists K_3, which only depends on M such that:

$$\sum_{i=1}^n \frac{1}{\lambda_1\lambda_i}\left(R^M_{i1\nu 1;i} + R^M_{i1\nu i;1}\right) \geqslant -K_3\left(1 + \sum_{i=1}^n \frac{1}{\lambda_i}\right).$$

Moreover

$$R^{\Sigma\,p}_{1ii} A_{p1} + R^{\Sigma\,p}_{1i1} A_{pi} = R^M_{1ii1}(\lambda_1 - \lambda_i) + \lambda_1\lambda_i(\lambda_1 - \lambda_i).$$

Bearing in mind that $\lambda_1 \geqslant 1$ and that $\lambda_1 \geqslant \lambda_i$ for all i, there exists K_2, which only depends on M such that:

$$\sum_{i,j=1}^n \frac{1}{\lambda_1\lambda_i}\left(R^{\Sigma\,p}_{1ii} A_{p1} + R^{\Sigma\,p}_{1i1} A_{pi}\right) \geqslant -K_2\left(1 + \sum_{i=1}^n \frac{1}{\lambda_i}\right).$$

Since $\nabla^\Sigma_{e_1} e_1 = 0$ at P:

$$\begin{aligned}
\nabla^M_{e_1} e_1 &= \nabla^\Sigma_{e_1} e_1 + \left\langle \nabla^M_{e_1} e_1, \mathsf{N}\right\rangle \mathsf{N} \\
&= -A(e_1,e_1)\mathsf{N} \\
&= -\lambda_1 \mathsf{N}.
\end{aligned}$$

Thus

$$\begin{aligned}\phi_{;11} &= \partial_1\partial_1\phi\\ &= \mathrm{Hess}^M(\phi)(e_1,e_1) - d\phi\big(\nabla^M_{e_1}e_1\big)\\ &= \mathrm{Hess}^M(\phi)(e_1,e_1) - \lambda_1\, d\phi(\mathsf{N}).\end{aligned}$$

Bearing in mind that $\lambda_1 \geqslant 1$, there thus exists K_3, which only depends on M and ϕ such that:

$$\frac{1}{\lambda_1}\phi_{;11} \geqslant -K_3.$$

The result now follows by combining the above relations. □

We recall that a function f is said to satisfy $\Delta f \geqslant g$ in the weak sense if and only if, for all $P \in \Sigma$, there exists a smooth function φ, defined near P such that:

(i) $f \geqslant \varphi$ near P;
(ii) $f = \varphi$ at P; and
(iii) $\Delta\varphi \geqslant g$ at P.

Corollary 6.7. *With the same K as in Proposition 6.6, if $\lambda_1 \geqslant 1$, then*

$$\Delta\,\mathrm{Log}(\lambda_1) \geqslant -K\left(1 + \sum_{i=1}^{n}\frac{1}{\lambda_i}\right),$$

in the weak sense.

Proof. Near $P \in \Sigma$, $\lambda_1 \geqslant a$ and $\lambda_1 = a$ at P. Since $P \in \Sigma_0$ is arbitrary, and since a is smooth at P, the result follows. □

Choose $x_0 \in M$. Define δ by

$$\delta = \frac{1}{2}d(x,x_0)^2.$$

Proposition 6.8. *There exists c, which only depends on M, Ω, ϕ and x_0 such that:*

$$\lambda_1 \geqslant c \quad\Rightarrow\quad \Delta^{\Sigma}\delta \geqslant \frac{1}{2}\left(1 + \sum_{i=1}^{n}\frac{1}{\lambda_i}\right).$$

Proof. Since M has non-positive curvature:

$$\begin{aligned}&\mathrm{Hess}^M(\delta) \geqslant \mathrm{Id}\\ &\quad\Rightarrow\quad \mathrm{Hess}^{\Sigma}(\delta) \geqslant \mathrm{Id} - d(x,x_0)\langle\mathsf{N},\nabla d\rangle A\\ &\quad\Rightarrow\quad \Delta\delta \geqslant \sum_{i=1}^{n}\frac{1}{\lambda_i} - nd(x,x_0).\end{aligned}$$

By compactness of Ω, there exists $K_1 > 0$ such that, throughout Ω:

$$e^{\phi} \leqslant K_1.$$

Thus

$$\begin{aligned} &\lambda_1\lambda_n^{n-1} \leqslant K_1 \\ &\quad\Rightarrow\quad \lambda_n \leqslant \left(K_1\lambda_1^{-1}\right)^{1/(n-1)} \\ &\quad\Rightarrow\quad \frac{1}{\lambda_n} \geqslant (\lambda_1/K_1)^{1/(n-1)} \\ &\quad\Rightarrow\quad \sum_{i=1}^{n}\frac{1}{\lambda_i} \geqslant (\lambda_1/K_1)^{1/(n-1)}. \end{aligned}$$

There thus exists $c_1 > 0$ such that, for $\lambda_1 \geqslant c_1$, and for $x \in \Omega$:

$$\begin{aligned} &\sum_{i=1}^{n}\frac{1}{\lambda_i} \geqslant 2n\, d(x, x_0) + 1 \\ &\quad\Rightarrow\quad \Delta^{\Sigma}\delta \geqslant \frac{1}{2}\left(1 + \sum_{i=1}^{n}\frac{1}{\lambda_i}\right). \end{aligned}$$

The result now follows. □

Corollary 6.9. *There exists $\lambda > 0$ and $c > 0$, which only depend on M, Ω, ϕ and x_0 such that:*

$$\lambda_1 \geqslant c \quad\Rightarrow\quad \Delta\big(\operatorname{Log}(a) + \lambda\delta\big) > 0,$$

in the weak sense.

Interior bounds now follow by the Maximum Principle:

Proof of Proposition 6.1. Consider the function $\|A\|e^{\lambda\delta} = \lambda_1 e^{\lambda\delta}$. If this function achieves its maximum along $\partial\Sigma$, then the result follows since $e^{\lambda\delta}$ is uniformly bounded above and below. Otherwise, it achieves its maximum in the interior of Σ, in which case, by Corollary 6.9 and the Maximum Principle, at this point:

$$\|A\| = \lambda_1 \leqslant c.$$

The result follows. □

7. Compactness

Let $M := M^{n+1}$ be an $(n+1)$-dimensional Hadamard manifold. Let $(\Sigma_0, \partial\Sigma_0) \subseteq M^{n+1}$ be a smooth, strictly convex hypersurface. Let N_0 and A_0 be the unit, exterior, normal vector field and the shape operator of Σ_0 respectively. Using the exponential map, we identify $\Sigma \times]-\infty, 0]$ with a subset of M.

Let $\mathrm{Conv} \subseteq C^\infty(\Sigma_0,]-\infty, 0])$ be the family of smooth, negative valued functions over Σ_0 which vanish along $\partial\Sigma_0$ and whose graphs are strictly convex. We define the Gauss Curvature Operator $K : \mathrm{Conv} \to C^\infty(\Sigma_0)$ such that, for all f, $K(f)(x)$ is the Gauss curvature of the graph of f at the point $(x, f(x))$. The formula for K is given by Proposition 3.1.

Let $\hat{f} \in \mathrm{Conv}$ be such that:

$$\hat{f} \leqslant 0, \qquad K(\hat{f}) - \epsilon > K(0) > 0,$$

for some $\epsilon > 0$. Denote $\phi_0 = K(0)$ and $\hat{\phi} = K(\hat{f})$. Denote by $\mathrm{Conv}(\hat{f})$ the set of all $f \in \mathrm{Conv}$ such that:

$$\hat{f} \leqslant f \leqslant 0 \quad \text{and} \quad \hat{\phi} - \epsilon \geqslant K(f) \geqslant \phi_0.$$

We prove a slightly stronger version of the assertion that the restriction of K to $\mathrm{Conv}(\hat{f})$ is a proper mapping:

Lemma 7.1. *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathrm{Conv}(\hat{f})$. Suppose there exists $(\phi_n)_{n\in\mathbb{N}} \in C^\infty(M)$ such that, for all n, and for all $x \in \Sigma_0$:*

$$K(f_n)(x) = \phi_n\big(x, f_n(x)\big).$$

If there exists $\phi_\infty \in C^\infty(M)$ to which $(\phi_n)_{n\in\mathbb{N}}$ converges, then there exists $f_\infty \in \mathrm{Conv}(\hat{f})$ to which $(f_n)_{n\in\mathbb{N}}$ subconverges.

Corollary 7.2. *The restriction of K to $\mathrm{Conv}(\hat{f})$ is a proper mapping.*

Proof of Lemma 7.1. By Lemma 2.1 and Propositions 5.1 and 6.1, there exists $C_1 > 0$ in $\mathcal{B}$ such that, for all n:

$$\|f_n\|_{C^2} \leqslant C_1.$$

By Proposition 3.1:

$$K(f) = F\big(\mathrm{Hess}(f), \nabla f, f, x\big),$$

where $F(M, p, t, x)$ is elliptic in the sense of [4] and concave in M. It follows by Theorem 1 of [4] that there exist $\alpha > 0$ and $C_2 > 0$ in $\mathcal{B}$ such that, for all n:

$$\|f_n\|_{C^{2,\alpha}} \leqslant C_2.$$

Thus, by the Schauder Estimates (see [7]), for all $k \in \mathbb{N}$, there exists $B_k > 0$ such that, for all n:

$$\|f_n\|_{C^k} \leqslant B_k.$$

The result now follows by the Arzela–Ascoli Theorem. $\square$

8. 1-dimensional families of solutions

Let $M := M^{n+1}$ be an $(n+1)$-dimensional Hadamard manifold. Let $(\Sigma_0, \partial\Sigma_0) \subseteq M^{n+1}$ be a smooth, convex hypersurface. Let N_0 and A_0 be the unit, exterior, normal vector field and the shape operator of Σ_0 respectively. Using the Exponential Map, we identify $\Sigma \times]-\infty, 0]$ with a subset of M.

Let $\hat{f}$, ϕ_0 and $\hat{\phi}$ be as in the previous section. Let $\gamma : [0,1] \to C^\infty(\Sigma_0)$ be a smooth family of smooth functions such that, for all τ:

$$\phi_0 + \epsilon < \gamma(\tau) < \hat{\phi} - \epsilon,$$

for some $\epsilon > 0$. As before, let $K : \mathrm{Conv} \to C^\infty(\Sigma_0)$ be the Gauss Curvature Operator. For all $\phi \in C^\infty(\Sigma_0)$, define $\Gamma_\phi \subseteq I \times \mathrm{Conv}(\hat{f})$ by

$$\Gamma_\phi = \big\{(t, f) \text{ s.t. } K(f) = \gamma(t) + \phi\big\}.$$

Viewing Conv as a Banach manifold (strictly speaking, the intersection of an infinite nested family of Banach manifolds), we will prove:

Proposition 8.1. *There exists $(\phi_n)_{n\in\mathbb{N}} \in C^\infty(\Sigma_0)$ which converges to 0 such that, for all n:*

(i) *$\Gamma_n := \Gamma_{\phi_n}$ is a (possibly empty) smooth, embedded, 1-dimensional submanifold of $I \times \mathrm{Conv}(\hat{f})$; and*
(ii) *$\partial\Gamma_n$ lies inside $\{0,1\} \times \mathrm{Conv}(\hat{f})$.*

We first prove:

Proposition 8.2.

(i) *For all ϕ, Γ_ϕ is compact; and*
(ii) *for any neighborhood Ω of Γ_0 in $I \times \mathrm{Conv}(\hat{f})$, there exists a neighborhood U of 0 in $C^\infty(\Sigma_0)$ such that if $\phi \in U$, then $\Gamma_\phi \subseteq \Omega$.*

Proof. (i) This assertion follows from Corollary 7.2.

(ii) Suppose the contrary. Let $(\tau_n)_{n\in\mathbb{N}} \in [0,1]$, $(\phi_n)_{n\in\mathbb{N}} \in C^\infty(\Sigma_0)$ and $(f_n)_{n\in\mathbb{N}} \in \mathrm{Conv}(\hat{f})$ be such that $(\tau_n)_{n\in\mathbb{N}}$ converges to $\tau_\infty \in [0,1]$, $(\phi_n)_{n\in\mathbb{N}}$ converges to 0 and for all n:

$$(\tau_n, f_n) \notin \Omega.$$

Suppose moreover that, for all n:

$$K(f_n) = \gamma(\tau_n) + \phi_n.$$

By Lemma 7.1, $(f_n)_{n\in\mathbb{N}}$ subconverges to $f_\infty \in \mathrm{Conv}(f_\infty, \hat{f})$ such that:

$$\begin{aligned} K(f_\infty) &= \gamma(\tau_\infty) \\ \Rightarrow \quad &(\tau_\infty, f_\infty) \in \Gamma_0. \end{aligned}$$

Thus, for sufficiently large n, $(\tau_n, f_n) \in \Omega$, which is absurd. The result follows. □

We denote by $C_0^\infty(\Sigma_0)$ the set of smooth functions on Σ_0 which vanish along $\partial\Sigma_0$, and we identify this with the tangent space of Conv in the natural manner. We consider the derivative of K:

Proposition 8.3. *At every point of* Conv*, DK defines a uniformly elliptic operator from $C_0^\infty(\Sigma_0)$ to $C^\infty(\Sigma_0)$.*

Proof. This follows by differentiating the formula for the Gauss Curvature Operator given by Proposition 3.1. □

DK is therefore Fredholm. Since it is defined on the space of smooth functions over a compact manifold with boundary, which themselves vanish over the boundary, it is of index zero.

Proof of Proposition 8.1. Define $\hat{K} : [0,1] \times \mathrm{Conv}(\hat{f}) \times C^\infty(\Sigma_0) \to C^\infty(\Sigma_0)$ by

$$\hat{K}(\tau, f, \phi) = \gamma(\tau) - K(f) + \phi.$$

By compactness, there exists a neighborhood Ω of Γ in $[0,1] \times \mathrm{Conv}(\hat{f})$, a subspace $E \subseteq C^\infty(\Sigma_0)$ of dimension $m < \infty$ and $\epsilon > 0$ such that the restriction of $D\hat{K}$ to $\Omega \times B_\epsilon(0) \subseteq \Omega \times E$ is always surjective. This restriction is Fredholm of index $(m+1)$. Define $\hat{\Gamma}$ by

$$\hat{\Gamma} = \hat{K}^{-1}(\{0\}).$$

By the Implicit Function Theorem for Banach manifolds, $\hat{\Gamma}$ is an $(m+1)$-dimensional smooth submanifold of $\Omega \times B_\epsilon(0)$. Let $\pi_3 : [0,1] \times \mathrm{Conv}(\hat{f}) \times B_\epsilon(0) \to E$ denote projection onto the third factor. By Sard's Lemma there exists a sequence $(\phi_n)_{n\in\mathbb{N}} \in B_\epsilon(0)$ which tends to 0 such that, for all n, ϕ_n is a regular value of the restriction of π_3 to $\hat{\Gamma}$. However, for all n:

$$\Gamma_n := \Gamma_{\phi_n} = \hat{\Gamma} \cap \pi_3^{-1}(\phi_n).$$

Moreover, since ϕ_n is a regular value of π_3, Γ_n is a (possibly empty) smooth 1-dimensional embedded submanifold. By Proposition 8.2, for all n, Γ_n is compact, and for sufficiently large n, Γ_n lies entirely inside $[0,1] \times \Omega$. Therefore

$$\partial\Gamma_n \subseteq \partial(I \times \Omega) \subseteq \big(\{0,1\} \times \Omega\big) \cup \big([0,1] \times \partial\,\mathrm{Conv}(\hat{f})\big).$$

It thus remains to show that $\partial\Gamma_n$ lies away from $[0,1]\times\partial\,\mathrm{Conv}(\hat{f})$. However, if $(\tau, f)\in\Gamma_n$, then

$$0\leqslant f\leqslant \hat{f},\qquad \phi_0+\epsilon<K(f)<K(\hat{f})-\epsilon.$$

Thus, by the Geometric Maximum Principle, away from $\partial\Sigma_0$:

$$0<f<\hat{f},$$

and by the geometric maximum principal along the boundary, a similar relation holds for the derivative of f in the internal normal direction along $\partial\Sigma_0$. It follows that Γ_n lies in the interior of $\mathrm{Conv}(\hat{f})$ and so

$$\partial\Gamma_n\subseteq\{0,1\}\times\Omega.$$

This completes the proof. □

9. Rigidity and local rigidity

Let $M:=M^{n+1}$ be an $(n+1)$-dimensional Hadamard manifold. Let $(\Sigma_0,\partial\Sigma_0)\subseteq M^{n+1}$ be a smooth, convex hypersurface. Let $K:\mathrm{Conv}\to C^\infty(\Sigma_0)$, $\hat{f}$, ϕ_0 and $\hat{\phi}$ be as in Section 7. Let $C^\infty_0(\Sigma_0)$ be the set of smooth functions over Σ_0 which vanish along $\partial\Sigma_0$, which, as in the preceding section, we identify with the tangent space of Conv. In particular, for all $f\in\mathrm{Conv}$ we denote by $DK_f:C^\infty_0(\Sigma_0)\to C^\infty(\Sigma_0)$ the derivative of K at f.

Definition 9.1.

(i) We say that $\phi\in C^\infty(\Sigma_0)$ is locally rigid over $\mathrm{Conv}(\hat{f})$ if and only if for all $f\in\mathrm{Conv}(\hat{f})$ such that $K(f)=\phi$, DK_f is invertible (in other words, ϕ is a regular value of K).
(ii) We say that $\phi\in C^\infty(\Sigma_0)$ is rigid over $\mathrm{Conv}(\hat{f})$ if and only if there exists at most one $f\in\mathrm{Conv}(\hat{f})$ such that $K(f)=\phi$.

Example. Let $\mathbb{H}^{n+1}$ be $(n+1)$-dimensional hyperbolic space. Let H be a totally geodesic hypersurface. For $D>0$, let $H(D)$ be the equidistant hypersurface at a distance D from H. $H(D)$ has constant Gaussian curvature equal to $\tanh(D)$. Let $\Omega\subseteq H(D)$ be any bounded open subset with smooth boundary and consider the hypersurface $(\Sigma_0,\partial\Sigma_0)=(\Omega,\partial\Omega)$. Define $f_0=0$ and $\phi_0=Kf_0=\tanh(D)$. By the strong Geometric Maximum Principle and the homogeneity of $\mathbb{H}^{n+1}$, we readily show that ϕ_0 is rigid for any choice of $\hat{\Sigma}$. Moreover, by calculating the Jacobi operator of $H(D)$ (or by calculating the derivative of K using the example in Section 3), we likewise show that ϕ_0 is locally rigid.

Example. The above example is a special case of a more general construction. Let M be a Riemannian manifold. Let $P\in M$ be a point, let $\mathsf{N}\in UM$ be a unit vector at P, let A be a positive-definite symmetric 2-form over $\mathsf{N}^\perp$ and let $k=\mathrm{Det}(A)$. There is no algebraic obstruction to the construction of a hypersurface Σ such that:

(i) $P\in\Sigma$;
(ii) N is normal to Σ at P;

(iii) the second fundamental form of Σ at P is equal to A; and
(iv) if $\psi = \mathrm{Det}(A)$ is the Gaussian curvature of Σ, then $\psi = k$ up to infinite order at P.

Since $\psi = k$ up to infinite order at P, for $\epsilon > 0$ small, there exists a smooth family $(\psi_t)_{t<\epsilon}$ of smooth functions such that:

(i) $\psi_0 = \psi$; and
(ii) for all t, $\psi_t = k$ over the geodesic ball of radius t about P.

Suppose moreover that M has negative sectional curvature bounded above by -1 and that $A = k\,\mathrm{Id}$ for $k < 1$. In this case, the derivative of the Gauss Curvature Operator is invertible over a geodesic ball of small radius about P (see [14] for details in the 2-dimensional case). We may therefore assume by the Inverse Function Theorem for Banach Manifolds that $\psi = k$ over a geodesic ball of small radius about P. Moreover, Σ may be extended to a foliation $(\Sigma_t)_{t\in]-\epsilon,\epsilon[}$ of a neighborhood of P in M by hypersurfaces of constant curvature equal to k. Now let $B \subseteq M$ be a geodesic ball in M centred on P which is covered by this foliation. Let $\Omega \subseteq \Sigma$ be an open set with smooth boundary contained in $B \cap \Sigma$. If Σ' is any other hypersurface of constant Gaussian curvature equal to k such that $\partial\Sigma' = \partial\Omega$, then, by the Geometric Maximum Principle, Σ' is contained inside B, and, by the strong Geometric Maximum Principle, Σ' coincides with a leaf of the foliation. It is therefore equal to Ω, and we have thus shown that $\psi = k$ is both rigid and locally rigid over Ω for any choice of $\hat{\Sigma}$.

Proposition 9.2.

(i) *If ϕ is locally rigid, then ϕ' is also locally rigid for all ϕ' sufficiently close to ϕ.*
(ii) *If ϕ is rigid and locally rigid, then ϕ' is rigid for all ϕ' sufficiently close to ϕ.*

Proof. (i) Suppose the contrary. Let $(\phi_n)_{n\in\mathbb{N}} \in C^\infty(\Sigma_0)$ be a sequence of non-locally rigid functions converging to ϕ. Since ϕ is locally rigid, DK is invertible at f for all $f \in K^{-1}(\{\phi\})$. There therefore exists a neighborhood Ω of $K^{-1}(\{\phi\})$ in $\mathrm{Conv}(\hat{f})$ such that DK is invertible at f for all $f \in \Omega$. However, by Corollary 7.2, for all sufficiently large n:

$$K^{-1}\big(\{\phi_n\}\big) \subseteq \Omega.$$

ϕ_n is therefore locally rigid for sufficiently large n, which is absurd, and the assertion follows.

(ii) Suppose the contrary. There exists a sequence $(\phi'_n)_{n\in\mathbb{N}}$ which converges to ϕ such that ϕ'_n is not globally rigid. Thus, for all n, there exists $f_{1,n} \neq f_{2,n} \in \mathrm{Conv}(\hat{f})$ such that:

$$K(f_{1,n}) = K(f_{2,n}) = \phi'_n.$$

By Corollary 7.2, there exist $f_1, f_2 \in \mathrm{Conv}(\hat{f})$ to which $(f_{1,n})_{n\in\mathbb{N}}$ and $(f_{2,n})_{n\in\mathbb{N}}$ respectively converge. In particular

$$K(f_1) = K(f_2) = \phi.$$

Since ϕ is rigid, it follows that:

$$f_1 = f_2 = f.$$

Since ϕ is locally rigid, DK is invertible at f and thus K is locally invertible over a neighborhood of f. In particular, for sufficiently large n:

$$f_{1,n} = f_{2,n}.$$

This is absurd, and the result follows. □

10. Stability and existence

Let $M := M^{n+1}$ be an $(n+1)$-dimensional Hadamard manifold. Let $(\Sigma_0, \partial\Sigma_0) \subseteq M^{n+1}$ be a smooth, convex hypersurface. Let $(\hat{\Sigma}, \partial\hat{\Sigma}) \subseteq M^{n+1}$ be another smooth, convex hypersurface which is a graph below Σ_0. Let $\hat{f} \in C_0^\infty(\Sigma_0)$ be the function of which $\hat{\Sigma}$ is a graph. As in Section 7, we denote $\hat{\phi} = K(\hat{f})$ and $\phi_0 = K(0)$, and we denote by $\mathrm{Conv}(\Sigma_0, \hat{\Sigma}) := \mathrm{Conv}(\hat{f})$ the set of all smooth functions in $C_0^\infty(\Sigma_0)$ such that:

$$\hat{f} \leqslant f \leqslant 0 \quad \text{and} \quad \hat{\phi} - \epsilon \geqslant K(f) \geqslant \phi_0.$$

We identify every function in $\mathrm{Conv}(\Sigma_0, \hat{\Sigma})$ with its graph.

Definition 10.1.

(i) We say that $(\Sigma_0, \hat{\Sigma})$ is stable if and only if for all $\psi \in C_0^\infty(\Sigma_0)$, if $DK_0\psi > 0$, then $\psi < 0$ over the interior of Σ_0.
(ii) We say that $(\Sigma_0, \hat{\Sigma})$ is rigid if and only if the only hypersurface $(\Sigma, \partial\Sigma) \in \mathrm{Conv}(\Sigma_0, \hat{\Sigma})$ such that $K(\hat{\Sigma}) = K(\Sigma_0)$ is Σ_0 itself.

Remark. In other words, $(\Sigma_0, \hat{\Sigma})$ is rigid if and only if $\phi_0 := K(0) \in C^\infty(\Sigma_0)$ is rigid over $\mathrm{Conv}(\hat{f})$.

Remark. Observe that if $(\Sigma_0, \hat{\Sigma})$ is both rigid and stable, then ϕ_0 is also locally rigid over $\mathrm{Conv}(\hat{f})$.

Example. Let N_0 and A_0 be respectively the outward pointing, unit, normal vector field over Σ_0 and its corresponding shape operator. Let JK be the Jacobi operator of Σ_0. JK measures the first order variation of the Gaussian curvature upon first order, normal perturbations of Σ_0 and is given by

$$JK\phi = \mathrm{Tr}\big(A_0^{-1}W - A_0\big)\phi - \mathrm{Tr}\big(A_0^{-1}\,\mathrm{Hess}(\phi)\big),$$

where the mapping W is given by

$$\big\langle W(X), Y\big\rangle = \langle R_{X\mathsf{N}_0}Y, \mathsf{N}_0\rangle,$$

and where R is the Riemann curvature tensor of M. It follows that if the sectional curvature of M is bounded above by $-\epsilon^2$ and if the principal curvatures of Σ_0 are bounded below by ϵ, then

$$JK\phi = h\phi - \mathrm{Tr}\big(A_0^{-1}\,\mathrm{Hess}(\phi)\big),$$

for some non-negative function h. Since DK is conjugate to JK, it follows from the maximum principal that $(\Sigma_0, \hat{\Sigma})$ is stable.

Lemma 10.2. *If $(\Sigma_0, \hat{\Sigma})$ is stable and rigid, then, for all ϕ such that:*

$$\phi_0 \leqslant \phi \leqslant \hat{\phi} - \epsilon,$$

for some $\epsilon > 0$, there exists a smooth, convex, immersed hypersurface Σ_ϕ such that:

(i) $\hat{\Sigma} \leqslant \Sigma_\phi \leqslant \Sigma_0$*; and*
(ii) *the Gaussian curvature of Σ_ϕ at the point p is equal to $\phi(p)$.*

Proof. Assume first that:

$$\phi_0 + \epsilon < \phi < \hat{\phi} - \epsilon.$$

By stability, reducing ϵ is necessary, there exists $f_0 \in \mathrm{Conv}(\hat{f})$ such that:

$$\phi_0' := K(f_0) > \phi_0 + \epsilon.$$

By Proposition 9.2, we may assume moreover that ϕ_0' is both rigid and locally rigid over $\mathrm{Conv}(\hat{f})$. Let $\gamma : [0,1] \to C^\infty(\Sigma_0)$ be a smooth family of smooth functions such that:

(i) $\gamma(0) = \phi_0'$, $\gamma(1) = \phi$; and
(ii) for all $t \in [0,1]$:

$$\phi_0 + \epsilon < \gamma(t) < \hat{\phi} - \epsilon.$$

By Proposition 8.1, there exists $(\phi_n)_{n\in\mathbb{N}} \in C^\infty(\Sigma_0)$ which converges to 0 such that, for all n, $\Gamma_n := \Gamma_{\phi_n}$ is a (possibly empty) smooth, 1-dimensional embedded submanifold of $[0,1] \times \mathrm{Conv}(\hat{f})$. Moreover, for all n, Γ_n is compact, and

$$\partial \Gamma_n \subseteq \{0,1\} \times \mathrm{Conv}(\hat{f}).$$

By Proposition 9.2, we may assume that, for all n, $\phi_0' + \phi_n$ is both rigid and locally rigid. In addition, since ϕ_0' is locally rigid, we may assume that, for all n, there exists $f_n \in \mathrm{Conv}(\hat{f})$ such that:

$$(0, f_n) \in \Gamma_n.$$

Γ_n is therefore non-empty for all n. Let Γ_n^0 be the connected component of Γ_n containing $(0, f_n)$. Since it is compact, it is either an embedded, compact interval or an embedded, closed loop. We claim that Γ_n^0 is not a closed loop. Indeed, by local rigidity, DK is invertible at $(0, f_n)$. Consequently, if $\pi_1 : [0,1] \times \mathrm{Conv}(\hat{f}) \to [0,1]$ is the projection onto the first factor, the restriction of $D\pi_1$ to $T\Gamma_n^0$ is invertible at f_n. Since $0 = \pi_1(f_n)$ is an end point of $[0,1]$, this would imply that $(0, f_n)$ is also an end point of Γ_n^0. This is absurd and the assertion follows.

For all n, let g_n by the other end of Γ_n^0. Since $(\phi_0' + \phi_n)$ is globally rigid:

$$g_n \in \{1\} \times \mathrm{Conv}(\hat{f}).$$

In other words:

$$K(g_n) = \phi + \phi_n.$$

By Corollary 7.2, there exists $g_0 \in \mathrm{Conv}(\hat{f})$ to which $(g_n)_{n\in\mathbb{N}}$ subconverges. In particular:

$$K(g_0) = \phi.$$

This proves existence in the case where $\phi_0 + \epsilon < \phi < \hat{\phi} - \epsilon$. The general case follows by taking limits. □

11. Space forms and the local geodesic condition

Let $M := M^{n+1}$ be an $(n+1)$-dimensional Riemannian manifold. Let $K \subseteq M$ be a convex set with non-trivial interior. For any $P \in \partial K$, we say that K satisfies the **local geodesic condition** at P if and only if there exists an open geodesic segment Γ such that:

(i) $P \in \Gamma$; and
(ii) $\Gamma \subseteq K$.

Observe that since K is convex, the second condition implies in particular that $\Gamma \subseteq \partial K$.

We henceforth restrict attention to the case where M is a space-form of non-positive curvature. In other words, up to rescaling, M is isometric to either $(n+1)$-dimensional Euclidean or Hyperbolic space. We obtain the following global consequence of the local geodesic condition (cf. [19]):

Lemma 11.1. *Let K be a bounded, convex set with non-trivial interior, let $X \subseteq \partial K$ be a closed subset and let $Y \subseteq \partial K$ be the set of all points in ∂K satisfying the local geodesic condition. If $X \cup Y$ is closed, then Y lies in the convex hull of X.*

Proof. Suppose that Y is not contained in the convex hull of X. Then there exists a point $Q \in Y$ and a supporting hyperplane H to K at Q such that $H \cap X$ is empty. Denote $K' = K \cap H$. Let $Y' \subseteq \partial K'$ be the set of points satisfying the local geodesic condition. In particular, $Q \in Y'$. Since $X \cup Y$ is closed, so is Y'. We now show that Y' is unbounded. Indeed, suppose the contrary. Choose any $P \in M$ and let d_P be the distance to P in M. Since Y' is closed and bounded, it is compact, and so there exists a point $Q' \in Y'$ maximizing d_P. Let Γ be the open geodesic segment in Y' passing through Q'. Trivially, $\Gamma \subseteq Y'$. However, the restriction of d_P to Γ is convex, and so it cannot have a local maximum at Q'. This is absurd, and the assertion follows. However, since K is bounded, so is Y'. This is absurd, and the result follows. □

In the current context, regularity follows from the following result:

Proposition 11.2. *Suppose that M is a space form of non-positive curvature. Choose $k > 0$ and let $(K_n)_{n\in\mathbb{N}} \subseteq M$ be a sequence of convex subsets of M with smooth boundary such that, for all n, the Gaussian curvature of ∂K_n is equal to k. Suppose that $(K_n)_{n\in\mathbb{N}}$ converges to $K_0 \subseteq M$ and that K_0 has non-empty interior. Then the set of points in ∂K_n satisfying the local geodesic condition is closed.*

Proof. We show that the complement is open. Indeed, let $Q \in \partial K_n$ be a point not satisfying the local geodesic condition. Then there exist a hyperplane H, a bounded, open, convex subset U of H and an open subset V of ∂K_0 such that:

(i) $Q \in V$;
(ii) Q lies at non-zero distance from H; and
(iii) V is a graph over U with $\partial V = \partial U$.

It follows from [1] that ∂K_0 is smooth over V and has constant Gaussian curvature equal to k (see also Appendix A). In particular, no point of V satisfies the local geodesic condition. This completes the proof. □

We thus refine Theorem 1.1 to obtain:

Lemma 11.3. *Let $\mathbb{H}^{n+1}$ be $(n+1)$-dimensional hyperbolic space, and let $H \subseteq \mathbb{H}^{n+1}$ be a totally geodesic hypersurface. Choose $k > 0$, and let $\Omega \subseteq H$ be a bounded open set such that there exists a convex hypersurface $\hat{\Sigma}$ such that:*

(i) $\hat{\Sigma}$ is a graph below Ω;
(ii) the second fundamental form of $\hat{\Sigma}$ is at least ϵ in the Alexandrov sense, for some $\epsilon > 0$; and
(iii) the Gaussian curvature of Σ is at least k in the Alexandrov sense.

There exists a unique convex, immersed hypersurface $(\Sigma, \partial\Sigma)$ such that:

(i) Σ is a graph below Ω and $\partial\Sigma = \partial\Omega$;
(ii) Σ lies above $\hat{\Sigma}$;
(iii) Σ has C^∞ interior and is $C^{0,1}$ up to the boundary; and
(iv) the Gaussian curvature of Σ is equal to k.

Moreover if $\partial\Omega$ is smooth, then Σ is smooth up to the boundary.

Proof. We begin by smoothing the upper barrier. Choose $k' < k$. As in Lemma 2.13 of [21], there exist a sequence $(\epsilon_n)_{n\in\mathbb{N}} \in]0, k - k'[$ of positive numbers and a sequence of smooth, convex, immersed hypersurfaces $(\hat{\Sigma}_n)_{n\in\mathbb{N}}$ such that:

(i) $(\epsilon_n)_{n\in\mathbb{N}}$ converges to 0 and $(\hat{\Sigma}_n)_{n\in\mathbb{N}}$ converges to $\hat{\Sigma}$ in the $C^{0,\alpha}$ sense for all α;
(ii) for all n, $\hat{\Sigma}_n$ is a graph over a bounded open subset of H; and
(iii) for all n, the Gaussian curvature of $\hat{\Sigma}_n$ is greater than $k - \epsilon_n$.

Let $(\delta_n)_{n\in\mathbb{N}} > 0$ be a sequence of positive numbers converging to 0. For all n, let H_n be the equidistant hypersurface at distance δ_n from H. We may assume that, for all n, a portion of $\hat{\Sigma}_n$ is a graph over H_n. Let Ω_n be the subset of H_n over which it as a graph.

For all n, since $(\Omega_n, \partial\Omega_n)$ is locally and globally rigid, it follows by Theorem 1.1 that there exists a smooth, convex hypersurface Σ_n which is a graph below Ω_n such that $\Sigma_n > \hat{\Sigma}_n$ and whose Gaussian curvature is equal to k'.

Suppose now that $\partial\Omega$ is smooth. There exists $\epsilon > 0$ such that, for all n and for all $P \in \partial\Omega_n$, there exists a geodesic ball $B \subseteq \Omega_n$ of radius ϵ such that $P \in \partial B$. For all such B, we consider the foliation of constant Gaussian curvature hypersurfaces which are graphs below B and whose boundary is ∂B (in the upper half space model of $\mathbb{H}^{n+1}$, these are merely intersections of spheres in $\mathbb{R}^{n+1}$ with $\mathbb{H}^{n+1}$). Using these foliations and the Geometric Maximum Principle, we find that there exists $\theta > 0$ such that, for all n, $T\Sigma_n$ makes an angle of at least θ with H_n along $\partial\Sigma_n$. Bearing in mind the remark following Proposition 5.1, this yields uniform lower bounds for the restriction to $\partial\Omega_n$ of the second fundamental form of Σ_n. Taking limits now yields the desired hypersurface, Σ.

Consider now the general case. By Lemma 2.1, we may nonetheless assume that $(\hat{\Sigma}_n)_{n\in\mathbb{N}}$ converges to a $C^{0,1}$, convex hypersurface Σ which is a graph below Ω such that $\Sigma \geqslant \hat{\Sigma}$. Let B be a geodesic ball such that $\overline{B} \subseteq \Omega$. Using the Geometric Maximum Principle, by considering the foliation of constant Gaussian curvature hypersurfaces which are graphs below B and whose boundary is B, we may show that Σ lies strictly below Ω over its interior. We now assert that no point of Σ satisfies the local geodesic condition. Indeed, suppose the contrary. By Proposition 11.2 the union of $\partial\Sigma$ with the set of points of $\Sigma \setminus \partial\Sigma$ which satisfy the local geodesic condition is closed. Thus, if $P \in \Sigma$ is such a point, it follows from Lemma 11.1 that P lies in the convex hull of $\partial\Sigma$. In particular, P lies in H and thus, by convexity, $\Sigma = \Omega$, which is absurd. The assertion follows and it now follows by [1] that Σ is smooth over its interior, and this proves existence (see also Appendix A).

Let Σ be a graph over Ω of constant Gaussian curvature equal to k. Let f be the graph function of Σ in conformal coordinates about H (see the example following Proposition 3.1). f satisfies the following equation:

$$\mathrm{Det}\big(f_{;ij} - \tan(f)(f_{;j}f_{;j} + \delta_{ij})\big)^{1/n} = k\frac{1}{\cos(f)^3}\big(1 + \|\nabla f\|^2\big)^{(n+2)/2n}.$$

Let Σ' be another such hypersurface and suppose that $\Sigma' \neq \Sigma$. Let f' be the graph function of Σ' in conformal coordinates about H. Without loss of generality, there exists $P \in H$ such that $f'(P) > f(P)$ and $f' - f$ is maximized at P. Define the field of matrices, A, by

$$A = \big(\mathrm{Hess}(f) - \tan(f)(\nabla f \otimes \nabla f + \mathrm{Id})\big)^{-1}.$$

(This matrix is invertible by convexity of Σ.) A is positive definite. Thus, near P, by concavity of $\mathrm{Det}^{1/n}$, and since $f' > f$, for some $\epsilon, \hat{k} > 0$ that we need not calculate:

$$\begin{aligned}&\hat{k}\,\mathrm{Tr}\big(A^{-1}\big(f'_{;ij} - f_{;ij}\big)\big) - \hat{k}\tan(f)\,\mathrm{Tr}\big(A^{-1}\big(f'_{;i}f'_{;j} - f_{;i}f_{;j}\big)\big)\\ &\quad\geqslant \epsilon + \frac{k}{\cos(f)^3}\big(\big(1 + \|\nabla f'\|^2\big)^{(n+2)/2} - \big(1 + \|\nabla f\|^2\big)^{(n+2)/2}\big).\end{aligned}$$

At P, since $(f' - f)$ is maximized, $\nabla f' = \nabla f$. Thus, near P,

$$\mathrm{Tr}\big(A^{-1}\big(f'_{;ij} - f_{;ij}\big)\big) > 0.$$

This yields a contradiction by the Maximum Principle. Uniqueness follows and this completes the proof. □

12. Relations to existing results

With small modifications, these techniques may be adapted to yield existing results. First, considering $\mathbb{R}^n$ as a subspace of $\mathbb{R}^{n+1}$ in the natural manner, we recover the following theorem of Guan (see [8]), which is the analogue in Euclidean space of Lemma 11.3:

Theorem 12.1. *(See [8].) Choose $k > 0$, and let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set. Suppose that there exists a convex hypersurface, $\hat{\Sigma}$ such that:*

(i) *$\hat{\Sigma}$ is a graph below Ω;*
(ii) *the second fundamental form of $\hat{\Sigma}$ is at least ϵ in the Alexandrov sense, for some $\epsilon > 0$; and*
(iii) *the Gaussian curvature of Σ is at least k in the Alexandrov sense.*

There exists a unique convex, immersed hypersurface $(\Sigma, \partial\Sigma)$ such that:

(i) *Σ is a graph below Ω and $\partial\Sigma = \partial\Omega$;*
(ii) *Σ lies above $\hat{\Sigma}$;*
(iii) *Σ has C^∞ interior and is $C^{0,1}$ up to the boundary; and*
(iv) *the Gaussian curvature of Σ is equal to k.*

Moreover, if $\partial\Omega$ is smooth, then Σ is smooth up to the boundary.

Remark. Although, as in Lemma 11.3, if we identify $(\Sigma_0, \partial\Sigma_0) = (\Omega, \partial\Omega)$, then the Gauss Curvature Equation is not elliptic at $f_0 = 0$, this, in itself, does not present a serious difficulty since there exist functions arbitrarily close to f_0 where the Gauss Curvature Equation is elliptic. The particular difficulty in Euclidean space lies in obtaining functions near f_0 for which the Gauss Curvature Equation is also stable. We circumvent this by approximating $\mathbb{R}^n$ by spaces of constant negative sectional curvature.

Proof of Theorem 12.1. Using polar coordinates for $\mathbb{R}^n$, we identify $\mathbb{R}^{n+1}$ with $\Sigma^{n-1} \times\ ]0,\infty[\ \times \mathbb{R}$, where Σ^{n+1} is the unit sphere. We thus denote a point in $\mathbb{R}^{n+1}$ by the coordinates $(\theta, r, t) \in \Sigma^{n-1} \times\ ]0,\infty[\ \times \mathbb{R}$. Let g^Σ denote the standard metric over Σ^{n-1}. For $\epsilon > 0$, we define the metric g_ϵ over $\mathbb{R}^{n+1}$ such that, at (θ, r, t):

$$g = \cosh^2(\epsilon t)\big(\sinh^2(\epsilon r) g^\Sigma \oplus dr^2\big) \oplus dt^2.$$

This metric is smooth and has constant curvature equal to $-\epsilon$. Indeed, this formula is obtained by using polar coordinates for $\mathbb{H}^n$ about a point and then identifying $\mathbb{H}^{n+1}$ with $\mathbb{H}^n \times \mathbb{R}$ using the foliation by geodesics normal to a totally geodesic hypersurface.

With respect to this metric, $\mathbb{R}^n$ is identified with a totally geodesic hypersurface, and, for all $k' < k$, there exists $\epsilon > 0$ such that $\hat{\Sigma}$ satisfies the hypotheses of Lemma 11.3, with k' instead of k. There therefore exists $\Sigma_\epsilon \subseteq \mathbb{R}^{n+1}$ possessing the desired properties and of constant Gaussian curvature equal to k' with respect to the metric g_ϵ. Existence follows by taking limits as in Lemma 11.3.

To prove uniqueness, let Σ_1 and Σ_2 be two solutions. Suppose that $\Sigma_1 \neq \Sigma_2$. Without loss of generality, there is a point of Σ_1 lying below Σ_2. There therefore exists a translate Σ_1' of Σ_1 in the vertical direction which lies strictly above Σ_1 and which is an interior tangent to Σ_2 at some point P'. Since $\partial\Sigma_1'$ lies strictly above $\partial\Sigma_2$, P' is an interior point of Σ_1' and Σ_2. It follows by the strong Geometric Maximum Principle that Σ_1' and Σ_2 coincide, which is absurd. Uniqueness follows and this completes the proof. □

If M is a Riemannian manifold, we say that a bounded open subset $\Omega \subseteq M$ satisfies a uniform exterior ball condition if and only if there exists $\epsilon > 0$ such that for every $P \in \partial\Omega$, there exists an open geodesic ball $B \subseteq \Omega^c$ of radius ϵ such that:

$$P \in \partial B \cap \partial\Omega.$$

By compactness, Ω satisfies a uniform exterior ball condition for a given metric over M if and only if it satisfies this condition for any metric over M, and we thus extend this condition to subsets of arbitrary C^∞ manifolds.

Example. Any compact, open subset with smooth boundary satisfies a uniform exterior ball condition.

Example. Any convex, open subset satisfies a uniform exterior ball condition.

We now recover the following theorem of Rosenberg and Spruck (see [17]), which has also recently been proven in a more general setting by Guan, Spruck and Szapiel (see [12]):

Theorem 12.2. *(See [17].) Let $\Omega \subseteq \partial_\infty\mathbb{H}^{n+1}$ be a non-trivial open subset whose boundary satisfies the uniform exterior ball condition. Then, for all $k \in]0,1[$, there exists a convex, immersed hypersurface $\Sigma_k \subseteq \mathbb{H}^{n+1}$ such that:*

(i) *identifying $\partial_\infty\mathbb{H}^{n+1}$ with $\mathbb{R}^n \cup \{\infty\}$ and viewing Ω as a subset of $\mathbb{R}^n$, Σ_k is a graph over Ω;*
(ii) *Σ_k is smooth and $C^{0,1}$ up to the boundary;*
(iii) *$\partial\Sigma_k = \partial\Omega$; and*
(iv) *Σ_k has constant Gaussian curvature equal to k.*

Moreover, if Ω is star-shaped, then Σ_k is unique.

Remark. In this case, we use horospheres as upper barriers. Since these have curvature equal to 1, we can only prove existence for hypersurfaces of curvature less than 1, hence the hypothesis on k.

Proof of Theorem 12.2. We identify $\mathbb{H}^{n+1}$ with the upper half space $\mathbb{R}^n \times]0, \infty[$ in the standard manner. We thus identify $\partial_\infty \mathbb{H}^{n+1}$ with $\mathbb{R}^n \cup \{\infty\}$ and view Ω as a subset of $\mathbb{R}^n$. For $\epsilon > 0$, let $H_\epsilon = \mathbb{R}^n \times \{\epsilon\}$ be the horosphere at height ϵ above $\mathbb{R}^n$. We define $\Omega_\epsilon \subseteq H_\epsilon$ by

$$\Omega_\epsilon = \big\{(x, \epsilon) \text{ s.t. } x \in \Omega\big\}.$$

By the uniform exterior ball condition, for ϵ sufficiently small, $\partial\Omega_\epsilon$ is uniformly strictly convex as a subset of $\mathbb{H}^{n+1}$ with respect to the outward pointing unit normal in H_ϵ.

Let K_ϵ be the complement of Ω_ϵ in H_ϵ. Let $\hat{K}_\epsilon$ be the convex hull of K_ϵ in $\mathbb{H}^{n+1}$. We denote by $\Sigma_{0,\epsilon}$ the portion of $\partial\hat{K}_\epsilon$ lying above H_ϵ. In other words:

$$\partial\hat{K}_\epsilon = (\partial\hat{K}_\epsilon \cap H_\epsilon) \cup \Sigma_{0,\epsilon}.$$

Since it is locally ruled, $\Sigma_{0,\epsilon}$ serves as a lower barrier for the problem (see [19]). We define $(\hat{\Sigma}_\epsilon, \partial\hat{\Sigma}_\epsilon) = (\Omega, \partial\Omega)$. The only difference between our current framework and that of Theorem 1.1 is that it is the upper barrier, $\hat{\Sigma}_\epsilon$ that is smooth and the lower barrier, $\Sigma_{0,\epsilon}$ that is not. The only change required to adapt the proof to our framework is therefore to replace $(f - f_0)$ in Corollary 5.5 with $(f - \hat{f})$. The uniform strict convexity of Ω_ϵ as a subset of $\mathbb{H}^{n+1}$ with respect to the normal in H_ϵ ensures uniform lower bounds of the restriction to $\partial\Omega$ of the second fundamental form of any surface Σ which is a graph above Ω such that $\partial\Sigma = \partial\Omega$. Thus proceeding as in Theorem 1.1, we show that there exists a graph Σ_ϵ over Ω_ϵ which is smooth up to the boundary and has constant Gaussian curvature equal to k.

Taking limits yields a $C^{0,1}$ graph Σ over Ω such that $\partial\Sigma = \partial\Omega$. Let $Y \subseteq \Sigma$ be the set of all points satisfying the local geodesic condition. By Proposition 11.2, $\partial\Sigma \cup Y$ is closed. It follows as in Lemma 11.1 that Y is contained in the convex hull of the intersection of some totally geodesic hyperplane H with $\partial\Omega$ (see [21] for details). In particular, if Y is non-empty, then, viewed as a graph, Σ is vertical at some point on the boundary. However, consider a point $P \in \partial\Omega_\epsilon$ and a geodesic ball $B \subseteq H_\epsilon$ such that $B \subseteq \Omega^c$ and $P \in \partial B$. Using the foliation of constant Gaussian curvature hypersurfaces in $\mathbb{H}^n$ whose boundary coincides with ∂B, we deduce by the Geometric Maximum Principle that there exists $\theta > 0$ such that, for ϵ sufficiently small, Σ_ϵ makes an angle at P of at least θ with the foliation of vertical geodesics along $\partial\Omega$. Moreover, θ may be chosen independent of P. Taking limits, it follows that Σ is everywhere strictly convex and is therefore smooth over the interior by [1]. This proves existence.

Suppose now that Ω is star-shaped, and let Σ_1 and Σ_2 be two solutions. Suppose that $\Sigma_1 \neq \Sigma_2$. Without loss of generality, there exists a point $P \in \Sigma_1$ lying below Σ_2. As before, we identify $\mathbb{H}^{n+1}$ with $\mathbb{R}^n \times]0, \infty[$. Without loss of generality, we may suppose that Ω is star-shaped about $(0, 0)$. Consider the family $(M_\lambda)_{\lambda>1}$ of isometries of $\mathbb{H}^{n+1}$ given by

$$M_\lambda(x, t) = (\lambda x, \lambda t).$$

There exists $\lambda > 1$ such that $M_\lambda\Sigma_1$ is an exterior tangent to Σ_2 at some point P'. Since $M_\lambda\partial\Sigma_1 \cap \partial\Sigma_2 = \emptyset$, P' is an interior point of Σ_1 and Σ_2. It follows by the strong Geometric Maximum Principle that $M_\lambda\Sigma_1 = \Sigma_2$, which is absurd. Uniqueness follows and this completes the proof. □

Appendix A. Regularity of limit hypersurfaces

Let M^{n+1} be an $(n+1)$-dimensional Riemannian manifold. Choose $k>0$ let $(\Sigma_m)_{m\in\mathbb{N}}$ be a sequence of smooth, convex, immersed hypersurfaces in M of constant Gaussian curvature equal to k. Suppose that there exists a $C^{0,1}$ locally convex, immersed hypersurface, Σ_0 to which $(\Sigma_m)_{m\in\mathbb{N}}$ converges in the $C^{0,\alpha}$ sense for all α. For all $m\in\mathbb{N}$, let N_m and A_m be the unit normal vector field and the second fundamental form respectively of Σ_m. Choose $p_0\in\Sigma_0$ and let $(p_m)_{m\in\mathbb{N}}\in(\Sigma_m)_{m\in\mathbb{N}}$ be a sequence converging to p_0. For all $r>0$, and for all $m\in\mathbb{N}\cup\{0\}$, let $B_{m,r}$ be the ball of radius r (with respect to the intrinsic metric) about p_m in Σ_m.

We will say that Σ_0 is functionally strictly convex at p_0 if there exists a smooth function, φ, defined on M near p_0 such that:

(i) φ is strictly convex;
(ii) $\varphi(p_0)>0$; and
(iii) the connected component of $\varphi^{-1}([0,\infty[)\cap\Sigma_0$ containing p_0 is compact.

Observe that if M is a space form, then Σ_0 is functionally strictly convex whenever it is strictly convex. We will prove:

Lemma A.1. *If Σ_0 is functionally strictly convex at p_0, then there exists $r>0$ such that $(B_{m,r},p_m)_{m\in\mathbb{N}}$ converges to $(B_{0,r},p_0)$ in the pointed C^∞-Cheeger Gromov sense. In particular, $B_{0,r}$ is a smooth, convex immersion of constant Gaussian curvature equal to k.*

As in Section 5, we denote by $\mathcal{B}$ the family of constants which depend continuously on the data: M, k, (Σ_0,p_0) and the C^1 jets of $(\Sigma_m,p_m)_{m\in\mathbb{N}}$. In this section, for any positive quantity, X, we denote by $O(X)$ any term which is bounded in magnitude by $K|X|$ for some K in $\mathcal{B}$.

The following elementary lemma will be of use in the proof:

Lemma A.2. *For $\lambda>0$ and for all $a,b\in\mathbb{R}$:*

$$(a+b)^2\leqslant(1+\lambda)a^2+\left(1+\lambda^{-1}\right)b^2.$$

Proof of Lemma A.1. Since $(\Sigma_m)_{n\in\mathbb{N}}$ converges to Σ_0 and since Σ_0 is functionally strictly convex at p_0, there exist $\epsilon,h>0$, open sets $\Omega_0,(\Omega_m)_{m\in\mathbb{N}}\subseteq M$ and, for every m, a smooth function $\varphi_m:\Omega_m\to[0,\infty[$ such that:

(i) for all m, Ω_m is a neighborhood of p_m and $(\Omega_m)_{m\in\mathbb{N}}$ converges to Ω_0 in the Hausdorff sense;
(ii) $(\varphi_m)_{m\in\mathbb{N}}$ converges to φ_0 in the C^∞ sense;
(iii) for all m, $\mathrm{Hess}(\varphi_m)\geqslant\epsilon\,\mathrm{Id}$;
(iv) for all m, $\varphi_m(p_0)=2h$; and
(v) for all m, the connected component of p_m in $\Sigma_m\cap\Omega_m$ is compact with smooth boundary and φ_m equals zero along the boundary: we denote this connected component by $\Sigma_{m,0}$.

We may assume that, for all m, $\varphi_m\leqslant1$ over $\Sigma_{m,0}$. Finally, after reducing ϵ if necessary, there exists a smooth, unit length vector field X defined over a neighborhood of p_0 such that, for all m, throughout $\Sigma_{m,0}$, $\langle X, \mathsf{N}_m\rangle \geqslant \epsilon$. We now follow an adaptation of reasoning presented by Pogorelov in [15].

Choose $\alpha \geqslant 1$. For all m, we define the function Φ_m by

$$\Phi_m = \alpha \operatorname{Log}(\varphi_m) - \langle X, \mathsf{N}_m\rangle + \operatorname{Log}\big(\|A_m\|\big),$$

where $\|A_m\|$ is the operator norm of A_m, which is equal to the highest eigenvalue of A_m. We aim to obtain a priori upper bounds for Φ_m for some α. We trivially obtain a priori bounds whenever $\|A_m\| \leqslant 1$. We thus consider the region where $\|A_m\| \geqslant 1$. Choose $m \in \mathbb{N}$ and $P \in \Sigma_{m,0}$. Let $\lambda_1 \geqslant \cdots \geqslant \lambda_n$ be the eigenvalues of A_m at P. In particular, $\lambda_1 = \|A_m\|$. Let $e_1, \ldots, e_n$ be the corresponding orthonormal basis of eigenvectors. In the sequel, we will suppress m.

Let the subscript ";" denote covariant differentiation with respect to the Levi-Civita covariant derivative of Σ. Thus, for example:

$$A_{ij;k} = \big(\nabla^{\Sigma}_{e_k} A\big)(e_i, e_j).$$

We consider the Laplacian, Δ, defined on functions by:

$$\Delta f = \sum_{i=1}^{n} \frac{1}{\lambda_i} f_{;ii}.$$

We aim to use the Maximum Principle in conjunction with Δ. Thus, in the sequel, we will only be interested in the orders of magnitude of potentially negative terms.

In analogy to Corollary 6.7, at P:

$$\Delta \operatorname{Log}(\lambda_1) \geqslant \sum_{i,j=1}^{n} \frac{1}{\lambda_1 \lambda_i \lambda_j} A_{ij;1} A_{ij;1} - \sum_{i=1}^{n} \frac{1}{\lambda_1 \lambda_1 \lambda_i} A_{11;i} A_{11;i} - O(1) - O\big(\big\|A^{-1}\big\|\big),$$

in the weak sense. However, by Lemma 6.3, for all i:

$$A_{11;i} = A_{i1;1} + R^{M}_{i1\nu 1},$$

where ν represents the exterior normal direction to Σ. Thus, bearing in mind Lemma A.2, and that $\lambda_1 \geqslant 1$, we obtain:

$$\Delta \operatorname{Log}(\lambda_1) \geqslant \sum_{i=2}^{n} \frac{1}{2\lambda_1 \lambda_1 \lambda_i} A_{i1;1} A_{i1;1} - O(1) - O\big(\big\|A^{-1}\big\|\big),$$

in the weak sense. Differentiating the Gauss Curvature Equation yields, for all j:

$$\sum_{i=1}^{n} \frac{1}{\lambda_i} A_{ii;j} = 0.$$

Thus

$$\begin{aligned}-\Delta\langle X, \mathsf{N}\rangle &\geqslant \langle X, \mathsf{N}\rangle \operatorname{Tr}(A) - O(1) - O\big(\|A^{-1}\|\big)\\ &\geqslant \epsilon\lambda_1 - O(1) - O\big(\|A^{-1}\|\big).\end{aligned}$$

Finally

$$\Delta\big(\alpha\operatorname{Log}(\varphi)\big) \geqslant \frac{\alpha}{\varphi}\epsilon\operatorname{Tr}\big(A^{-1}\big) - \frac{\alpha}{\varphi^2}\sum_{i=1}^{n}\frac{1}{\lambda_i}\varphi_{;i}\varphi_{;i} - O(\alpha).$$

However, bearing in mind Lemma 6.3,

$$\Phi_{;i} = \frac{\alpha}{\varphi}\varphi_{;i} - X^{\nu}{}_{;i} - X^i\lambda_i + \frac{1}{\lambda_1}A_{i1;1} + \frac{1}{\lambda_1}R^M_{i1\nu1},$$

where ν is the exterior normal direction over Σ. Thus, by induction on Lemma A.2, modulo $\nabla\Phi$:

$$\left|\frac{\alpha}{\varphi}\varphi_{;i}\right|^2 \leqslant \frac{4}{\lambda_1^2}A_{i1;1}A_{i1;1} + \frac{4}{\lambda_1^2}\big(R^M_{i1\nu1}\big)^2 + 4\big(X^i\lambda_i\big)^2 + 4\big(X^{\nu}_{;i}\big)^2.$$

Thus, bearing in mind that $\lambda_1 \geqslant \lambda_i$ for all i and that $\lambda_1 \geqslant 1$, we obtain, modulo $\nabla\Phi$:

$$\frac{\alpha}{\varphi^2}\sum_{i=2}^{n}\frac{1}{\lambda_i}\varphi_{;i}\varphi_{;i} = O\big(\alpha^{-1}\|A^{-1}\|\big) + O\big(\alpha^{-1}\lambda_1\big) + \sum_{i=2}^{n}\frac{4}{\alpha\lambda_1^2\lambda_i}A_{i1;1}A_{i1;1}.$$

Since φ is bounded above (and thus φ^{-1} is bounded below), for sufficiently large α we obtain, modulo $\nabla\Phi$:

$$\begin{aligned}\Delta\Phi &\geqslant \frac{\epsilon}{2}\lambda_1 - O\big(\lambda_1^{-1}\varphi^{-2}\big) - O(1)\\ &= \big(\varphi^{2\alpha}\|A\|\big)^{-1}\left(\frac{\epsilon}{2}\big(\varphi^{\alpha}\|A\|\big)^2 - O\big(\varphi^{\alpha}\|A\|\big) - O(1)\right).\end{aligned}$$

There therefore exists $K_1 > 0$ in $\mathcal{B}$ such that if $(\varphi^{\alpha}\|A\|) \geqslant K$, then the right-hand side is positive. However, for all $m \in \mathbb{N}$, $\Phi_m = -\infty$ along $\partial\Sigma_{m,0}$. There thus exists a point $P \in \Sigma_{m,0}$ where Φ_m is maximized. By the Maximum Principle, at this point, either $\|A\| \leqslant 1$ or $\varphi^{\alpha}\|A\| \leqslant K_1$. Taking exponentials, there therefore exists $K_2 > 0$ in $\mathcal{B}$ such that, for all $m \in \mathbb{N}$, throughout $\Sigma_{m,0}$:

$$\varphi^{\alpha}\langle X, \mathsf{N}_m\rangle^{-1}\|A_m\| \leqslant K_2.$$

Since $\langle X, \mathsf{N}_m\rangle \leqslant 1$, this yields a priori bounds for $\|A_m\|$ over the intersection of $\Sigma_{m,0}$ with $\varphi_m \geqslant h$. Using, for example, an adaptation of the proof of Theorem 1.2 of [20] in conjunction with the Bernstein Theorem [5,13,15] of Calabi, Jörgens, Pogorelov, we obtain a priori C^k bounds for $\Sigma_{m,0}$ over the region $\varphi_m \geqslant 3h$ for all k. The result now follows by the Arzela–Ascoli Theorem. □

References

[1] L. Caffarelli, Monge–Ampère equation, div–curl theorems in Lagrangian coordinates, compression and rotation, Lecture Notes, 1997.
[2] L. Caffarelli, L. Nirenberg, J. Spruck, The Dirichlet problem for nonlinear second-order elliptic equations. I. Monge–Ampère equation, Comm. Pure Appl. Math. 37 (3) (1984) 369–402.
[3] L. Caffarelli, L. Nirenberg, J. Spruck, Nonlinear second-order elliptic equations. V. The Dirichlet problem for Weingarten hypersurfaces, Comm. Pure Appl. Math. 41 (1) (1988) 47–70.
[4] L. Caffarelli, J.J. Kohn, L. Nirenberg, J. Spruck, The Dirichlet problem for nonlinear second-order elliptic equations. II. Complex Monge–Ampère, and uniformly elliptic equations, Comm. Pure Appl. Math. 38 (2) (1985) 209–252.
[5] E. Calabi, Improper affine hypersurfaces of convex type and a generalization of a theorem by K. Jörgens, Michigan Math. J. 5 (1958) 105–126.
[6] C.F. Gauss, General Investigations of Curved Surfaces (translated from the Latin and German by Adam Hiltebeitel and James Morehead), Raven Press, Hewlett, NY, 1965.
[7] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, Classics Math., Springer, Berlin, 2001.
[8] B. Guan, The Dirichlet problem for Monge–Ampère equations in non-convex domains and spacelike hypersurfaces of constant Gauss curvature, Trans. Amer. Math. Soc. 350 (1998) 4955–4971.
[9] B. Guan, J. Spruck, Boundary value problems on $\mathbb{S}^n$ for surfaces of constant Gauss curvature, Ann. of Math. 138 (1993) 601–624.
[10] B. Guan, J. Spruck, The existence of hypersurfaces of constant Gauss curvature with prescribed boundary, J. Differential Geom. 62 (2) (2002) 259–287.
[11] B. Guan, J. Spruck, Hypersurfaces of constant curvature in hyperbolic space II, arXiv:0810.1781.
[12] B. Guan, J. Spruck, M. Szapiel, Hypersurfaces of constant curvature in hyperbolic space I, J. Geom. Anal.
[13] K. Jörgens, Über die Lösungen der Differentialgleichung $rt - s^2 = 1$, Math. Ann. 127 (1954) 130–134.
[14] F. Labourie, Un lemme de Morse pour les surfaces convexes, Invent. Math. 141 (2) (2000) 239–297 (in French).
[15] A.V. Pogorelov, On the improper affine hyperspheres, Geom. Dedicata 1 (1972) 33–46.
[16] A.V. Pogorelov, Extrinsic Geometry of Convex Surfaces, Transl. Math. Monogr., vol. 35, American Mathematical Society, Providence, RI, 1973. Translated from the Russian by Israel Program for Scientific Translations.
[17] H. Rosenberg, J. Spruck, On the existence of convex hypersurfaces of constant Gauss curvature in hyperbolic space, J. Differential Geom. 40 (2) (1994) 379–409.
[18] G. Smith, Compactness results for immersions of prescribed Gaussian curvature II – Geometric aspects, arXiv: 1002.2982.
[19] G. Smith, Moduli of flat conformal structures of hyperbolic type, arXiv:0804.0744.
[20] G. Smith, Special Lagrangian curvature, arXiv:math/0506230.
[21] G. Smith, The non-linear Dirichlet problem in Hadamard manifolds, arXiv:0908.3590.
[22] G. Smith, The Plateau problem for general curvature functions, arXiv:1008.3545.
[23] J. Spruck, Fully nonlinear elliptic equations and applications to geometry, in: Proceedings of the International Congress of Mathematicians, vols. 1, 2, Zürich, 1994, Birkhäuser, Basel, 1995, pp. 1145–1152.
[24] N.S. Trudinger, X. Wang, On locally convex hypersurfaces with boundary, J. Reine Angew. Math. 551 (2002) 11–32.