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CW simplicial resolutions of spaces with an application to loop spaces

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Abstract

We show how a certain type of CW simplicial resolutions of spaces by wedges of spheres may be constructed, and how such resolutions yield an obstruction theory for a given space to be a loop space. © 2000 Published by Elsevier Science B.V. All rights reserved.

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1. Introduction

A simplicial resolution of a space $\boldsymbol{X}$ by wedges of spheres is a simplicial space $\boldsymbol{W}_\bullet$ such that (a) each space $\boldsymbol{W}_n$ is homotopy equivalent to a wedge of spheres, and (b) for each $k \geqslant 1$, the augmented simplicial group $\pi_k \boldsymbol{W}_\bullet \to \pi_k \boldsymbol{X}$ is acyclic (see Definition 3.5 below). Such resolutions, first constructed by Chris Stover in [36, Section 2], are dual to the “unstable Adams resolutions” of [11, I, Section 2], and have a number of applications: see Section 3 below and [36,13,14,1,5–7].

However, the Stover construction yields very large resolutions, which do not lend themselves readily to computation, and no other construction was hitherto available. In particular, it was not clear whether one could find minimal resolutions of this type. The purpose of this note is to show that any space X has simplicial resolutions by wedges of spheres, which may be constructed from purely algebraic data, consisting of an (arbitrary) simplicial resolution of $\pi_* X$ as a Π-algebra—that is, as a graded group with an action on the primary homotopy operations on it (see Definition 3.1 below):

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Theorem A. *Every free simplicial Π-algebra resolution of a realizable Π-algebra $\pi_* X$ is realizable topologically as a simplicial resolution by wedges of spheres.*

In fact such resolutions can be given a convenient "CW structure" (Definition 3.11). There is an analogous result for maps (Theorem 3.17).

Since *no* such resolution of a non-realizable Π-algebra can be realized (see Remark 3.12 below), this completely determines which free simplicial Π-algebra resolutions are realizable.

Theorem A implies that in the spectral sequences of [36,1,13] we can work with minimal resolutions, and allows us to identify the higher homotopy operations of [5–7] as lying in appropriate cohomology groups (compare [6, 4.17] and [8, Section 6]). A generalization of Theorem A to other model categories appears in [9].

As an application of such CW resolutions, we describe an obstruction theory for deciding whether a given space $\boldsymbol{X}$ is a loop space, in terms of higher homotopy operations. One such theory was given in [7], but the present approach does not require a given H-space structure on $\boldsymbol{X}$, and may be adapted also to the existence of A_n-structures (and thus subsumes [6]). It is summarized in

Theorem B. *A space $\boldsymbol{X}$ with trivial Whitehead products is homotopy equivalent to a loop space if and only if the higher homotopy operations of Definition* 5.5 *below vanish coherently.*

Notation and conventions. $\mathcal{G}p$ will denote the category of groups, $\mathcal{T}$ that of topological spaces, and $\mathcal{T}_*$ that of pointed topological spaces with base-point preserving maps. The full subcategory of 0-connected spaces will be denoted by $\mathcal{T}_c \subset \mathcal{T}_*$. The category of simplical sets will be denoted by $\mathcal{S}$ and that of pointed simplicial sets by $\mathcal{S}_*$; we shall use boldface letters: $\boldsymbol{X}, \boldsymbol{S}^n, \ldots$ to denote objects in any of these four categories. If $f: \boldsymbol{X} \to \boldsymbol{Y}$ is a map in one of these categories, we denote by $f_\#: \pi_* \boldsymbol{X} \to \pi_* \boldsymbol{Y}$ the induced map in the homotopy groups.

Organization. In Section 2 we review some background on simplicial objects and bisimplicial groups, and in Section 3 we recall some facts on Π-algebras, and prove our main results on CW resolutions of spaces by wedges of spheres: Theorem A (= Theorem 3.16) and Theorem 3.17. In Section 4 we define a certain cosimplicial simplicial space up-to-homotopy, which can be rectified if and only if $\boldsymbol{X}$ is a loop space. In Section 5 we construct a certain collection of *face-codegeneracy polyhedra*, which are used to define the higher homotopy operations refered to in Theorem B (= Theorem 5.6). We also show how the theorem may be used in reverse to calculate a certain tertiary operation in $\pi_* \boldsymbol{S}^7$.

2. Simplicial objects

We first provide some definitions and facts on simplicial objects:

Definition 2.1. Let Δ denote the category of ordered sequences $\boldsymbol{n} = \langle 0, 1, \ldots, n\rangle$ ($n \in \mathbb{N}$), with order-preserving maps. A *simplicial object* over a category $\mathcal{C}$ is a functor $X : \Delta^{op} \to \mathcal{C}$, usually written $X_\bullet$, which may be described explicitly as a sequence of objects $\{X_k\}_{k=0}^{\infty}$ in $\mathcal{C}$, equipped with face maps $d_i^k : X_k \to X_{k-1}$ and degeneracies $s_j^k : X_k \to X_{k+1}$ (usually written simply d_i, s_j, for $0 \leqslant i, j \leqslant k$), satisfying the usual simplicial identities [26, Section 1.1]. If $I = (i_1, i_2, \ldots, i_r)$ is some multi-index, we write d_I for $d_{i_1} \circ d_{i_2} \circ \cdots \circ d_{i_r}$, with $d_\emptyset := id$; and similarly for s_I. An *augmented* simplicial object is one equipped with an augmentation $\varepsilon : X_0 \to Y$ (for $Y \in \mathcal{C}$), with $\varepsilon d_0 = \varepsilon d_1$.

The category of simplicial objects over $\mathcal{C}$ is denoted by $s\mathcal{C}$. We write $s_{\langle n\rangle}\mathcal{C}$ for the category n-*simplicial objects* over $\mathcal{C}$ (that is, objects of the form $\{X_k\}_{k=0}^{n}$, with the relevant face maps and degeneracies), and denote the truncation functor $s\mathcal{C} \to s_{\langle n\rangle}\mathcal{C}$ by τ_n.

For technical convenience in the next two sections we shall be working mainly in the category of simplicial groups, denoted by $\mathcal{G}$ (rather than $s\mathcal{G}p$); objects in $\mathcal{G}$ will be denoted by capital letters X, Y, and so on. A simplicial object $X_\bullet = (X_0, X_1, \ldots)$ in $s\mathcal{G}$ is thus a bisimplicial group, which has an *external* simplicial dimension (the n in $X_n \in \mathcal{G}$), as well as the *internal* simplicial dimension k (inside $\mathcal{G}$), which we shall denote by $(X_n)_k^{int}$, if necessary.

Simplicial sets and groups. The standard n-simplex in $\mathcal{S}$ is denoted by $\Delta[n]$, generated by $\sigma_n \in \Delta[n]_n$. $\dot{\Delta}[n]$ denotes the sub-object of $\Delta[n]$ generated by $d_i\sigma_n$ ($0 \leqslant i \leqslant n$). The simplicial n-sphere is

$$\boldsymbol{S}^n := \Delta[n]/\dot{\Delta}[n],$$

and the n-disk is

$$\boldsymbol{D}^n := C\boldsymbol{S}^{n-1}.$$

Let $F : \mathcal{S} \to \mathcal{G}$ denote the (dimensionwise) free group functor of [28, Section 2], and $G : \mathcal{S} \to \mathcal{G}$ be Kan's simplicial loop functor (cf. [26, Definition 26.3]), with $\overline{W} : \mathcal{G} \to \mathcal{S}$ the Eilenberg–MacLane classifying space functor (cf. [26, Section 21]). Recall that if $S : \mathcal{T} \to \mathcal{S}$ is the singular set functor and $\| - \| : \mathcal{S} \to \mathcal{T}$ the geometric realization functor (see [26, Section 1,14]), then the adjoint pairs of functors

$$\mathcal{T} \underset{\|-\|}{\overset{S}{\rightleftarrows}} \mathcal{S} \underset{\overline{W}}{\overset{G}{\rightleftarrows}} \mathcal{G} \tag{2.1}$$

induce isomorphisms of the corresponding homotopy categories (see [29, I, Section 5]), so that for the purposes of homotopy theory we can work in $\mathcal{G}$ rather than $\mathcal{T}$.

Definition 2.2. In particular, $\mathbb{S}^n := F\boldsymbol{S}^{n-1} \in \mathcal{G}$ for $n \geqslant 1$ (and $\mathbb{S}^0 := G\boldsymbol{S}^0$ for $n = 0$) will be called the *n-dimensional $\mathcal{G}$-sphere*, in as much as

$$[\mathbb{S}^n, G\boldsymbol{X}]_{\mathcal{G}} \cong \pi_n \boldsymbol{X} = [\boldsymbol{S}^n, \boldsymbol{X}]$$

for any Kan complex $\boldsymbol{X} \in \mathcal{S}$. Similarly, $\mathbb{D}^n := F\boldsymbol{D}^{n-1}$ will be called the *n-dimensional $\mathcal{G}$-disk*.

Definition 2.3. In any complete category $\mathcal{C}$, the *matching object* functor $M : \mathcal{S}^{op} \times s\mathcal{C} \to \mathcal{C}$, written $M_{\boldsymbol{A}} X_\bullet$ for a (finite) simplicial set $\boldsymbol{A} \in \mathcal{S}$ and $X_\bullet \in s\mathcal{C}$, is defined by requiring: (a) $M_{\Delta[n]} X_\bullet := X_n$, and (b) if $\boldsymbol{A} = \mathrm{colim}_i\, \boldsymbol{A}_i$, then $M_{\boldsymbol{A}} X_\bullet = \lim_i M_{\boldsymbol{A}_i} X_\bullet$ (see [15, Section 2.1]). In particular, if $\boldsymbol{A}_n^k$ is the subcomplex of $\dot{\Delta}[n]$ generated by the last $(n-k+1)$ faces $(d_k\sigma_n, \dots, d_n\sigma_n)$, we write $M_n^k X_\bullet$ for $M_{\boldsymbol{A}_n^k} X_\bullet$: explicitly, in $\mathcal{G}$ we have

$$M_n^k X_\bullet = \left\{(x_k, \dots, x_n) \in (X_{n-1})^{n+1} \mid d_i x_j = d_{j-1} x_i \text{ for all } k \leqslant i < j \leqslant n\right\} \tag{2.2}$$

and the map $\delta_n^k : X_n \to M_n^k X_\bullet$ induced by the inclusion $\boldsymbol{A}_n^k \hookrightarrow \Delta[n]$ is defined $\delta_n^k(x) = (d_k x, \dots, d_n x)$. The original matching object of [11, X, Section 4.5] was $M_n^0 X_\bullet = M_{\dot{\Delta}[n]} X_\bullet$, which we shall further abbreviate to $M_n X_\bullet$; each face map $d_k : X_{n+1} \to X_n$ factors through $\delta_n := \delta_n^0$. See also [20, XVII, 87.17].

Remark 2.4. Note that for $X \in \mathcal{G}$ and $\boldsymbol{A} \in \mathcal{S}$ we have $M_{\boldsymbol{A}} X \cong \mathrm{Hom}_{\mathcal{G}}(F\boldsymbol{A}, X) \in \mathcal{G}p$ (cf. Section 2), so for $X_\bullet \in s\mathcal{G}$ also

$$(M_{\boldsymbol{A}} X)_k \cong \mathrm{Hom}_{\mathcal{G}}\left(F\boldsymbol{A}, (X_\bullet)_k^{int}\right)$$

in each simplicial dimension k.

Definition 2.5. $X_\bullet \in s\mathcal{G}$ is called *fibrant* if each of the maps $\delta_n : X_n \to M_n X_\bullet$ $(n \geqslant 0)$ is a fibration in $\mathcal{G}$ (that is, a surjection onto the identity component—see [29, II, 3.8]). This is just the condition for fibrancy in the Reedy model category (see [31]), as well as in that of [14], but we shall not make explicit use of either.

By analogy with Moore's normalized chains (cf. [26, 17.3]) we have:

Definition 2.6. Given $X_\bullet \in s\mathcal{G}$, we define the *n-cycles* object of $X_\bullet$, written $Z_n X_\bullet$, to be the fiber of $\delta_n : X_n \to M_n X_\bullet$, so $Z_n X_\bullet = \{x \in X_n \mid d_i x = 0 \text{ for } i = 0, \dots, n\}$ (cf. [29, I, Section 2]). Of course, this definition really makes sense only when $X_\bullet$ is fibrant (Definition 2.5). Similarly, the *n-chains* object of $X_\bullet$, written $C_n X_\bullet$, is defined to be the fiber of $\delta_n^1 : X_n \to M_n^1 X_\bullet$.

If $X_\bullet \in s\mathcal{G}$ is fibrant, the map $d_0^n = d_0|_{C_n X_\bullet} : C_n X_\bullet \to Z_{n-1} X_\bullet$ is the pullback of $\delta_n : X_n \to M_n X_\bullet$ along the inclusion $\iota : Z_{n-1} X_\bullet \to M_n X_\bullet$ (where $\iota(z) = (z, 0, \dots, 0)$), so d_0^n is a fibration (in $\mathcal{G}$), fitting into a fibration sequence

$$Z_n X_\bullet \xrightarrow{j_n} C_n X_\bullet \xrightarrow{d_0^n} Z_{n-1} X_\bullet. \tag{2.3}$$

Proposition 2.7. *For any fibrant $X_\bullet \in s\mathcal{C}$, the inclusion $\iota : C_n X_\bullet \hookrightarrow X_n$ induces an isomorphism $\iota_\star : \pi_* C_n X_\bullet \cong C_n(\pi_* X_\bullet)$ for each $n \geqslant 0$.*

Proof. (a) First note that if $j : \boldsymbol{A} \hookrightarrow \boldsymbol{B}$ is a trivial cofibration in $\mathcal{S}$, then $j^* : M_{\boldsymbol{B}} X_\bullet \to M_{\boldsymbol{A}} X_\bullet$ has a natural section $r : M_{\boldsymbol{A}} X_\bullet \to M_{\boldsymbol{B}} X_\bullet$ (with $j^* \circ r = id$) for any $X_\bullet \in s\mathcal{G}$. This is because by Remark 2.4, $(M_{\boldsymbol{A}} X_\bullet)_k \cong \mathrm{Hom}_{\mathcal{G}}(F\boldsymbol{A}, (X_\bullet)_k^{int})$ for $\boldsymbol{A} \in \mathcal{S}$; since $F\boldsymbol{A}$

is fibrant in $\mathcal{G}$, we can choose a left inverse $\rho : F\boldsymbol{B} \to F\boldsymbol{A}$ for $Fj : F\boldsymbol{A} \hookrightarrow F\boldsymbol{B}$, so $j^* : (M_{\boldsymbol{B}} X_\bullet)_k^{int} \to (M_{\boldsymbol{A}} X_\bullet)_k^{int}$ has a right inverse ρ^*, which is natural in $(X_\bullet)_k^{int}$; so these maps ρ^* fit together to yield the required map r.

This need not be true in general if j is not a weak equivalence, as the example of $M_2^1 X_\bullet \to M_1^0 X_\bullet$ shows.

(b) Given $\eta \in C_n \pi_m X_\bullet$ represented by $h : \mathcal{S}^m \to X_n$ with $d_k h \sim 0$ ($1 \leqslant k \leqslant n$), consider the diagram:

$$\begin{tikzcd}
\mathcal{S}^m \arrow[dr, "h"] \arrow[ddrrr, "\sim 0"] \arrow[dddrr, "\sim 0"'] & & & \\
& X_n \arrow[dr, "\delta_n^k"] \arrow[drr, "\delta_n^{k+1}"] \arrow[ddr, "d_k"'] & & \\
& & M_n^k X_\bullet \arrow[r] \arrow[d, "\pi_k"'] \arrow[dr, phantom, "\boxed{\mathrm{PB}}"] & M_n^{k+1} X_\bullet \arrow[d, "j^*=(d_k,\ldots,d_k)"] \\
& & X_{n-1} \arrow[r, "\delta_{n-1}^k"] & M_{n-1}^k X_\bullet
\end{tikzcd}$$

in which j^* is a fibration by (a) if $k \geqslant 1$, so the lower left-hand square is in fact a homotopy pullback square (see [25, Section 1]). By descending induction on $1 \leqslant k \leqslant n-1$ (starting with $\delta_n^n = d_n$), we may assume $\delta_n^{k+1} \circ h : \mathcal{S}^m \to M_n^{k+1} X_\bullet$ is nullhomotopic in $\mathcal{C}$, as is $d_k \circ h$, so the induced pullback map $\delta_n^k \circ h : \mathcal{S}^m \to M_n^k X_\bullet$, is also nullhomotopic by the universal property. We conclude that $\delta_n^1 \circ h \sim 0$, and since $\delta_n^1 : X_n \to M_n^1 X_\bullet$ is a fibration by (a), we can choose $h : \mathcal{S}^m \to X_n$ so that $\delta_n^1 h = 0$. Thus h lifts to $C_n X_\bullet = \mathrm{Fib}(\delta_n^1)$, and $\iota_\star$ is surjective.

(c) Finally, the long exact sequence in homotopy for the fibration sequence

$$C_n X_\bullet \xrightarrow{\iota} X_n \xrightarrow{\delta_n^1} M_n^1 X_\bullet$$

implies that $\iota_\# : \pi_* C_n X_\bullet \to \pi_* X_n$ is monic, so $\iota_\star : \pi_* C_n X_\bullet \to C_n(\pi_* X_\bullet)$ is, too. $\square$

Definition 2.8. The dual construction to that of Definition 2.3 yields the colimit

$$L_n X_\bullet := \coprod_{0 \leqslant i \leqslant n-1} X_{n-1}/\sim,$$

where for any $x \in X_{n-2}$ and $0 \leqslant i \leqslant j \leqslant n-1$ we set $s_j x$ in the ith copy of X_{n-1} equivalent under $\sim$ to $s_i x$ in the $(j+1)$st copy of X_{n-1}. $L_n X_\bullet$ has sometimes been called the "nth latching object" of $X_\bullet$. The map $\sigma_n : L_n X_\bullet \to X_n$ is defined $\sigma_n x_{(i)} = s_i x$, where $x_{(i)}$ is in the ith copy of X_{n-1}.

3. Π-algebras and resolutions

In this section we recall some definitions and prove our main results on Π-algebras and resolutions:

Definition 3.1. A *Π-algebra* is a graded group $G_* = \{G_k\}_{k=1}^{\infty}$ (abelian in degrees > 1), together with an action on G_* of the primary homotopy operations (i.e., compositions and Whitehead products, including the "π_1-action" of G_1 on the higher G_n's, as in [38, X, Section 7]), satisfying the usual universal identities. See [3, Section 2.1] for a more explicit description. These are algebraic models of the homotopy groups $\pi_* \boldsymbol{X}$ of a space (or Kan complex) $\boldsymbol{X}$, in the same way that an algebra over the Steenrod algebra models its cohomology ring. The category of Π-algebras is denoted by Π-$\mathcal{A}lg$.

We say that a space (or Kan complex, or simplicial group) $\boldsymbol{X}$ *realizes* an (abstract) Π-algebra G_* if there is an isomorphism of Π-algebras $G_* \cong \pi_* \boldsymbol{X}$. (There may be non-homotopy equivalent spaces realizing the same Π-algebra—cf. [5, Section 7.18].) Similarly, an abstract morphism of Π-algebras $\phi : \pi_* \boldsymbol{X} \to \pi_* \boldsymbol{Y}$ (between realizable Π-algebras) is *realizable* if there is a map $f : \boldsymbol{X} \to \boldsymbol{Y}$ such that $\pi_* f = \phi$.

Definition 3.2. The *free Π-algebra generated* by a graded set $T_* = \{T_k\}_{k=1}^{\infty}$ is $\pi_* \boldsymbol{W}$, where

$$\boldsymbol{W} = \bigvee_{k=1}^{\infty} \bigvee_{\tau \in T_k} \boldsymbol{S}^k_{(\tau)}$$

(and we identify $\tau \in T_k$ with the generator of $\pi_k \boldsymbol{W}$ representing the inclusion $\boldsymbol{S}^k_{(\tau)} \hookrightarrow \boldsymbol{W}$).

If we let $\mathcal{F} \subset \Pi$-$\mathcal{A}lg$ denote the full subcategory of free Π-algebras, and Π the homotopy category of wedges of spheres (inside $ho\mathcal{T}_*$ or $ho\mathcal{S}_*$—or equivalently, the homotopy category of coproducts of $\mathcal{G}$-spheres in $ho\mathcal{G}$), then the functor $\pi_* : \Pi \to \mathcal{F}$ is an equivalence of categories. Thus any Π-algebra morphism $\varphi : G_* \to H_*$ is realizable (uniquely, up to homotopy), if G_* and H_* are free Π-algebras (actually, only G_* need be free).

Definition 3.3. Let $T : \Pi$-$\mathcal{A}lg \to \Pi$-$\mathcal{A}lg$ be the "free Π-algebra" comonad (cf. [24, VI, Section 1]), defined

$$TG_* = \coprod_{k=1}^{\infty} \coprod_{g \in G_k} \pi_* \boldsymbol{S}^k_{(g)}.$$

The counit

$$\varepsilon = \varepsilon_{G_*} : TG_* \twoheadrightarrow G_*$$

is defined by $\iota^k_{(g)} \mapsto g$ (where $\iota^k_{(g)}$ is the canonical generator of $\pi_* \boldsymbol{S}^k_{(g)}$), and the comultiplication $\vartheta = \vartheta_{G_*} : TG_* \hookrightarrow T^2 G_*$ is induced by the natural transformation $\bar{\vartheta} : id_{\mathcal{F}} \to T|_{\mathcal{F}}$ defined by $x_k \mapsto \iota^k_{(x_k)}$.

Definition 3.4. An *abelian* Π-algebra is one for which all Whitehead products vanish.

These are indeed the abelian objects of Π-$\mathcal{A}lg$—see [3, Section 2]. In particular, if $\boldsymbol{X}$ is an H-space, then $\pi_*\boldsymbol{X}$ is an abelian Π-algebra (cf. [38, X, (7.8)]).

Definition 3.5. A simplicial Π-algebra $A_\bullet$ is called *free* if for each $n \geqslant 0$ there is a graded set $T_*^n \subseteq A_n$ such that A_n is the free Π-algebra generated by T_*^n (Definition 3.2), and each degeneracy map $s_j : A_n \to A_{n+1}$ takes T_*^n to T_*^{n+1}.

A *free simplicial resolution* of a Π-algebra G_* is defined to be an augmented simplicial Π-algebra $A_\bullet \to G_*$, such that

(i) $A_\bullet$ is a free simplicial Π-algebra,

(ii) in each degree $k \geqslant 1$, the homotopy groups of the simplicial group $(A_\bullet)_k$ vanish in dimensions $n \geqslant 1$, and the augmentation induces an isomorphism $\pi_0(A_\bullet)_k \cong G_k$.

Such resolutions always exist, for any Π-algebra G_*—see [29, II, Section 4], or the explicit construction in [1, Section 4.3].

Definition 3.6. For any $X \in \mathcal{G}$, a simplical object $\boldsymbol{W}_\bullet \in s\mathcal{G}$ equipped with an augmentation $\varepsilon : W_0 \to X$ is called a *resolution of* X *by spheres* if each $\boldsymbol{W}_n$ is homotopy equivalent to a wedge of $\mathcal{G}$-spheres, and $\pi_*\boldsymbol{W}_\bullet \to \pi_*X$ is a free simplicial resolution of Π-algebras.

Example 3.7. One example of such a resolution by spheres is provided by Stover's construction; we shall need a variant in $\mathcal{G}$ (as in [7, Section 5]), rather than the original version of [36, Section 2], in $\mathcal{T}_*$. (The argument from this point on would actually work equally well in $\mathcal{T}_*$; but we have already chosen to work in $\mathcal{G}$, in order to facilitate the proof of Proposition 2.7.)

Define a comonad $V : \mathcal{G} \to \mathcal{G}$ for $G \in \mathcal{G}$ by

$$VG = \coprod_{k=0}^{\infty} \coprod_{\phi \in \mathrm{Hom}_{\mathcal{G}}(\mathcal{S}^k, G)} \mathcal{S}_\phi^k \cup \coprod_{k=0}^{\infty} \coprod_{\Phi \in \mathrm{Hom}_{\mathcal{G}}(\mathcal{D}^{k+1}, G)} \mathcal{D}_\Phi^{k+1}, \tag{3.1}$$

where $\mathcal{D}_\Phi^{k+1}$, the $\mathcal{G}$-disc indexed by $\Phi : \mathcal{D}^{k+1} \to G$, is attached to $\mathcal{S}_\phi^k$, the $\mathcal{G}$-sphere indexed by $\phi = \Phi|_{\partial\mathcal{D}^{k+1}}$, by identifying $\partial\mathcal{D}^{k+1} := F\partial\boldsymbol{D}^k$ with $\mathcal{S}^k$ (see Definition 2.2 above). The coproduct here is just the (dimensionwise) free product of groups; the counit $\varepsilon : VG \to G$ of the comonad V is "evaluation of indices", and the comultiplication $\vartheta : VG \hookrightarrow V^2G$ is as in Definition 3.3.

Now given $X \in \mathcal{G}$, define $Q_\bullet \in s\mathcal{G}$ by setting $Q_n = V^{n+1}X$, with face and degeneracy maps induced by the counit and comultiplication, respectively (cf. [17, App., Section 3]). The counit also induces an augmentation $\varepsilon : Q_\bullet \to X$; and this is in fact a resolution of X by spheres (see [36, Proposition 2.6]).

Remark 3.8. Note that we need not use the $\mathcal{G}$-sphere and disk $\mathcal{S}^k$ and $\mathcal{D}^k$ of Definition 2.2 in this construction; we can replace it by any other homotopy equivalent cofibrant pair of simplicial groups, so in particular by $(F\widehat{\boldsymbol{D}}^k, F\widehat{\boldsymbol{S}}^{k-1})$ for any pair of simplicial sets $(\widehat{\boldsymbol{D}}^k, \widehat{\boldsymbol{S}}^{k-1}) \simeq (\boldsymbol{D}^k, \boldsymbol{S}^{k-1})$.

The Quillen spectral sequence. A resolution by spheres $\boldsymbol{W}_\bullet \to X$ is in fact a resolution (i.e., cofibrant replacement) for the constant simplicial object $cX_\bullet \in s\mathcal{G}$ (i.e., $c(X)_n = X$, $d_i = s_j = id_X$) in an appropriate model category structure on $s\mathcal{G}$—see [14,9]. However, we shall not need this fact; for our purposes it suffices to recall that for any bisimplicial group $\boldsymbol{W}_\bullet \in s\mathcal{G}$, there is a first quadrant spectral sequence with

$$E^2_{s,t} = \pi_s(\pi_t \boldsymbol{W}_\bullet) \Rightarrow \pi_{s+t}\,\mathrm{diag}\,\boldsymbol{W}_\bullet \tag{3.2}$$

converging to the diagonal $\mathrm{diag}\,\boldsymbol{W}_\bullet \in \mathcal{G}$, defined $(\mathrm{diag}\,\boldsymbol{W}_\bullet)_k = (\boldsymbol{W}_k)^{int}_k$ (see [30]). Thus if $\boldsymbol{W}_\bullet \to X$ is a resolution by spheres, the spectral sequence collapses, and the natural map $\boldsymbol{W}_0 \to \mathrm{diag}\,\boldsymbol{W}_\bullet$ induces an isomorphism $\pi_*\boldsymbol{X} \cong \pi_*(\mathrm{diag}\,\boldsymbol{W}_\bullet)$. Combined with the fact that $\pi_*\boldsymbol{W}_\bullet$ is a resolution (in $s\Pi\text{-}\mathcal{A}lg$) of $\pi_*\boldsymbol{X}$, this simple result has many applications—see, for example, [1,13,36].

Definition 3.9. A *CW complex* over a pointed category $\mathcal{C}$ is a simplicial object $R_\bullet \in s\mathcal{C}$, together with a sequence of objects $\overline{R}_n$ $(n = 0, 1, \ldots)$ such that $R_n \cong \overline{R}_n \amalg L_n R_\bullet$ (Definition 2.3), and $d^n_i|_{\overline{R}_n} = 0$ for $1 \leqslant i \leqslant n$. The objects $(\overline{R}_n)^\infty_{n=0}$ are called a *CW basis* for $R_\bullet$, and $\bar{d}^n_0 := d_0|_{\overline{R}_n}$ is called the nth attaching map for $R_\bullet$.

One may then describe $R_\bullet$ explicitly in terms of its CW basis by

$$R_n \cong \coprod_{0\leqslant\lambda\leqslant n} \coprod_{I\in\mathfrak{I}_{\lambda,n}} \overline{R}_{n-\lambda}, \tag{3.3}$$

where $\mathfrak{I}_{\lambda,n}$ is the set of sequences I of λ non-negative integers $i_1 < i_2 < \cdots < i_\lambda$ $(< n)$, with $s_I = s_{i_\lambda} \circ \cdots \circ s_{i_0}$ the corresponding λ-fold degeneracy (if $\lambda = 0$, $s_I = id$). See [2, 5.2.1] and [26, p. 95(i)].

Such CW bases are convenient to work with in many situations; but they are most useful when each basis object is *free*, in an appropriate sense. In particular, if $\mathcal{C} = \Pi\text{-}\mathcal{A}lg$, we have the following

Definition 3.10. A *CW resolution* of a Π-algebra G_* is a CW complex $A_\bullet \in s\Pi\text{-}\mathcal{A}lg$, with CW basis $(\overline{A}_n)^\infty_{n=0}$ and attaching maps $\bar{d}^n_0 : \overline{A}_n \to Z_{n-1}A_\bullet$, such that each $\overline{A}_n$ is a free Π-algebra, and each attaching map $d^n_0|_{C_nA_\bullet}$ is onto $Z_{n-1}A_\bullet$ (for $n \geqslant 0$, where we let $\bar{d}^0_0$ denote the augmentation $\varepsilon : A_\bullet \to G_*$ and $Z_{-1}A_\bullet := G_*$). Compare [2, Section 5].

Every Π-algebra has a CW resolution (Definition 3.10), as was shown in [1, 4.4]: for example, one could take the graded set of generators $\overline{T}^n_*$ for $\overline{A}_n$ to be equal to the graded *set* $\pi_* Z_{n-1}A_\bullet$.

Definition 3.11. $Q_\bullet \in s\mathcal{G}$ is called a *CW resolution by spheres* of $X \in \mathcal{G}$ if $Q_\bullet \to X$ is a resolution by spheres (Definition 3.6), and $Q_\bullet$ is a CW complex with CW basis $(\overline{Q}_n)^\infty_{n=0}$), such that each $\overline{Q}_n \in \mathcal{F}$ (i.e., $\overline{Q}_n$ is homotopy equivalent to a wedge of spheres). The concept is defined analogously for $X \in \mathcal{S}$ or $X \in \mathcal{T}_*$.

Remark 3.12. Closely related to the problem of realizing abstract Π-algebras (Definition 3.1) is that of realizing a free simplicial Π-algebra $A_\bullet \in s\Pi\text{-}\mathcal{A}lg$: this is because, as noted in Definition 3.5, every $G_* \in \Pi\text{-}\mathcal{A}lg$ has a free simplicial resolution $A_\bullet \to G_*$; if it can be realized by a simplicial space $\boldsymbol{W}_\bullet \in s\mathcal{T}_c$—or equivalently, via (2.1), by a bisimplicial space or group—then the spectral sequence (3.2) implies that $\pi_* \operatorname{diag} \boldsymbol{W}_\bullet \cong G_*$. However, not every Π-algebra is realizable (see [5, Section 8] or [4, Proposition 4.3.6]).

It would nevertheless be very useful to know the converse: namely, that any free resolution of a *realizable* Π-algebra is itself realizable. This was mistakenly quoted as a theorem in [5, Section 6], where it was needed to make the obstruction theory for realizing Π-algebras described there of any practical use—and appeared as a conjecture in [6, Section 4], in the context of an obstruction theory for a space to be an H-space.

In order to show that this conjecture is in fact true, we need several preliminary results:

Proposition 3.13. *Every CW resolution $A_\bullet \to \pi_* X$ of a realizable Π-algebra embeds in $\pi_* Q_\bullet$ for some resolution by spheres $Q_\bullet \to X$.*

Proof. To simplify the notation, we work here with topological spaces, rather than simplicial groups, changing back to $\mathcal{G}$ if necessary via the adjoint pairs of Section 2.

Given a free simplicial Π-algebra resolution $A_\bullet \to J_*$ with CW basis $(\overline{A}_n)_{n=0}^{\infty}$, where $J_* = \pi_* \boldsymbol{X}$ for some $\boldsymbol{X} \in \mathcal{T}_*$, and $\overline{A}_n$ is the free Π-algebra generated by the graded set T_*^n, let μ denote the cardinality of $\coprod_{n=0}^{\infty} \coprod_{k=0}^{\infty} T_k^n$, and set

$$\boldsymbol{X}' := \boldsymbol{X} \vee \bigvee_{n=0}^{\infty} \bigvee_{\lambda<\mu} \boldsymbol{D}^n.$$

Define new “spheres” and “disks” of the form

$$\widehat{\boldsymbol{S}}^n := \boldsymbol{S}^n \vee \bigvee_{n=0}^{\infty} \bigvee_{\lambda<\mu} \boldsymbol{D}^n \quad \text{and} \quad \widehat{\boldsymbol{D}}^n := \widehat{\boldsymbol{S}}^n \vee \boldsymbol{D}^n.$$

(This is to ensure that there will be at least μ different representatives for each homotopy class in $\pi_* \boldsymbol{X}'$ or $\pi_* \widehat{\boldsymbol{S}}^n$.)

By Remark 3.8 above, if we use the construction of Example 3.7 in $\mathcal{T}_*$ (or in $\mathcal{G}$, *mutatis mutandis*) with these “spheres” and “disks”, and apply it to the space $\boldsymbol{X}'$, rather than to $\boldsymbol{X}$, we obtain a resolution by spheres $Q_\bullet \to \boldsymbol{X}'$.

We define $\phi : A_\bullet \hookrightarrow \pi_* Q_\bullet$ by induction on the simplicial dimension; it suffices to produce for each $n \geqslant 0$ an embedding $\bar{\phi}_n : \overline{A}_n \hookrightarrow C_n \pi_* Q_\bullet$ commuting with d_0. If we denote $\varepsilon^A : A_0 \to \pi_* \boldsymbol{X} \cong \pi_* \boldsymbol{X}'$ by $\bar{d}_0^0 : C_0 A_\bullet \to Z_{-1} A_\bullet =: A_{-1}$ and set $\phi_{-1} = id_{\pi_* X}$, then we may assume by induction we have a monomorphism $\phi_{n-1} : A_{n-1} \hookrightarrow \pi_* Q_{n-1}$ (taking generators to generators, and commuting with face and degeneracy maps).

For each Π-algebra generator ι_α in $(\overline{A}_n)_k$, if $d_0(\iota_\alpha) \neq 0$ then $\phi_{n-1}(d_0(\iota_\alpha)) \in Z_{n-1}\pi_k Q_\bullet$ is represented by some $g : \widehat{\boldsymbol{S}}^k \to Q_{n-1}$, and we can choose distinct (though perhaps homotopic) maps g for different generators ι_α by our choice of $\widehat{\boldsymbol{S}}^k$. Then by (3.1) there is a wedge summand $\widehat{\boldsymbol{S}}^k_g$ in $Q_n = VQ_{n-1}$ (with no disks attached), and the corresponding free Π-algebra coproduct summand $\pi_*\widehat{\boldsymbol{S}}^k_g$ in π_*Q_n, generated by ι_g, has $d_0(\iota_g) = [g] \in \pi_k Q_{n-1}$ and $d_i(\iota_g) = \iota_{d_{i-1}g} = 0 \in \pi_k Q_{n-1}$ for $1 \leqslant i \leqslant n$ by Example 3.7, since $[g] = \phi_{n-1}(d_0(\iota_\alpha)) \in Z_{n-1}\pi_k Q_\bullet$ and thus $d_i[g] = [d_i g] = 0$, and spheres indexed by nullhomotopic maps have disks attached to them. We see that $\iota_g \in C_n\pi_k Q_\bullet$, so we may define $\bar{\phi}_n(\iota_\alpha) = \iota_g$.

If $d_0(\iota_\alpha) = 0$, then all we need are enough distinct Π-algebra generators in $Z_n\pi_*Q_\bullet$: we cannot simply take ι_g for nullhomotopic $g : \boldsymbol{S}^k \to Q_{n-1}$, because of the attached disks; but we can proceed as follows:

Since

$$\widehat{\boldsymbol{D}}^k = C\widehat{\boldsymbol{S}}^k \vee \boldsymbol{D}^k \quad \text{and} \quad \boldsymbol{X}' = \boldsymbol{X} \vee \bigvee_{i=0}^{\infty} \bigvee_{\lambda<\mu} \boldsymbol{D}^i,$$

we have μ distinct nonzero maps

$$F_\lambda : \widehat{\boldsymbol{D}}^k \to \boldsymbol{X}' \quad \text{with} \quad F_\lambda|_{C\widehat{\boldsymbol{S}}^k} = *.$$

Define $H_+ = F_\lambda$, $H_- = *$; then

$$\boldsymbol{S}^k_H := \widehat{\boldsymbol{D}}^k_{H^+} \cup_{\widehat{\boldsymbol{S}}^{k-1}_*} \widehat{\boldsymbol{D}}^k_{H^-}$$

is, up to homotopy, a sphere wedge summand in Q_0, and thus $\iota_{H_\lambda} \in \pi_k Q_0$ is a Π-algebra generator mapping to 0 under the augmentation. Similalry, define

$$\boldsymbol{S}^k_{G_\lambda} := \widehat{\boldsymbol{D}}^k_{G^+} \cup_{\widehat{\boldsymbol{S}}^{k-1}_*} \widehat{\boldsymbol{D}}^k_{G^-}$$

in Q_1 by $G^+ = *$, $G_- = * \perp \iota^k$ where ι^k is a homoeomorphism onto the summand $\boldsymbol{D}^k$ in $\widehat{\boldsymbol{D}}^k_{H^-_\lambda}$. Then $G_\lambda \sim *$ and $G_\lambda \neq *$ but $H \circ G = *$; thus ι_{H_λ} is a Π-algebra generator in $Z_1\pi_k Q_\bullet$. By thus alternating the $+$ and $-$ we produce μ distinct Π-algebra generators in $Z_n\pi_*Q_\bullet$ for each n. $\square$

Remark 3.14. The referee has suggested an alternative proof of this proposition, which may be easier to follow: rather than “fattening” the spheres and disks, we can modify the Stover construction of (3.1) by using μ copies of each sphere or disk for each $\phi \in \mathrm{Hom}_{\mathcal{G}}(\mathbb{S}^k, G)$ or $\Phi \in \mathrm{Hom}_{\mathcal{G}}(\mathbb{D}^{k+1}, G)$, respectively. The proof of [36, Proposition 2.6] still goes through, and so does the argument for embedding $A_\bullet$ in $\pi_*Q_\bullet$ above.

Proposition 3.15. *Any free simplicial Π-algebra $A_\bullet$ has a (free) CW basis $(\overline{A}_n)_{n=0}^{\infty}$.*

Proof. Start with $\overline{A}_0 = A_0$. For $n \geqslant 1$, assume

$$A_n = \coprod_{k=0}^{\infty} \coprod_{\tau \in T^n_k} \pi_*\mathbb{S}^k.$$

By Definition 3.5,

$$T^n_* \cong \overline{T}^n_* \cup \bigcup_{0\leqslant\lambda\leqslant n} \bigcup_{I\in\mathfrak{I}_{\lambda,n}} \widehat{T}^{n-\lambda}_*$$

(as in Definition 3.3), so we can set

$$\widehat{A}_n = \coprod_{k=0}^{\infty} \coprod_{\tau\in\widehat{T}^n_k} \pi_*\mathcal{S}^k;$$

but $d_i|_{\widehat{A}_n}$ need not vanish for $i \geqslant 1$.

However, given $\tau \in \widehat{T}^n_k$, we may define $\tau_i \in (A_n)^{int}_k$ inductively, starting with $\tau_0 = \tau$, by $\tau_{i+1} = \tau_i s_{n-i-1} d_{n-i} \tau_i^{-1}$ (face and degeneracy maps taken in the external direction); we find that $\bar{\tau} := \tau_n$ is in $C_n A_\bullet$. If we define $\bar{\varphi}: \widehat{T}^n_* \to A_n$ by $\varphi(\tau) = \bar{\tau}$, by the universal property of free Π-algebras this extends to a map $\varphi: \widehat{A}_n \to A_n$, which together with the inclusion $\sigma_n: L_n A_\bullet \hookrightarrow A_n$ yields a map $\psi: A_n \to A_n$ which is an isomorphism by the Hurewicz Theorem (cf. [7, Lemma 2.5]). Thus we may set $\overline{A}_n := \varphi(\widehat{A}_n)$, that is, the free Π-algebra generated by $\{\bar{\tau}\}_{\tau\in\widehat{T}^n_*}$. Compare [23, Section 3]. $\square$

Theorem 3.16. *Every free simplicial Π-algebra resolution $A_\bullet \to \pi_* X$ of a realizable Π-algebra $\pi_* X$ is itself realizable by a CW resolution $R_\bullet \to X$ in $s\mathcal{G}$.*

Proof. By Propositions 3.13 and 3.15 we may assume $A_\bullet$ has a (free) CW basis $(\overline{A}_n)_{n=0}^{\infty}$, and that there is a resolution by spheres $Q_\bullet \to X$ (in $s\mathcal{G}$) and an embedding of simplicial Π-algebras $\phi: A_\bullet \to Q_\bullet$. We may also assume that $Q_\bullet$ is fibrant (Definition 2.5), with $\varepsilon^Q: Q_0 \to X$ a fibration. We shall actually realize ϕ by a map of bisimplicial groups $f: R_\bullet \to Q_\bullet$.

Note that once $R_\bullet$ has been defined through simplicial dimension n, for any $k \geqslant 0$ we have a commutative diagram

$$\begin{array}{ccccccccc}
\pi_k C_n R_\bullet & \xrightarrow{(d_0)_\#} & \pi_k Z_{n-1} R_\bullet & \xrightarrow{(j_{n-1})_\#} & \pi_k C_{n-1} R_\bullet & \xrightarrow{(d_0)_\#} & \pi_k Z_{n-2} R_\bullet & \xrightarrow{(j_{n-2})_\#} & \pi_k C_{n-2} R_\bullet \\
{\scriptstyle\iota_\star}\downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\rho_{n-1}} & & {\scriptstyle\iota_\star}\downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\rho_{n-2}} & & {\scriptstyle\iota_\star}\downarrow{\scriptstyle\cong} \\
C_n \pi_k R_\bullet & \xrightarrow{d_0^n} & Z_{n-1}\pi_k R_\bullet & \xrightarrow{\text{inc.}} & C_{n-1}\pi_k R_\bullet & \xrightarrow{d_0^{n-1}} & Z_{n-2}\pi_k R_\bullet & \xrightarrow{\text{inc.}} & C_{n-2}\pi_k R_\bullet
\end{array}$$

(obtained by fitting together three of the long exact sequences of the fibrations (2.3)). The vertical maps are induced by the inclusions $C_n R_\bullet \hookrightarrow R_n$, and so on—see Proposition 2.7.

The only difficulty in constructing $R_\bullet$ is that Proposition 2.7 does not hold for Z_n—i.e., the maps ρ_n in the above diagram in general need not be isomorphisms—so we may have an element in $Z_n A_\bullet$ represented by $\alpha \in C_n \pi_* R_\bullet = \pi_* C_n R_\bullet$ with $(d_0^n)_\#(\alpha) \neq 0$ (but of course $(j_{n-1})_\#(d_0^n)_\#(\alpha) = 0$). In this case we could not have $\beta \in \pi_* C_{n+1} R_\bullet = C_{n+1} A_\bullet$ with $(j_n)_\#(d_0^{n+1})_\#(\beta) = \alpha$, so $\pi_* R_\bullet$ would not be acyclic.

It is in order to avoid this difficulty that we need the embedding ϕ, since by definition this cannot happen for $Q_\bullet$: we know that $d_0^n: C_n \pi_* Q_\bullet \to Z_{n-1}\pi_* Q_\bullet$ is surjective for each $n > 0$, so $\rho_{n-1}: \pi_* Z_{n-1} Q_\bullet \to Z_{n-1}\pi_* Q_\bullet$ is, too, which implies that for each $n > 0$:

$$\operatorname{Im}\bigl\{(d_0^{n+1})_\#:\pi_*C_{n+1}Q_\bullet\to\pi_*Z_nQ_\bullet\bigr\}\cap\operatorname{Ker}\bigl\{(j_n)_\#:\pi_*Z_nQ_\bullet\to\pi_*C_nQ_\bullet\bigr\}=0 \qquad (3.4)$$

which we shall call *Property* (3.4) *for* $Z_nQ_\bullet$. (This implies in particular that $Z_n\pi_*Q_\bullet = \operatorname{Ker}\{(d_0^n)_\#:\pi_*C_nQ_\bullet\to Z_{n-1}Q_\bullet\}$.)

Note that given any fibrant $K_\bullet\in s\mathcal{G}$ having Property (3.4) for $Z_mK_\bullet$ for each $0<m\leqslant n$, if we consider the long exact sequence of the fibration $d_0^m:C_mK_\bullet\to Z_{m-1}K_\bullet$:

$$\cdots\pi_{k+1}C_mK_\bullet\xrightarrow{(d_0^m)_\#}\pi_{k+1}Z_{m-1}K_\bullet\xrightarrow{\partial^{m-1}}\pi_kZ_mK_\bullet\xrightarrow{(j_{m-1})_\#}\pi_kC_{m-1}K_\bullet\cdots, \qquad (3.5)$$

we may deduce that

$$\partial^m|_{\operatorname{Im}(\partial^{m-1})}\text{ is one-to-one, and surjects onto}\operatorname{Im}(\partial^m) \qquad (3.6)$$

for $0<m\leqslant n$.

We now construct $R_\bullet$ by induction on the simplicial dimension:

(i) First, choose a fibration $\varepsilon^R:R_0\to X$ realizing $\varepsilon^A:A_0\to\pi_*X$. By Definition 3.2, there is a map $f_0':R_0'\to Q_0$ realizing ϕ_0, so $\varepsilon^Q\circ f_0'\sim\varepsilon^R$; since ε^Q is a fibration, we can change f_0' to $f_0:R_0\to Q_0$ with $\varepsilon^Q\circ f_0'=\varepsilon^R$.

(ii) Let $Z_0R_\bullet$ denote the fiber of ε^R. Since $\varepsilon^R_\#=\varepsilon^A$ is a surjection, we have $\pi_*Z_0R_\bullet=\operatorname{Ker}(\varepsilon^R_\#)=Z_0A_\bullet$, and d_0^A maps $C_1A_\bullet$ onto $Z_0A_\bullet$, so $\bar d_0^A:\overline{A}_1\to A_0$ factors through $\pi_*Z_0R_\bullet$, and we can thus realize it by a map $\bar d_0^R:\overline{R}_1\to Z_0R_\bullet$. Set $R_1':=\overline{R}_1\amalg L_1R_\bullet$ (so $\pi_*R_1'\cong A_1$), with $\delta_1':R_1'\to M_1R_\bullet=R_0\times R_0$ equal to $(\bar d_0^R,0)\perp\varDelta$, and change δ_1' to a fibration $\delta_1:R_1\to M_1R_\bullet$. Again we can realize $\phi_1:A_1\to\pi_*Q_1$ by $f_1:R_1\to Q_1$ with $\delta_1^Q\circ f_1=f_0\circ\delta_1^R$, since δ_1^Q is a fibration; so we have defined $\tau_1f:\tau_1R_\bullet\to\tau_1Q_\bullet$ realizing $\tau_1\phi$.

(iii) Now assume we have $\tau_nf:\tau_nR_\bullet\to\tau_nQ_\bullet$ realizing $\tau_n\phi$, with Property (3.4) holding for $Z_mR_\bullet$ for $0<m<n$.
For each $\varPi$-algebra generator $\alpha\in\overline{A}_{n+1}$ (in degree k, say), (3.4) implies that

$$d_0^{n+1}(\alpha)\in\operatorname{Ker}(d_0^n)=\operatorname{Ker}\bigl((d_0^{R_n})_\#\bigr)\subset(C_nA_\bullet)_k=\pi_kC_nR_\bullet,$$

so by the exactness of (3.5) we can choose $\beta\in\pi_k\boldsymbol{Z}_nR_\bullet$ such that $(j_n)_\#\beta=d_0^{n+1}(\alpha)$. This allows us to define $\bar d_0^R:\overline{R}_{n+1}\to Z_nR_\bullet$ so that $(j_n)_\#(\bar d_0^R)_\#$ realizes $(\text{inc.})\circ\bar d_0^A:\overline{A}_{n+1}\to C_nA_\bullet$, as well as $\bar f_{n+1}:\overline{R}_{n+1}\to C_nQ_\bullet$ realizing $\phi_{n+1}|_{\overline{A}_{n+1}}$. Because $\overline{A}_{n+1}=\pi_*\overline{R}_{n+1}$ is a free $\varPi$-algebra, this implies the homotopy-commutativity of the outer rectangle in

$$\begin{tikzcd}
\overline{R}_{n+1} \arrow[r, "\bar f_{n+1}"] \arrow[d, "\bar d_0^R"'] & C_{n+1}Q_\bullet \arrow[d, "d_0^Q"] \\
Z_nR_\bullet \arrow[r, dotted, "Z_nf"] \arrow[d, "j_n^R"'] & Z_nQ_\bullet \arrow[d, "j_n^Q"] \\
C_nR_\bullet \arrow[r, "C_nf"] & C_nQ_\bullet
\end{tikzcd}$$

(as well as the lower square, by the induction hypothesis). Thus $j_n^Q \circ Z_n f \circ \bar{d}_0^R \sim j_n^Q \circ d_0^Q \circ \bar{f}_{n+1}$, so

$$\left(j_n^Q\right)_\# \circ (Z_n f)_\# \circ \left(\bar{d}_0^R\right)_\# = \left(j_n^Q\right)_\# \circ \left(d_0^Q\right)_\# \circ \left(\bar{f}_{n+1}\right)_\#.$$

By (3.4) this implies $(Z_n f)_\# \circ (\bar{d}_0^R)_\# = (d_0^Q)_\# \circ (\bar{f}_{n+1})_\#$, so (since $\pi_* \overline{R}_{n+1}$ is a free Π-algebra) also $Z_n f \circ \bar{d}_0^R \sim d_0^Q \circ \bar{f}_{n+1}$—which means that we can choose $\bar{f}_{n+1}$ so that $Z_n f \circ \bar{d}_0^R = d_0^Q \circ \bar{f}_{n+1}$ (since d_0^Q is a fibration). Thus if we set

$$\bar{\delta}_{n+1}^R : \overline{R}_{n+1} \to M_{n+1} R_\bullet$$

to be $(\bar{d}_0^R, 0, \ldots, 0)$, we have

$$M_{n+1} f \circ \bar{\delta}_{n+1}^R = \delta_{n+1}^Q \circ \bar{f}_{n+1}.$$

If $\psi_{n+1}^R := \delta_{n+1}^R \circ \sigma_{n+1}^R$ (in the notation of Definition 2.3 and 2.8) we set

$$R'_{n+1} := \overline{R}_{n+1} \amalg L_{n+1} R_\bullet,$$

and define

$$\delta'_{n+1} : R'_{n+1} \to M_{n+1} R_\bullet, \quad \text{and} \quad f'_{n+1} : R'_{n+1} \to Q_{n+1},$$

respectively by

$$\delta'_{n+1} := \left(\bar{\delta}_{n+1}^R \perp \psi_{n+1}^R\right) \quad \text{and} \quad f'_{n+1} := \left(\bar{f}_{n+1} \perp L_{n+1} f\right).$$

We see that $(f'_{n+1})_\# = \phi_{n+1}$ and $M_{n+1} f \circ \delta'_{n+1} = \delta_{n+1}^Q \circ f'_{n+1}$, and this will still hold if we change δ'_{n+1} into a fibration, and extend f'_{n+1} to $f_{n+1} : R_{n+1} \to Q_{n+1}$. This defines $\tau_{n+1} f : \tau_{n+1} R_\bullet \to \tau_{n+1} Q_\bullet$ realizing $\tau_{n+1}\phi$.

(iv) It remains to verify that $\tau_{n+1} R_\bullet$ so defined satisfies (3.4). However, (3.6) implies that we have a map of short exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Im}(\partial_R^{n-1}) & \xrightarrow{\text{inc.}} & \pi_k Z_n R_\bullet & \longrightarrow & \mathrm{Im}((j_n^R)_\#) \cong Z_n A_\bullet & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle (Z_n f)_\#} & & \downarrow{\scriptstyle Z_n \phi} & & \\
0 & \longrightarrow & \mathrm{Im}(\partial_Q^{n-1}) & \xrightarrow{\text{inc.}} & \pi_k Z_n Q_\bullet & \longrightarrow & \mathrm{Im}((j_n^Q)_\#) \cong Z_n \pi_k Q_\bullet & \longrightarrow & 0
\end{array}$$

in which the left vertical map is an isomorphism and the right map is one-to-one, so $(Z_n f)_\#$ is one-to-one, too. Therefore,

$$\mathrm{Ker}\big((j_n^R)_\#\big) = \mathrm{Ker}\big((j_n^R)_\#\big) \cap \pi_* Z_n R_\bullet,$$

which implies that Property (3.4) holds for $Z_n R_\bullet$, too.

This completes the inductive construction of $R_\bullet$. □

We also have an analogous result for maps:

Theorem 3.17. *If $K_\bullet \xrightarrow{\varepsilon^K} \pi_* X$ and $L_\bullet \xrightarrow{\varepsilon^L} \pi_* Y$ are two free simplicial Π-algebra resolutions, $g: X \to Y$ is a map in $\mathcal{G}$, and $\varphi: K_\bullet \to L_\bullet$ is a morphism of simplicial Π-algebras such that $\varepsilon^L \circ \varphi_0 = \pi_* g \circ \varepsilon^K$, then φ is realizable by a map $f: A_\bullet \to B_\bullet$ in $s\mathcal{G}$.*

Proof. Choose free CW bases for $K_\bullet$ and $L_\bullet$, and realize the resulting CW resolutions by $A_\bullet$ and $B_\bullet$, respectively, where (as in the proof of Theorem 3.16) we may assume

$$d_0: C_n B_\bullet \to Z_{n-1} B_\bullet$$

is a fibration for each $n \geqslant 0$. $f_n: A_n \to B_n$ will be defined by induction on n: $\varphi_0: K_0 \to L_0$ may be realized by a map $f_0': A_0 \to B_0$ (Definition 3.2), and since ε^B is a fibration and $\varepsilon^B \circ f_0' \sim g \circ \varepsilon^A$, we can choose a realization f_0 for φ_0 such that $\varepsilon^B \circ f_0 = g \circ \varepsilon^A$.

In general, $\bar{\varphi}_n = \varphi_n|_{\overline{K}_n}: \overline{K}_n \to C_n L_\bullet$ may be realized by a map $\bar{f}_n: \overline{A}_n \to C_n B_\bullet$ (Proposition 2.7), and since $d_0: C_n B_\bullet \to Z_{n-1} B_\bullet$ is a fibration, we may choose $\bar{f}_n$ so $d_0 \circ \bar{f}_n = Z_{n-1} f \circ d_0: \overline{A}_n \to Z_{n-1} B_\bullet$. By induction this yields a map

$$f_n = L_n f \perp \bar{f}_n : A_n = L_n A_\bullet \amalg \overline{A}_n \to L_n B_\bullet \amalg \overline{B}_n = B_n$$

such that

$$\delta_n^B \circ f_n = M_n f \circ \delta_n^A : A_n \to M_n B_\bullet,$$

so f is indeed a simplicial morphism (realizing ϕ). $\square$

4. The simplicial bar construction

As an application of Theorem 3.16, we describe an obstruction theory for determining whether a given space X is, up to homotopy, a loop space (and thus a topological group—see [27, Section 3]). In the next two sections we no longer need to work with simplicial groups, so we revert to the more familiar category of topological spaces; we can still utilize the results of the previous section via the adjoint pairs of (2.1).

Definition 4.1. A *Δ-cosimplicial object $E_\Delta^\bullet$* over a category $\mathcal{C}$ is a sequence of objects $E^0, E^1, \ldots,$ together with *coface maps* $d^i: E^n \to E^{n+1}$ for $1 \leqslant 1 \leqslant n$ satisfying $d^j d^i = d^i d^{j-1}$ for $i < j$ (cf. [32]). Given an ordinary cosimplicial object $E^\bullet$ (cf. [11, X, 2.1]), we let $E_\Delta^\bullet$ denote the underlying Δ-cosimplicial object (obtained by forgetting the codegeneracies).

The cosimplicial James construction. Given a space $\boldsymbol{X} \in \mathcal{T}_*$, we define a Δ-cosimplicial space $\boldsymbol{U}_\Delta^\bullet = U(\boldsymbol{X})_\Delta^\bullet$ by setting $\boldsymbol{U}^n = \boldsymbol{X}^{n+1}$ (the Cartesian product), and $d^i(x_0, \ldots, x_n) = (x_0, \ldots, x_{i-1}, *, x_i, \ldots, x_n)$. Note that $\operatorname{colim} \boldsymbol{U}(X)_\Delta^\bullet \cong J\boldsymbol{X}$ (the James reduced product construction), and

Fact 4.2. *If $\langle \boldsymbol{X}, m\rangle$ is a (strictly) associative H-space, we can extend $\boldsymbol{U}_\Delta^\bullet$ to a full cosimplicial space $\boldsymbol{U}^\bullet$ by setting $s^j(x_0, \ldots, x_n) = (x_0, \ldots, m(x_j, x_{j-1}), \ldots, x_n)$.*

Definition 4.3. Let $A_\bullet$ be a CW resolution of the Π-algebra $\pi_* \boldsymbol{X} = \pi_* \boldsymbol{U}^0$, as in Definition 3.10. We construct a Δ-cosimplicial augmented simplicial Π-algebra $(E_\bullet)^\bullet_\Delta \to \pi_* \boldsymbol{U}^\bullet_\Delta$, such that each $E^n_\bullet$ is a CW resolution of $\pi_* \boldsymbol{U}^n = \pi_*(\boldsymbol{X}^{n+1})$, with CW basis $\{\overline{E}^n_r\}^\infty_{r=0}$. We start by setting $\overline{E}^0_r = \overline{C}^0_r = \overline{A}_r$ for all $r \geqslant 0$, and then define $\overline{E}^n_r$ by a double induction (on $r \geqslant 0$ and then on $n \geqslant 0$) as

$$\overline{E}^n_r = \coprod_{0 \leqslant \lambda \leqslant n} \coprod_{I \in \mathfrak{I}_{\lambda,n}} [\overline{C}^{n-\lambda}_r]_I, \tag{4.1}$$

where $\mathfrak{I}_{\lambda,n}$ is as in (3.3) and $\overline{C}^m_0 = 0 = \overline{C}^0_r$ for all $m, r \geqslant 0$.

The coface maps $d^i : E^{n-1}_r \to E^n_r$ are determined by the cosimplicial identities and the requirement that $d^i|_{[\overline{C}^{n-\lambda}_r]_{(i_1,\dots,i_n)}}$ be an isomorphism onto $[\overline{C}^{n-\lambda}_r]_{(i_1,\dots,i_n,i)}$ if $i > i_n$.

The only summand in (4.1) which is not defined is thus $[\overline{C}^n_r]_\emptyset$, which we denote simply by $\overline{C}^n_r$. We require that it be an nth *cross-term* in the sense that $\bar{d}_0|_{\overline{C}^n_r}$ does not factor through the image of any coface map $d^i : E^{n-1}_{r-1} \to E^n_{r-1}$. Other than that, $\overline{C}^n_r$ may be any free Π-algebra which ensures that (4.1) defines a CW basis for a CW resolution $E^n_\bullet \to \pi_* \boldsymbol{U}^n$. We shall call the double sequence $((\overline{C}^n_r)^\infty_{n=1})^\infty_{r=1}$ a *cross-term basis* for $(E_\bullet)^\bullet_\Delta$.

Note that $A_\bullet$ is a retract of $E^2_\bullet$ in two different ways (under the two coface maps d^0, d^1), corresponding to the fact that $\boldsymbol{X}$ is a retract of $\boldsymbol{X} \times \boldsymbol{X}$ in two different ways; the presence of the cross-terms $\overline{C}^2_r$ indicates that $A_\bullet \times A_\bullet$ is a resolution of $\pi_* \boldsymbol{X}^2$, but not a free one, while $A_\bullet \amalg A_\bullet$ is a free simplicial Π-algebra, but not a resolution.

Similarly, $\boldsymbol{X} \times \boldsymbol{X}$ embeds in $\boldsymbol{X}^3$ in three different ways, and so on.

Example 4.4. For any $A_\bullet \to \pi_* \boldsymbol{X}$ we may set

$$\overline{C}^2_1 = \coprod_{S^p_x \hookrightarrow A^{(0)}_0} \coprod_{S^q_y \hookrightarrow A^{(1)}_0} S^{p+q-1}_{(x,y)},$$

with $\bar{d}_0|_{S^{p+q-1}_{(x,y)}} = [\iota_x, \iota_y]$ (in the notation of Definition 3.3). The higher cross-terms $\overline{C}^n_1 = 0$ for $n \geqslant 3$, since any kth order cross-term element z in $\coprod^n_{j=0} A^{(j)}_0$ ($k \geqslant 3$) is a sum of elements of the form

$$z = \zeta^\# \big[\dots \big[[\iota^{r_1}_{(x_1)}, \iota^{r_2}_{(x_2)}], \iota^{r_3}_{(x_3)} \big], \dots, \iota^{r_k}_{(x_k)} \big],$$

and then

$$z = d_0 \big(\zeta^\# \big[\dots \big[\iota^{r_1+r_2-1}_{(x_1,x_2)}, s_0 \iota^{r_3}_{(x_3)} \big], \dots, s_0 \iota^{r_k}_{(x_k)} \big] \big).$$

Definition 4.5. Let ${}^h(\boldsymbol{W}_\bullet)^\bullet_\Delta \to \boldsymbol{U}^\bullet_\Delta$ be the Δ-cosimplicial augmented simplicial space up-to-homotopy which corresponds to $(E_\bullet)^\bullet_\Delta \to \pi_* \boldsymbol{U}^\bullet_\Delta$ via Definition 3.2. Thus the various (co)simplicial morphisms exist, and satisfy the (co)simplicial identities, only in the homotopy category (we may choose representatives in $\mathcal{T}_*$, but then the identities are satisfied only up to homotopy). Each $\boldsymbol{W}^n_r$ is homotopy equivalent to a wedge of spheres, and has a wedge summand $\overline{\boldsymbol{W}}^n_r \hookrightarrow \boldsymbol{W}^n_r$ corresponding to the CW basis free Π-algebra summand $\overline{E}^n_r \hookrightarrow E^n_r$. We let $\overline{C}^n_r$ denote the wedge summand of $\overline{\boldsymbol{W}}^n_r$ corresponding to $\overline{C}^n_r \hookrightarrow \overline{E}^n_r$.

Definition 4.6. An simplicial space $\boldsymbol{V}_\bullet \in s\mathcal{T}_*$ is called a *rectification* of a simplicial space up-to-homotopy ${}^h\boldsymbol{W}_\bullet$ if $\boldsymbol{V}_n \simeq \boldsymbol{W}_n$ for each $n \geqslant 0$, and the face and degeneracy maps of $\boldsymbol{V}_\bullet$ are homotopic to the corresponding maps of ${}^h\boldsymbol{W}_\bullet$. See [12, Section 2.2], e.g., for a more precise definition; for our purposes all we require is that $\pi_*\boldsymbol{V}_\bullet$ be isomorphic (as a simplicial Π-algebra) to $\pi_*({}^h\boldsymbol{W}_\bullet)$. Similarly for rectification of (Δ-)cosimplicial objects, and so on.

By considering the proof of Theorem 3.16, we see that we can make the following

Assumption 4.7. $(E_\bullet)^\bullet_\Delta$ maps monomorphically into $\pi_*\boldsymbol{V}_\bullet(U^\bullet_\Delta)$, and ${}^h(\boldsymbol{W}_\bullet)^\bullet_\Delta \to \boldsymbol{U}^\bullet_\Delta$ can be rectified so as to yield a strict Δ-cosimplicial augmented simplicial space $(\boldsymbol{W}_\bullet)^\bullet_\Delta \to \boldsymbol{U}^\bullet_\Delta$ realizing $(E_\bullet)^\bullet_\Delta \to \pi_*\boldsymbol{U}^\bullet_\Delta$.

Definition 4.8. Now assume that $\pi_*\boldsymbol{X}$ is an abelian Π-algebra (Definition 3.4)—this is the necessary Π-algebra condition in order for $\boldsymbol{X}$ to be an H-space—and let

$$\mu : \pi_*\boldsymbol{X} \times \pi_*\boldsymbol{X} \to \pi_*\boldsymbol{X}$$

be the morphism of Π-algebras defined levelwise by the group operation (see [6, Section 2]). This μ is of course associative, in the sense that

$$\mu \circ (\mu, id) = \mu \circ (id, \mu) : \pi_*(\boldsymbol{X}^3) \to \pi_*\boldsymbol{X},$$

so it allows one to extend the Δ-cosimplicial Π-algebra $F^\bullet_\Delta := \pi_*(\boldsymbol{U}^\bullet_\Delta)$ to a full cosimplicial Π-algebra $F^\bullet$, defined as in Fact 4.2.

Since $E^n_\bullet \to F^n = \pi_*\boldsymbol{U}^n$ is a free resolution of Π-algebras, the codegeneracy maps $s^j : F^n \to F^{n-1}$ induce maps of simplicial Π-algebras $s^j_\bullet : E^n_\bullet \to E^{n-1}_\bullet$, unique up to simplicial homotopy, by the universal property of resolutions (cf. [29, I, p. 1.14]; [II, Section 2, Proposition 5]). Note, however, that the individual maps $s^j_r : E^n_r \to E^{n-1}_r$ are not unique, in general; in fact, different choices may correspond to different H-multiplications on $\boldsymbol{X}$.

These maps s^j make $(E_\bullet)^\bullet_\Delta \to F^\bullet_\Delta$ into a full cosimplicial augmented simplicial Π-algebra $E^\bullet_\bullet \to F^\bullet$, and thus ${}^h\boldsymbol{W}^\bullet_\bullet \to \boldsymbol{U}^\bullet_\Delta$ into a cosimplicial augmented simplicial space up-to-homotopy (for which we may assume by Assumption 4.7 that all simplicial identities, and all the cosimplicial identities involving only the coface maps, hold precisely).

Proposition 4.9. *The cosimplicial simplicial space up-to-homotopy ${}^h\boldsymbol{W}^\bullet_\bullet$ of Definition* 4.8 *may be rectified if and only if $\boldsymbol{X}$ is homotopy equivalent to a loop space.*

Proof. If $\boldsymbol{X}$ is a loop space, it has a strictly associative H-multiplication $m : \boldsymbol{X} \times \boldsymbol{X} \to \boldsymbol{X}$ which induces μ on $\pi_*(-)$ (cf. [18, Proposition 9.9]), so $\boldsymbol{U}^\bullet_\Delta$ extends to a cosimplicial space $\boldsymbol{U}^\bullet$ by Fact 4.2. Applying the functorial construction of [36, Section 2] to $\boldsymbol{U}^\bullet$ yields a (strict) cosimplicial augmented simplicial space $(\boldsymbol{V}_\bullet)^\bullet_\Delta \to \boldsymbol{U}^\bullet$, and since we assumed $\pi_*\boldsymbol{W}^n_\bullet$ embeds in $\pi_*\boldsymbol{V}^n_\bullet$ for each n, ${}^h\boldsymbol{W}^\bullet_\bullet$ may also be rectified.

Conversely, if $\boldsymbol{W}^\bullet_\bullet$ is a (strict) cosimplicial simplicial space realizing $E^\bullet_\bullet$, then we may apply the realization functor for simplicial spaces in each cosimplicial dimension $n \geqslant 0$ to obtain $\|\boldsymbol{W}^n_\bullet\| \simeq \boldsymbol{U}^n = \boldsymbol{X}^{n+1}$ (by Section 3). The realization of the codegeneracy map

$$\|s^0\| : \|\boldsymbol{W}^1_\bullet\| \to \|\boldsymbol{W}^0_\bullet\|$$

induces

$$\mu : \pi_*(\boldsymbol{X}^2) \to \pi_*\boldsymbol{X},$$

so it corresponds to an H-space multiplication $m : \boldsymbol{X}^2 \to \boldsymbol{X}$ (see [6, Proposition 2.7]).

The fact that $\|\boldsymbol{W}^\bullet_\bullet\|$ is a (strict) cosimplicial space means that all composite codegeneracy maps

$$\|s^0 \circ s^{j_1} \circ \cdots s^{j_{n-1}}\| : \|\boldsymbol{W}^n_\bullet\| \to \|\boldsymbol{W}^0_\bullet\|$$

are equal, and thus all possible composite multiplications $\boldsymbol{X}^{n+1} \to \boldsymbol{X}$ (i.e., all possible bracketings in (2.2)) are homotopic, with homotopies between the homotopies, and so on—in other words, the H-space $\langle \boldsymbol{X}, m\rangle$ is an A_∞ space (see [35, Definition 11.2])—so that $\boldsymbol{X}$ is homotopy equivalent to loop space by [35, Theorem 11.4]. Note that we only required that the codegeneracies of ${}^h\boldsymbol{W}^\bullet_\bullet$ be rectified; after the fact this ensures that the full cosimplicial simplicial space is rectifiable. □

In summary, the question of whether $\boldsymbol{X}$ is a loop space reduces to the question of whether a certain diagram in the homotopy category, corresponding to a diagram of free Π-algebras, may be rectified—or equivalently, may be made ∞-homotopy commutative.

5. Polyhedra and higher homotopy operations

As in [5, Section 4], there is a sequence of higher homotopy operations which serve as obstructions to such a rectification, and these may be described combinatorially in terms of certain polyhedra, as follows:

Definition 5.1. The *N-permutohedron* $\boldsymbol{P}^N$ is defined to be the convex hull in $\mathbb{R}^N$ of the points $p_\sigma = (\sigma(1), \sigma(2), \ldots, \sigma(N))$, where σ ranges over all permutations $\sigma \in \Sigma_N$ (cf. [39, Section 9]). It is $(N-1)$-dimensional.

For any two integers $0 \leqslant n < N$, the corresponding *(N,n)-face-codegeneracy polyhedron* $\boldsymbol{P}^N_n$ is a quotient of the N-permutohedron $\boldsymbol{P}^N$ obtained by identifying two vertices p_σ and $p_{\sigma'}$ to a single vertex $\bar{p}_\sigma = \bar{p}_{\sigma'}$ of $\boldsymbol{P}^N_n$ whenever $\sigma = (i, i+1)\sigma'$, where $(i, i+1)$ is an adjacent transposition and $\sigma(i), \sigma(i+1) > n$.

Since each facet A of $\boldsymbol{P}^N$ is uniquely determined by its vertices (see below), the facets in the quotient $\boldsymbol{P}^N_n$ are obtained by collapsing those of $\boldsymbol{P}^N$ accordingly.

Note that $\boldsymbol{P}^N_{N-1}$ is the N-permutohedron $\boldsymbol{P}^N$, and in fact the quotient map $q:\boldsymbol{P}^N \twoheadrightarrow \boldsymbol{P}^N_n$ is homotopic to a homeomorphism (though not a combinatorial isomorphism, of course) for $n \geqslant 1$. On the other hand, $\boldsymbol{P}^N_0$ is a single point. For non-trivial examples of face-codegeneracy polyhedra, see Figs 1 and 2 below.

Fact 5.2. *From the description of the facets of the permutohedron given in* [16], *we see that $\boldsymbol{P}^N_n$ has an edge connecting a vertex p_σ to any vertex of the form $p_{(i,i+1)\sigma}$ (unless $\sigma(i), \sigma(i+1) > n$, in which case the edge is degenerate).*

More generally, let $\bar{p}_\sigma$ be any vertex of $\boldsymbol{P}^N_n$. The facets of $\boldsymbol{P}^N_n$ containing $\bar{p}_\sigma$ are determined as follows:

Let

$$\mathbb{P} = \langle 1, 2, \ldots, \ell_1 \mid \ell_1+1, \ldots, \ell_2 \mid \cdots \mid \ell_{i-1}+1, \ldots, \ell_i \mid \cdots \mid \ell_{r-1}+1, \ldots, N \rangle$$

be a partition of $1, \ldots, N$ into r consecutive blocs, subject to the condition that for each $1 \leqslant j < r$ at least one of $\sigma(\ell_i)$, $\sigma(\ell_{i+1})$ is $\leqslant n$. Denote by n_i the number of j's in the ith bloc (i.e., $\ell_{i-1}+1 \leqslant j \leqslant \ell_i$) such that $\sigma(j) \leqslant n$. Then $\boldsymbol{P}^N_n$ will have a subpolyhedron $Q(\mathbb{P})$ (containing p_σ) which is isomorphic to the product

$$\boldsymbol{P}^{\ell_1}_{n_1} \times \boldsymbol{P}^{\ell_2-\ell_1}_{n_2} \times \cdots \times \boldsymbol{P}^{\ell_i-\ell_{i-1}}_{n_i} \times \cdots \times \boldsymbol{P}^{N-\ell_{r-1}}_{n_r}.$$

This follows from the description of the facets of the N-permutohedron in [5, Section 4.3].

We denote by $(\boldsymbol{P}^N_n)^{(k)}$ the union of all facets of $\boldsymbol{P}^N_n$ of dimension $\leqslant k$. In particular, for $n \geqslant 1$ we have $\partial \boldsymbol{P}^N_n := (\boldsymbol{P}^N_n)^{(N-2)} = \boldsymbol{S}^{N-2}$, since the homeomorphism $\tilde{q}:\boldsymbol{P}^N \to \boldsymbol{P}^N_n$ preserves $\partial \boldsymbol{P}^N$.

Factorizations. Given a cosimplicial simplicial object $E^\bullet_\bullet$ as in Definition 4.8, any composite face-codegeneracy map $\psi: E^{n+k}_{m+\ell} \to E^k_\ell$ has a (unique) canonical factorization of the form $\psi = \phi \circ \theta$, where $\theta: E^{n+k}_{m+\ell} \to E^k_{m+\ell}$ may be written $\theta = s^{j_1} \circ s^{j_2} \circ \cdots s^{j_n}$ for $0 \leqslant j_1 < j_2 < \cdots < j_n < n+k$ and $\phi: E^k_{m+\ell} \to E^k_\ell$ may be written $\phi = d_{i_1} \circ d_{i_2} \circ \cdots d_{i_n}$ for $0 \leqslant i_1 < i_2 < \cdots < i_n \leqslant m+\ell$.

Let $\mathcal{D}(\psi)$ denote the set of all possible (not necessarily canonical) factorizations of ψ as a composite of face and codegeneracy maps: $\psi = \lambda_{n+m} \circ \cdots \circ \lambda_1$. We define recursively a bijective correspondence between $\mathcal{D}(\psi)$ and the vertices of an $(n+m)$-permutohedron $\boldsymbol{P}^{n+m}$, as follows (compare [5, Lemma 4.7]):

The canonical factorization $\psi = d_{i_1} \circ d_{i_2} \circ \cdots d_{i_n} \circ s^{j_1} \circ s^{j_2} \circ \cdots s^{j_n}$ corresponds to the vertex p_{id}. Next, assume that the factorization $\psi = \lambda_{n+m} \circ \cdots \circ \lambda_1$ corresponds to p_σ. Then the factorization corresponding to $p_{\sigma'}$, for $\sigma = (i, i+1)\sigma'$, is obtained from $\psi = \lambda_1 \circ \cdots \circ \lambda_{n+m}$ by switching λ_i and λ_{i+1}, using the identity $s^j \circ s^i = s^{i-1} \circ s^j$ for $i > j$ if λ_i and λ_{i+1} are both codegeneracies, and the identity $d_i \circ d_j = d_{j-1} \circ d_i$ for $i < j$ if they are both face maps.

Passing to the quotient face-codegeneracy polyhedron, we see that the vertices of $\boldsymbol{P}^{n+m}_n$ are now identified with factorizations of ψ of the form

$$\begin{aligned} &E^{n+k}_{m+\ell} \xrightarrow{s^{j^t_{n_t}}} E^{n+k-1}_{m+\ell} \cdots E^{n_t+1}_{m+\ell} \xrightarrow{s^{j^t_1}} E^{n_t}_{m+\ell} \xrightarrow{\theta_t} E^{n_t}_{m_t} \cdots E^{n_1}_{m_1} \\ &\qquad \xrightarrow{s^{j^0_{n_1}}} \cdots E^{n+1}_{m_1} \xrightarrow{s^{j^0_{n_0}}} E^n_{m_1} \xrightarrow{\theta_0} E^n_m, \end{aligned} \tag{5.1}$$

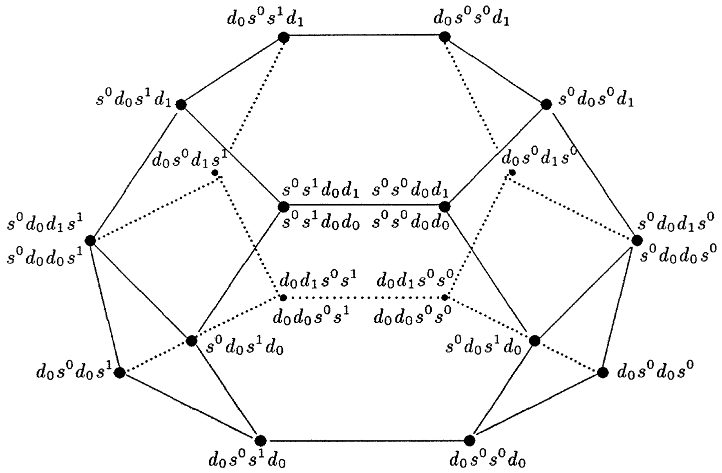

Fig. 1. The face-codegeneracy polyhedron $\boldsymbol{P}_2^4(d_0d_1s^0s^1)$.

where θ_i is a composite of face maps (i.e., we do not distinguish the different ways of decomposing θ_i as $d_{k_1} \circ \cdots d_{k_r}$). The collection of such factorizations of ψ will be denoted by $D(\psi)/{\sim}$, where $\sim$ is the obvious equivalence relation on $D(\psi)$. We shall denote the face-codegeneracy polyhedron $\boldsymbol{P}_n^{n+m}$ with its vertices so labelled by $\boldsymbol{P}_n^{n+m}(\psi)$. An example for $\psi = d_0d_1s^0s^1$ appears in Fig. 1.

Notation. For $\psi : E_{m+\ell}^{n+k} \to E_\ell^k$ as above, we denote by $\mathcal{C}(\psi)$ the collection of all composite face-codegeneracy maps

$$\rho : E_{m(\rho)+\ell(\rho)}^{n(\rho)+k(\rho)} \to E_{\ell(\rho)}^{k(\rho)}$$

such that ρ is of the form $\rho = \xi_t \circ \cdots \circ \xi_s$ $(1 \leqslant s \leqslant t \leqslant \nu)$ for some decomposition $\psi = \xi_\nu \circ \cdots \circ \xi_1 = \theta_0 \circ s^{j_{n_0}^0} \circ \cdots \circ s^{j_{n_1}^0} \circ \theta_1 \circ \cdots \circ \theta_t \circ s^{j_1^t} \circ \cdots \circ s^{j_{n_t}^t}$ of (5.1). That is, we allow only those subsequences $\lambda_b, \ldots, \lambda_a$ of a factorization $\psi = \lambda_{n+m} \circ \cdots \circ \lambda_1$ in $\mathcal{D}(\psi)$ which are compatible with the equivalence relation $\sim$ in the sense that λ_{b+1} and λ_b are not both face maps, and similarly for λ_{a-1} and λ_a. Such a ρ will be called *allowable*.

Higher homotopy operations. Given a cosimplicial simplicial space up-to-homotopy ${}^h\boldsymbol{W}^\bullet_\bullet$ as in Section 4, we now define a certain sequence of higher homotopy operations. First recall that the *half-smash* of two spaces $\boldsymbol{X}, \boldsymbol{Y} \in \mathcal{T}_*$ is

$$\boldsymbol{X} \ltimes \boldsymbol{Y} := (\boldsymbol{X} \times \boldsymbol{Y})/\big(\boldsymbol{X} \times \{*\}\big);$$

if $\boldsymbol{X}$ is a suspension, there is a (non-canonical) homotopy equivalence $\boldsymbol{X} \ltimes \boldsymbol{Y} \simeq \boldsymbol{X} \wedge \boldsymbol{Y} \vee \boldsymbol{X}$.

Definition 5.3. Given a composite face-codegeneracy map $\psi : \boldsymbol{W}_{m+\ell}^{n+k} \to \boldsymbol{W}_\ell^k$ as above, a *compatible collection for* $\mathcal{C}(\psi)$ and ${}^h\boldsymbol{W}^\bullet_\bullet$ is a set $\{g^\rho\}_{\rho \in \mathcal{C}(\psi)}$ of maps

$$g^\rho : \boldsymbol{P}_{m(\rho)}^{n(\rho)+m(\rho)}(\rho) \ltimes \boldsymbol{W}_{m(\rho)+\ell(\rho)}^{n(\rho)+k(\rho)} \to \boldsymbol{W}_{\ell(\rho)}^{k(\rho)}$$

for each $\rho \in \mathcal{C}(\psi)$, satisfying the following condition:

Assume that for such a $\rho \in \mathcal{C}(\psi)$ we have some decomposition

$$\rho = \xi_\nu \circ \cdots \circ \xi_1 = \theta_0 \circ s^{j^0_{n_0}} \circ \cdots \circ s^{j^0_{n_1}} \circ \theta_1 \circ \cdots \circ \theta_t \circ s^{j^t_1} \circ \cdots \circ s^{j^t_{n_t}}$$

in $\mathcal{D}(\rho)/\sim$, as in (5.1), and let

$$\mathbb{P} = \langle 1, \ldots, \ell_1 \mid \cdots \mid \ell_{i-1}+1, \ldots, \ell_i \mid \cdots \mid \ell_{r-1}+1, \ldots, \nu \rangle$$

be a partition of $(1, \ldots, \nu)$ as in Fact 5.2, yielding a sequence of composite face-codegeneracy maps $\rho_i \in \mathcal{C}(\rho) \subseteq \mathcal{C}(\psi)$ for $i = 1, \ldots, r$.

Let

$$Q(\mathbb{P}) \cong \boldsymbol{P}^{\ell_1}_{n_1}(\rho_1) \times \cdots \times \boldsymbol{P}^{\ell_i-\ell_{i-1}}_{n_i}(\rho_i) \times \cdots \times \boldsymbol{P}^{\nu-\ell_{r-1}}_{n_r}(\rho_r)$$

be the corresponding sub-polyhedron of $\boldsymbol{P}^{n(\rho)+m(\rho)}_{m(\rho)}(\rho)$. Then we require that

$$g^\rho|_{Q(\mathbb{P}) \ltimes \boldsymbol{W}^{n(\rho)+k(\rho)}_{m(\rho)+\ell(\rho)}}$$

be the composite of the corresponding maps g^{ρ_i} in the sense that

$$g^\rho(x_1, \ldots, x_r, w) = g^{\rho_1}\big(x_1, g^{\rho_2}\big(x_2, \ldots, g^{\rho_r}(x_r, w)\ldots\big)\big) \tag{5.2}$$

for $x_i \in \boldsymbol{P}^{\ell_i-\ell_{i-1}}_{n_i}(\rho_i)$ and $w \in \boldsymbol{W}^{n(\rho)+k(\rho)}_{m(\rho)+\ell(\rho)}$.

We further require that if $\rho = \lambda_1$ is of length 1, then g^ρ must be in the prescribed homotopy class of the face or codegeneracy map λ_1. Thus in particular, for each vertex $\bar{p}_\sigma$ of $\boldsymbol{P}^{n+m}_n(\psi)$, indexed by a factorization $\psi = \xi_\nu \circ \cdots \circ \xi_1$ in $\mathcal{D}(\psi)/\sim$, the map $g^\rho|_{\{\bar{p}_\sigma\} \times \boldsymbol{W}^{n+\ell}_{m+k}}$ represents the class $[\xi_\nu \circ \cdots \circ \xi_1]$.

Fact 5.4. *Any compatible collection of maps $\{g^\rho\}_{\rho \in \mathcal{C}(\psi)}$ for $C(\psi)$ induces a map*

$$f = f^\psi : \partial \boldsymbol{P}^{n+m}_n \ltimes \boldsymbol{W}^{n+k}_{m+\ell} \to \boldsymbol{W}^k_\ell$$

(*since all the facets of $\partial \boldsymbol{P}^{n+m}_n$ are products of face-codegeneracy polyhedra of the form $\boldsymbol{P}^{n(\rho)+m(\rho)}_{n(\rho)}(\rho)$ for $\rho \in \mathcal{C}(\psi)$, and condition* (5.2) *guarantees that the maps g^ρ agree on intersections*).

Definition 5.5. Given ${}^h\boldsymbol{W}_\bullet^\bullet$ as in Definition 4.8, for each $k \geqslant 2$ and each composite face-codegeneracy map $\psi : \boldsymbol{W}^{n+k}_{m+\ell} \to \boldsymbol{W}^k_\ell$, the *kth order homotopy operation* associated to ${}^h\boldsymbol{W}_\bullet^\bullet$ and ψ is a subset $\langle\psi\rangle$ of the track group $[\Sigma^{n+m-2}\boldsymbol{W}^{n+k}_{m+\ell}, \boldsymbol{W}^k_\ell]$, defined as follows:

Let $S \subseteq [\partial \boldsymbol{P}^{n+m}_n \ltimes \boldsymbol{W}^{n+k}_{m+\ell}, \ \boldsymbol{W}^k_\ell]$ be the set of homotopy classes of maps

$$f = f^\psi : \partial \boldsymbol{P}^{n+m}_n \ltimes \boldsymbol{W}^{n+k}_{m+\ell} \to \boldsymbol{W}^k_\ell$$

which are induced as above by some compatible collection $\{g^\rho\}_{\rho \in \mathcal{C}(\psi)}$ for $\mathcal{C}(\psi)$.

Now choose a splitting

$$\partial \boldsymbol{P}^{n+m}_n(\psi) \ltimes \boldsymbol{W}^{n+k}_{m+\ell} \cong \boldsymbol{S}^{n+m-2} \ltimes \boldsymbol{W}^{n+k}_{m+\ell} \simeq (\boldsymbol{S}^{n+m-2} \wedge \boldsymbol{W}^k_\ell) \vee \boldsymbol{W}^k_\ell \tag{5.3}$$

and let $\langle\psi\rangle \subseteq [\Sigma^{n+m-2}\boldsymbol{W}^{n+k}_{m+\ell}, \boldsymbol{W}^k_\ell]$ be the image of the subset S under the resulting projection.

It is clearly a necessary condition in order for the subset $\langle\psi\rangle$ to be non-empty that all the lower order operations $\langle\rho\rangle$ *vanish* (i.e., contain the null class) for all $\rho \in \mathcal{C}(\psi) \setminus \{\psi\}$—because otherwise the various maps

$$g^{\rho} : \boldsymbol{P}^{n(\rho)+m(\rho)}_{m(\rho)}(\rho) \ltimes \boldsymbol{W}^{n(\rho)+k(\rho)}_{m(\rho)+\ell(\rho)} \to \boldsymbol{W}^{k(\rho)}_{\ell(\rho)}$$

cannot even extend over the interior of $\boldsymbol{P}^{n(\rho)+m(\rho)}_{m(\rho)}(\rho)$. A *sufficient* condition is that the operations $\langle\rho\rangle$ vanish *coherently*, in the sense that the choices of compatible collections for the various ρ be consistent on common subpolyhedra (see [5, Section 5.7] for the precise definition, and [5, Section 5.9] for the obstructions to coherence).

On the other hand, if ${}^{h}\boldsymbol{W}^{\bullet}_{\bullet}$ is the cosimplicial simplicial space up-to-homotopy of Definition 4.3 (corresponding to the cosimplicial simplicial Π-algebra $(E_{\bullet})^{\bullet}_{\Delta}$ with the CW basis $\{\overline{E}^{n}_{r}\}^{\infty}_{r,n=0}$), then the vanishing of the homotopy operation $\langle\psi|_{\overline{\mathcal{C}}^{n}_{r}}\rangle$—with ψ restricted to the (n,r)-cross-term—implies the vanishing of $\langle\psi\rangle$, for any $\psi : \boldsymbol{W}^{n+k}_{m+\ell} \to \boldsymbol{W}^{k}_{\ell}$ (assuming lower order vanishing). This is because outside of the wedge summand $\overline{\mathcal{C}}^{n}_{r}$, the map ψ is determined by the maps $\rho \in \mathcal{C}(\psi)$ and the coface and degeneracy maps of ${}^{h}\boldsymbol{W}^{\bullet}_{\bullet}$, which we may assume to ∞-homotopy commute by induction and Assumption 4.7, respectively.

We may thus sum up the results of this section, combined with Proposition 4.9, in:

Theorem 5.6. *A space $\boldsymbol{X} \in \mathcal{T}_{*}$, for which $\pi_{*}\boldsymbol{X}$ is an abelian Π-algebra, is homotopy equivalent to a loop space if and only if all the higher homotopy operations $\langle\psi|_{\overline{\mathcal{C}}^{n}_{r}}\rangle$ defined above vanish coherently.*

Remark 5.7. As observed in Section 4, for any $\boldsymbol{X} \in \mathcal{T}_{*}$ the space $J\boldsymbol{X}$ is the colimit of the Δ-cosimplicial space $\boldsymbol{U}(X)^{\bullet}_{\Delta}$, and in fact the nth stage of the James construction, $J_{n}\boldsymbol{X}$, is the (homotopy) colimit of the $(n-1)$-coskeleton of $\boldsymbol{U}^{\bullet}_{\Delta}$. Thus if we think of the sequence of higher homotopy operations "in the simplicial direction" as obstructions to the validity of the identity [7, Theorem 5.7(∗)] (up to ∞-homotopy commutativity), then the nth cosimplicial dimension corresponds to verifying this identity for $f \circ i_{A} : \boldsymbol{A} \to F\boldsymbol{B}$ of James filtration $n+1$ (cf. [22, Section 2]).

In particular, if we fix $k = \ell = 0$, $n = 1$ and proceed by induction on m, we are computing the obstructions for the existence of an H-multiplication on $\boldsymbol{X}$, as in [6]. (Thus if $\boldsymbol{X}$ is endowed with an H-space structure to begin with, they must all vanish.) Observe that the face-codegeneracy polyhedron $\boldsymbol{P}^{n}_{1}$ is an $(n-1)$-cube, as in Fig. 2, rather than the $(n-1)$-simplex we had in [6, Section 4]—so the homotopy operations we obtain here are more complicated. This is because they take value in the homotopy groups of spheres, rather than those of the space $\boldsymbol{X}$.

As a corollary to Theorem 5.6 we may deduce the following result of Hilton (cf. [19, Theorem C]):

Corollary 5.8. *If $\langle\boldsymbol{X}, m\rangle$ is a $(p-1)$-connected H-space with $\pi_{i}\boldsymbol{X} = 0$ for $i \geqslant 3p$, then $\boldsymbol{X}$ is a loop space, up to homotopy.*

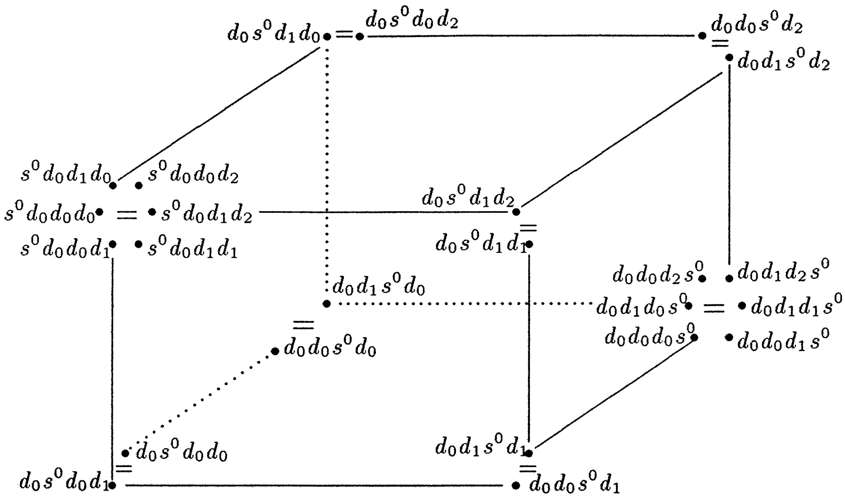

Fig. 2. The face-codegeneracy polyhedron $\boldsymbol{P}_1^4(d_0d_1d_2s^0)$.

Proof. Choose a CW resolution of $\pi_*\boldsymbol{X}$ which is $(p-1)$-connected in each simplicial dimension, and let $E_\bullet^\bullet$ be as in Definition 4.3. By definition of the cross-term Π-algebras C_r^n in Definition 4.3, they must involve Whitehead products of elements from *all* lower order cross-terms; but since $\boldsymbol{X}$ is an H-space by assumption, all obstructions of the form $\langle\psi|_{\overline{C}_r^1}\rangle$ vanish (see Remark 5.7). Thus, the lowest-dimensional obstruction possible is a third-order operation $\langle\psi|_{\overline{C}_r^2}\rangle$ $(r \geqslant 2)$, which involves a triple Whitehead product and thus takes value in $\pi_i\boldsymbol{W}_\ell^k$ for $i \geqslant 3p$. If we apply the $(3p-1)$-Postnikov approximation functor to ${}^h\boldsymbol{W}_\bullet^\bullet$ in each dimension, to obtain ${}^h\boldsymbol{Z}_\bullet^\bullet$, all obstructions to rectification vanish, and from the spectral sequence of Section 3 we see that the obvious map $\boldsymbol{X} = \|\boldsymbol{W}_\bullet^1\| \to \|\boldsymbol{Z}_\bullet^1\|$ induces an isomorphism in π_i for $i < 3p$. Since $\|\boldsymbol{Z}_\bullet^1\|$ is a loop space by Theorem 5.6, so is its $(3p-1)$-Postnikov approximation, namely $\boldsymbol{X}$. $\square$

Example 5.9. The 7-sphere is an H-space (under the Cayley multiplication, for example), but none of the 120 possible H-multiplications on $\boldsymbol{S}^7$ are homotopy-associative; the first obstruction to homotopy-associativity is a certain “separation element” in $\pi_{21}\boldsymbol{S}^7$ (cf. [21, Theorem 1.4 and Corollary 2.5]).

Since $\pi_*\boldsymbol{S}^7$ is a free Π-algebra, it has a very simple CW resolution $A_\bullet \to \pi_*\boldsymbol{S}^7$, with $\overline{A}_0 \cong \pi_*\boldsymbol{S}^7$ (generated by ι^7), and $\overline{A}_r = 0$ for $r \geqslant 1$. A cross-term basis (Definition 4.3) for the cosimplicial simplicial Π-algebra $E_\bullet^\bullet$ of Definition 4.8 is then given in dimensions < 24 by:

- $\overline{C}_1^1 \cong \pi_*\boldsymbol{S}^{13}$, with $\bar{d}_0\iota^{13} = [d^0\iota^7, d^1\iota^7]$;
- $\overline{C}_2^2 \cong \pi_*\boldsymbol{S}^{19}$, with $\bar{d}_0\iota^{19} = [d^0\iota^{13}, s_0d^2d^1\iota^7] - [d^1\iota^{13}, s_0d^2d^0\iota^7] + [d^2\iota^{13}, s_0d^1d^0\iota^7]$;
- $\overline{C}_r^n$ is at least 24-connected for all other n, r.

We set $s_r^j|_{\overline{C}_r^n} = 0$ for all $n \leqslant 2$; this determines $E_\bullet^\bullet$ in degrees $\leqslant 21$ and cosimplicial dimensions $\leqslant 2$.

By Remark 5.7, the two secondary operations $\langle d_0 s^0|_{\overline{C}_1^1}\rangle$ and $\langle d_1 s^0|_{\overline{C}_1^1}\rangle$ must vanish; on the other hand, by Corollary 5.8 all obstructions to $\boldsymbol{S}^7$ being a loop space are in degrees $\geqslant 21$, so the only relevant cross-term is $\overline{C}_2^2$, with three possible third-order operations $\langle\psi|_{\overline{C}_2^2}\rangle$, for $\psi = d_0 d_1 s^0 s^1$, $d_0 d_2 s^0 s^1$, or $d_1 d_2 s^0 s^1$. The corresponding face-codegeneracy polyhedra $P_2^4(\psi)$ is as in Fig. 2.

It is straightforward to verify that the operations $\langle\psi|_{\overline{C}_2^2}\rangle$ are trivial for $\psi = d_0 d_2 s^0 s^1$ or $d_1 d_2 s^0 s^1$ (in fact, many of the maps g^ρ, for $\rho \in C(\psi)$, may be chosen to be null). On may also show that there is a compatible collection $\{g^\rho\}_{\rho\in C(\varphi)}$ for $\varphi = d_0 d_1 s^0 s^1$, in the sense of Definition 5.3, so that the corresponding subset $\langle\varphi|_{\overline{C}_2^2}\rangle \subseteq \pi_{21}\boldsymbol{S}^7$ is non-empty; in fact, it contains the only possible obstruction to the 21-Postnikov approximation for $\boldsymbol{S}^7$ to be a loop space.

The existence of the tertiary operation $\langle\varphi|_{\overline{C}_2^2}\rangle$ corresponds to the fact that the element $[[\iota^7,\iota^7],\iota^7] - [[\iota^7,\iota^7],\iota^7] + [[\iota^7,\iota^7],\iota^7] \in \pi_{21}\boldsymbol{S}^7$ is trivial "for three different reasons": because of the Jacobi identity, because all Whitehead products vanish in $\pi_*\boldsymbol{S}^7$, and because of the linearity of the Whitehead product—i.e., $[0,\alpha] = 0$.

On the other hand, we know that there *is* a 3-primary obstruction to the homotopy-associativity of any H-multiplication on $\boldsymbol{S}^7$, namely the element $\sigma_{14}^{\#}\tau_7 \in \pi_{21}\boldsymbol{S}^7$ (see [21, Theorem 2.6]). We deduce that $0 \notin \langle\varphi|_{\overline{C}_2^2}\rangle$, and in fact (modulo 3) this tertiary operation consists exactly of the elements $\pm\sigma_{14}^{\#}\tau_7$.

For a detailed calculation of such higher order operations using simplicial resolutions of Π-algebras, see [6, Section 4.13].

Remark 5.10. Our approach to the question of whether $\boldsymbol{X}$ is a loop space is clearly based on, and closely related to, the classical approaches of Sugawara and Stasheff (cf. [33,34, 37]. One might wonder why Stasheff's associahedra K_i (cf. [33, Section 2,6]) do not show up among the face-codegeneracy polyhedra we describe above. Apparently this is because we do not work directly with the space $\boldsymbol{X}$, but rather with its simplicial resolution, which may be thought of as a "decomposition" of $\boldsymbol{X}$ into wedges of spheres.

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