Further Remarks on Nonlinear $P$-Compact Operators in Banach Space

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1. INTRODUCTION

In recent papers [1, 2] the author, using rather elementary arguments, derived a number of results concerning the existence and the construction of solutions and of fixed points for a class of nonlinear equations involving bounded projectionally-compact ($P$-compact) operators defined on a real Banach or Hilbert space with a basis. These results extended and simplified similar results for quasicompact operators obtained by Kaniel [3]. At the same time we deduced from our theorems certain basic results on bounded monotone operators obtained previously by Minty [4], Browder [5, 6] and others.

The purpose of this paper is two-fold: First, we extend our main results derived in [1, 2] to $P$-compact operators without assuming their boundedness; second, using our extended results, we generalize to our class of operators the recent results of Granas [7, 8] concerning the solvability of nonlinear equations and the geometrical intersection theorem involving quasibounded completely continuous operators. At the same time we obtain some new results for $P$-compact and monotone operators in Hilbert space.

2. EXTENDED RESULTS FOR $P$-COMPACT OPERATORS

Let $X$ be a real Banach space with the property that there exists a sequence $\{X_n\}$ of finite dimensional subspaces $X_n$ of $X$, a sequence of linear projections $\{P_n\}$ defined on $X$, and a constant $K > 0$ such that

\[ P_nX = X_n, \quad X_n \subset X_{n+1}, \quad n = 1, 2, 3, \cdots, \quad \bigcup_{n}^{\infty} X_n = X, \quad (1) \]

\[ \|P_n\| \leq K, \quad n = 1, 2, 3, \cdots. \quad (2) \]

For the precise definitions of the concepts and the statements of the results mentioned in the Introduction, see the succeeding sections of this paper.
Let $B_r$ denote the closed ball in $X$ of radius $r > 0$ about the origin and let $S_r$ denote the boundary of $B_r$. Let the symbols "−" and "→" denote the strong and weak convergence in $X$, respectively. In what follows we consider the class of $P$-compact operators which were studied by the author in [8, 9] and which are defined as follows:

**Definition 1.** A nonlinear operator $A$ mapping $X$ into itself is called projectionally-compact ($P$-compact) if $P_n A$ is continuous in $X_n$ for all sufficiently large $n$ and if for any constant $p > 0$ and any bounded sequence $(x_n)$ with $x_n \in X_n$ the strong convergence of the sequence $(\xi_n) = (P_n Ax_n - px_n)$ implies the existence of a strongly convergent subsequence $(x_{n_i})$ and an element $x$ in $X$ such that $x_{n_i} \to x$ and $P_{n_i} Ax_{n_i} \to Ax$.

The results of this paper will be based essentially on the following theorem which for bounded $P$-compact operators was proved by the author in [1, 2].

**Theorem 1.** Suppose that $A$ is $P$-compact. Suppose further that for given $r > 0$ and $\mu > 0$ the operator $A$ satisfies both of the following conditions:

$(A)$: There exists a number $c(r) > 0$ such that if, for any $n$, $P_n Ax = \lambda x$ holds for $x$ in $S_r$ with $\lambda > 0$, then $\lambda \leq c(r)$.

$(\Pi_\mu)$: If for some $x$ in $S_r$, the equation $Ax = \alpha x$ holds then $\alpha < \mu$.

Then there exists at least one element $u$ in $(B_r - S_r)$ such that

$$Au - \mu u = 0. \tag{3}$$

**Proof.** The proof Theorem 1 follows the same line of argument as the proof of the corresponding Theorem 2 in [2]. As in [2], all we need is to show that the present conditions $(A)$-$(\Pi_\mu)$ imply the validity of the following lemma.

**Lemma 1.** If $A$ satisfies the conditions of Theorem 1, then there exists an integer $n_0 > 0$ such that if $n \geq n_0$ and $P_n Ax = \beta x$ for some $x$ in $S_r \cap X_n$ then $\beta < \mu$.

**Proof of Lemma 1.** If the assertion of Lemma 1 were not true for any $n_0$, we could find a sequence $(x_n)$ with $x_n \in X_n \cap S_r$ and a sequence of numbers $(\beta_n)$ such that

$$P_n Ax_n = \beta_n x_n, \quad (\beta_n \geq \mu). \tag{4}$$

Hence our condition $(A)$ implies that

$$\|P_n Ax_n\| = \beta_n r \leq c(r) r,$$
i.e., $\beta_n \in [\mu, c(r)]$ for each $n$. Passing to a subsequence, we may assume that $\beta_n \to \beta$ and $\beta \in [\mu, c(r)]$. This and (4) show that
\[ P_n A x_n - \beta x_n = (\beta_n - \beta) x_n \to 0, \quad (n \to \infty). \tag{5} \]
Since $A$ is $P$-compact, (5) implies the existence of a strongly convergent subsequence, which we again denote by $\{x_n\}$, and an element $x$ in $S_r \cap X$ such that $x_n \to x$ and $P_n A x_n \to A x$. This and (5) imply that $A x - \beta x = 0$ for $x$ in $S_r$ and $\beta \geq \mu$ in contradiction to condition (II$_\mu$) of Theorem 1.

The proof of Theorem 1 is then completed in exactly the same way as in [2].

**Remark 1.** It is obvious that if in Definition 1 we require $\rho < 0$ instead of $\rho > 0$, then we obtain a theorem analogous to Theorem 1. We need only consider $-A$ instead of $A$ and assume that for given $r > 0$ and $\mu < 0$ instead conditions (A) and (II$_\mu$) the operator $A$ satisfied the conditions

(A$^-$): There exists a number $c(r) > 0$ such that if, for any $n$, $P_n(Ax) = \lambda x$ holds for $x$ in $S_r$ with $\lambda < 0$, then $-c(r) \leq \lambda$.

(II$_\mu^-$) If for some $x$ in $S_r$ the equation $A x = \alpha x$ holds then $\mu < \alpha$.

**Remark 2.** Let us remark that condition (A) is in no way a condition on the size of $A x$ or even on the size of $P_n A x$. All it says is that when for any $x$ in $S_r$ and any $n$ the vector $P_n A x$ is in the same direction as $x$ then $P_n A x$ are uniformly bounded.

**Remark 3.** The assertion of Theorem 1 remains valid if condition (A) is replaced, for example, by any one of the following stronger conditions whose degree of generality increases in the given order:

(a) $A$ is bounded, i.e., $A$ maps bounded sets in $X$ into bounded sets.

(b) For any given $r > 0$ the set $A(S_r)$ is bounded.

(c) $X$ is a Hilbert space $H$ and, for any given $r > 0$, $(A x, x) \leq c \| x \|^2$ for all $x$ in $S_r$ and some $c > 0$.

**Corollary 1.** If $A$ is $P$-compact and for some $r > 0$ the conditions (A) and (II$_\mu$) are satisfied on $S_r$, then $A$ has a fixed point in $(B_r - S_r)$.

**Theorem 2.** Suppose that $A$ is $P$-compact. Suppose further that there exists a sequence of spheres $\{S_p\}$ with $r_p \to \infty$, as $p \to \infty$, and two sequences of positive numbers $c_p = c(r_p)$ and $k_p = k(r_p)$ with $k_p \to \infty$, as $r_p \to \infty$, such that the following conditions hold:

(A$_\lambda$): Whenever for any given $f$ in $B_k$ and any $n$ the equation $P_n A x - \lambda x = P_n f$ holds for $x$ in $S_{r_p}$ with $\lambda > 0$ then $\lambda \leq c_p$. 

(17): \[ \| Ax - \eta x \| \geq k_p \text{ for any } \eta \geq \mu > 0 \text{ and any } x \in S_{r_p}. \]

Then for every \( f \) in \( X \) there exists an element \( u \) in \( X \) such that

\[ Au - \mu u = f. \] (6)

**Proof.** For any given \( f \) in \( X \) choose \( r_p \) so large that \( \| f \| < k_p \). If we define \( Cx = Ax - f \), then obviously \( C \) is \( P \)-compact. Furthermore, on \( S_{r_p} \) the operator \( C \) satisfies conditions (\( A \)) and (\( II_p \)) of Theorem 1. Indeed, if \( P_n Cx = \lambda x \) for any \( x \) in \( S_{r_p} \) with \( \lambda > 0 \), then, by the definition of \( C, f \in B_{k_p} \) and \( P_n Ax - \lambda x = P_n f \). Hence, by (\( A_f \)), \( \lambda \leq c_p \), i.e., (\( A \)) is fulfilled. Suppose now that \( Cx = \alpha x \) for some \( x \) in \( S_{r_p} \). Then \( \alpha < \mu \) since, in virtue of (\( II_p \)), the assumption that \( \alpha \geq \mu \) would lead to the contradiction

\[ \| Cx - \alpha x \| = \| Ax - \alpha x - f \| \geq \| Ax - \alpha x \| - \| f \| \geq k_p - \| f \| > 0. \]

Hence, by Theorem 1, there exists an element \( u \) in \( (B_{r_p} - S_{r_p}) \) such that \( Cu = \mu u \), i.e., \( u \) is a solution of Eq. (6).

In case \( X \) is a Hilbert space we have the following interesting theorem.

**Theorem 3.** If \( A \) is a \( P \)-compact mapping of \( H \) into itself such that

\[ (Ax, x) \leq (A(0), x), \quad x \in H, \] (7)

then for any given \( \mu > 0 \) the operator \( (A - \mu I) \) is onto.

**Proof.** Let \( \{S_{r_p}\} \) be a sequence of spheres in \( H \) with

\[ r_p = p + \frac{\| A(0) \|}{\mu} \]

and let \( \{c_p\} \) and \( \{k_p\} \) be two sequences given by \( c_p = \mu \) and

\[ k_p = \mu p = \mu r_p - \| A(0) \|. \]

For any given \( f \) in \( H \) choose \( r_p \) so large that \( \| f \| < k_p \). If we put \( Cx = Ax - f \), then (7) implies that for all \( x \) in \( H \)

\[ (Cx, x) \leq (C(0), x). \] (8)

It is not hard to show that, in virtue of (8), the above defined sequences \( \{c_p\} \)

and \( \{k_p\} \) satisfy conditions (\( A_f \)) and (\( II_p \)) of Theorem 2. In fact, suppose that for any \( f \) in \( B_{k_p} \) and any \( n \) the equation \( P_n Ax - \lambda x = P_n f \) holds for \( x \) in \( S_{r_p} \) with \( \lambda > 0 \). Then, by our definition of \( C, P_n Cx = \lambda x \) and \( x \in H_n \cap S_r \).

This and (8) imply that

\[ \lambda(x, x) = (P_n Cx, x) = (Cx, x) \leq (C(0), x) \leq \| C(0) \|_{} \| x \| \]

\[ \leq (\| A(0) \|_{} + \| f \|) \| x \|. \]
from which we derive the inequality $\lambda \| x \| \leq \| A(o) \| + k_p = \mu r_p$. Thus, $\lambda \leq c_p$, i.e., condition $(A_j)$ is satisfied. To verify condition $(II_p)$ note that if for $\eta \geq \mu$ we put $Qx = Ax - \eta x$, then by (7)

\[
(Q(o) - Qx, x) - (A(o) - Ax + \eta x, x)
= (A(o), x) - (Ax, x) + \eta(x, x) \geq \eta(x, x).
\]

Hence $\| Qx - Q(o) \| \geq \eta \| x \| \geq \mu \| x \|$ for each $x$ in $H$ and any $\eta \geq \mu$. Thus for any $x$ in $S_{r_p}$ and $\eta \geq \mu$

\[
\| Ax - \eta x \| = \| Qx \| \geq \| Qx - Q(o) \| - \| Q(o) \| \geq \mu \| x \| - \| A(o) \| = \mu \| - \| A(o) \| = \mu \rho = k_p
\]

which is precisely condition $(II_p)$. Consequently Theorem 3 follows from Theorem 2.

**Corollary 2.** If $A$ is $P$-compact and monotone decreasing, i.e.,

\[
(Ax - Ay, x - y) \leq 0, \quad x, y \in H,
\]

then for any given $\mu > 0$ the mapping $P = A - \mu I$ is one-to-one and onto.

**Proof.** The onto part of the assertion of Corollary 2 follows from Theorem 3, since (9) clearly implies (7) while the one-to-one part follows from the fact that $\| Px - Py \| \geq \mu \| x - y \|$ for any $x$ and $y$ in $H$ and any $\mu > 0$.

### 3. Applications to Quasibounded Mappings

Let $Y$ and $Z$ be any two real Banach spaces. Following Granas [7] we say that a nonlinear mapping $A$ of $Y$ into $Z$ is *quasibounded* if there exist two constants $M > 0$ and $q_0 > 0$ such that

\[
\| Ax \| \leq M \| x \| \quad \text{for all } x \text{ in } Y \text{ with } \| x \| \geq q_0.
\]

If $A$ is quasibounded, then the number $| A |$ defined by

\[
| A | = \inf_{q_0 \leq q < \infty} \left\{ \sup_{\| x \| \geq q} \frac{\| Ax \|}{\| x \|} \right\}
\]

is called the *quasinorm* of $A$. It follows that every bounded linear operator is quasibounded and that its norm coincides with its quasinorm. Furthermore, as was observed by Granas, every nonlinear mapping of $Y$ into $Z$ which is asymptotically differentiable in the sense of Krasnoselsky [9] is quasibounded.
In fact, if \( A \) is asymptotically differentiable, then there exists a linear operator \( A' \) mapping \( Y \) into \( Z \), called the asymptotic derivative of \( A \), such that
\[
\lim_{x \to \infty} \frac{\| A(x) - A'x \|}{\| x \|} = 0;
\]
hence it follows easily from (12) that \( A \) is quasibounded and that \( |A| \leqslant \|A'\| \). Let us add that the class of asymptotically differentiable operators was thoroughly studied in [9].

The purpose of this section is to apply our theorems proved in Section 2 to the generalization of results obtained by Granas [7, 8] for completely continuous quasibounded operators by means of topological arguments.

**Theorem 4.** Suppose that \( A \) is a \( P \)-compact and quasibounded mapping of \( X \) into itself. If \( \mu > M \), then \( (A - \mu I) \) is onto.

**Proof.** Let \( \{r_n\} \) be a sequence of real numbers such that \( r_n \geq q_0 \) for all \( n \) and such that \( r_n \to \infty \), as \( n \to \infty \). Then, in view of our conditions, for all \( x \in S_{r_n} \) and any \( \eta \geq \mu, \)
\[
\|Ax - \eta x\| \geq \eta \|x\| - \|Ax\| \geq \mu \|x\| - M \|x\| = (\mu - M) \|x\|
\]
Thus condition (II) of Theorem 2 is satisfied with \( k_p = (\mu - M) r_n \). Now suppose that for any \( f \in B_{q_0} \) and any \( n \) the equation \( P_nAx - \lambda x = P_nf \) holds for \( x \in S_{r_p} \) with \( \lambda > 0 \). Then, by (2) and (10), the latter equation implies that
\[
\lambda r_p = \lambda \|x\| = \|P_n(Ax - f)\| \leq K \|Ax - f\| \leq K(\|Ax\| + \|f\|)
\]
Hence, \( \lambda \leq \mu K \), i.e., condition (A) is satisfied with \( c_p = \mu K \) for each \( p \). Consequently, Theorem 4 follows from Theorem 2.

**Remark 4.** It is not hard to see that Theorem 4 remains valid if instead of assuming that \( \mu > M \) we assume that \( \mu > |A| \).

**Corollary 3.** Suppose that \( A \) is quasibounded and \( P \)-compact with \( p < 0 \). If \( \mu > M \), then \( (\mu I + A) \) maps \( X \) onto itself.

**Proof.** The conditions of Corollary 3 imply that \( \bar{A} = -A \) is quasibounded and \( P \)-compact with \( \bar{p} = -p > 0 \). Hence, by Theorem 4, \( (\bar{A} - \mu I) \) or equivalently the operator \( (\mu I + A) \) is onto.
REMARK 5. When \( A \) is completely continuous (i.e., \( A \) is continuous and compact) and \( \mu = 1 \) Corollary 3 was proved by Granas [7] by the application of the topological fixed point theorem of Rothe [10].

AN INTERSECTION THEOREM IN \( X \)

Suppose that \( X \) is a direct sum of the subspaces \( V \subset X \) and \( W \subset X \), i.e., \( X = V \oplus W \), and suppose that \( P_V \) and \( P_W \) denote the projections of \( X \) onto \( V \) and \( W \), respectively. It is obvious that \( P_V \) and \( P_W \) are linear and that

\[
\| P_V x \| \leq \| P_V \| \| x \| , \quad \| P_W x \| \leq \| P_W \| \| x \| , \quad x \in X. \tag{13}
\]

Suppose further that \( f(v) = z + F(v) \) maps \( V \) into \( X \) and that \( g(w) = w + G(w) \) maps \( W \) into \( X \), where \( F \) and \( G \) are quasibounded and completely continuous nonlinear mappings. Using Rothe's theorem, Granas [8] obtained an interesting intersection theorem by proving that if

\[
\| F \| \| P_V \| + \| G \| \| P_W \| < 1, \tag{14}
\]

then the images \( f(V) \) and \( g(W) \) have a nonempty intersection, i.e., \( f(V) \cap g(W) \neq \emptyset \).

Here we consider the intersection theorem when either \( F \) or \( G \) is \( P \)-compact and when condition (14) is replaced by a much weaker condition. Our result is based on the application of Theorem 1.

THEOREM 5. Let \( G \) be a nonlinear mapping of \( W \) into \( X \) such that the operator \( G(-P_W) \) is \( P \)-compact and such that to a given \( r > 0 \) there corresponds a number \( c(r) > 0 \) with the property that for all \( x \) in \( S_r \)

\[
\| G(-P_W x) \| \leq c(r). \tag{15}
\]

Let \( F \) be a completely continuous nonlinear mapping of \( V \) into \( X \) and let \( f_\mu(v) \) and \( g_\mu(w) \) be the mappings defined respectively from \( V \) and \( W \) to \( X \) by \( f_\mu(v) = \mu v + F v \) and \( g_\mu(w) = \mu w + G w \). If for given \( r > 0 \) and \( \mu > 0 \) the operators \( F \) and \( G \) satisfy the condition

\[
(\Pi): \text{If } F v + \alpha w = G w + \alpha v \text{ for some } v \text{ in } V \text{ and } w \text{ in } W \text{ with } \| v - w \| = r, \text{ then } \alpha < \mu,
\]

then \( f_\mu(V) \cap g_\mu(W) \neq \emptyset \).

PROOF. Let us define a nonlinear mapping \( A \) of \( X \) into \( X \) by

\[
Ax = G(w) - F(v) \quad \text{with} \quad w = -P_W x \quad \text{and} \quad v = P_V x, \quad x \in X, \tag{16}
\]
and observe that \( f_\mu(V) \cap g_\mu(W) \neq \emptyset \) if and only if the equation

\[
Ax = \mu x
\]  

has a solution in \( X \). Indeed, if \( x \) is a solution of (17), then \( x \) has a unique representation \( x = P_\nu x + P_\mu x = v - w \) and, in view of (16), (17) implies that \( Gw - Fv = \mu(v - w) \) or that \( \mu v + Gw = Fv + Fv \), i.e., \( f_\mu(V) \cap g_\mu(W) \neq \emptyset \).

On the other hand, if \( v \in V \) and \( w \in W \) are two elements such that \( f_\mu(v) = g_\mu(w) \), then \( \mu(v - w) = Gw - Fv \). Hence, if we put \( x = v - w \), (16) implies that \( x \) is a solution of (17).

Thus, to prove Theorem 5 it is sufficient, in view of the above observation and Theorem 1, to show that the operator \( A \) defined by (16) is \( P \)-compact and satisfies conditions (A) and (\( \Pi_\mu \)).

Let us first show that \( A \) is \( P \)-compact. Now, by our conditions on \( G \) and \( F \), \( P_n A \) is certainly continuous in \( X_n \) for all sufficiently large \( n \). Further, let \( \{x_n\} \) be a bounded sequence so that for any \( p > 0 \)

\[
g_n = P_n Ax_n - px_n = P_n G(-P_\mu x_n) - px_n - P_n F(P_\nu x_n) \to g, \quad x_n \in X_n.
\]  

(18)

Since \( \{v_n\} = \{P_\nu x_n\} \) is bounded and \( F \) is completely continuous, there exists a subsequence, which we again denote by \( \{x_n\} \), such that \( F(v_n) = F(P_\nu x_n) \to v \) and \( P_n F(v_n) \to v \), where \( v \) is some element in \( X \). This and (18) imply that

\[
g_n = P_n G(-P_\mu x_n) - px_n = g_n + P_n F(P_\nu x_n) \to g + v, \quad (n \to \infty).
\]

Since \( G(-P_\mu) \) is \( P \)-compact there exists a subsequence, again denoted by \( \{x_n\} \), such that \( x_n \to x \) and \( P_n G(-P_\mu x_n) \to G(-P_\nu x) \). This and the continuity of \( F \) imply that

\[
P_n Ax_n = P_n G(-P_\mu x_n) - P_n F(P_\nu x_n) \to G(P_\nu x) - F(P_\nu x) = Ax
\]
i.e., \( A \) is \( P \)-compact.

Suppose now that \( Ax = \alpha x \) for some \( x \) in \( S_\nu \). This then means that \( Gw + \alpha w = Fv + \alpha v \) with \( \|v - w\| = \|P_\nu x + P_\mu x\| = r \). Hence our condition (\( \Pi_\mu \)) implies that \( \alpha < \mu \), i.e., \( A \) satisfies condition (\( \Pi_\mu \)). Finally we see that for any \( x \) in \( S_\nu \) condition (15) and the complete continuity of \( F \) imply the inequality

\[
\|Ax\| \leq \|G(-P_\mu x)\| + \|F(P_\nu x)\| \leq c(r) + c, \quad x \in S_\nu,
\]

where \( c > 0 \) is such that \( \|F(P_\nu x)\| \leq c \) for all \( x \) in \( S_\nu \). Thus the set \( A(S_\nu) \) is bounded and therefore, by Remark 3(b), \( A \) satisfies condition (A). Hence, by Theorem 1, Eq. (17) has at least one solution in \( B_\nu \) or, equivalently, the intersection \( f_\mu(V) \cap g_\mu(W) \neq \emptyset \).
COROLLARY 4. Suppose that $G$ and $F$ satisfy all conditions of Theorem 5 except that condition (II) is replaced by the condition

$$\| \mu w + G w - (\mu v + F v) \|^2 \geq \| F v - G w \|^2 - \mu^2 \| v - w \|^2$$

(19)

for $v \in V$, $w \in W$ with $\| v - w \| = r$. Then $f_{\mu}(V) \cap g_{\mu}(W) \neq \emptyset$.

PROOF. We may assume, without loss of generality, that there are no elements $z_J$ in $V$ and $w$ in $W$ with $\| v - w \| = r$ such that $f_{\mu}(v) = g_{\mu}(w)$. Suppose now that for some $x$ in $S$, or equivalently for some $v$ in $V$ and $w$ in $W$ with $\| v - w \| = r$ we have $F v + \alpha v = G w + \alpha w$. Then

$$\| G w - F v \|^2 = \| \alpha(v - w) - (\alpha - \mu)(v - w) \|^2 = (\alpha - \mu)^2 \| v - w \|^2$$

and

$$\| G w - F v \|^2 - \mu^2 \| v - w \|^2 = (\alpha^2 - \mu^2) \| v - w \|^2.$$ 

Hence, by (19), $(\alpha - \mu)^2 \geq (\alpha^2 - \mu^2)$ or $2\mu^2 \geq 2\mu\alpha$. Since $\mu > 0$, our assumption then implies that $\alpha < \mu$ and, consequently, (18) implies condition (II). Corollary 4 then follows from Theorem 5.

REMARK 6. In case $X$ is a Hilbert space condition (19) is equivalent to the requirement

$$(G(- P_w x) - F(P_v x), x) \leq \mu \| x \|^2, \quad x \in S_r.$$ 

(20)

COROLLARY 5. Suppose that $F$ and $G$ are completely continuous and quasi-bounded: i.e., there exists four constants $M_1 > 0$, $M_2 > 0$, $r_1 > 0$ and $r_2 > 0$ such that

$$\| F v \| \leq M_1 \| v \| \quad \text{for every } v \in V \text{ with norm } \| v \| \geq r_1$$

(21)

$$\| G(w) \| \leq M_2 \| w \| \quad \text{for every } w \in W \text{ with norm } \| w \| \geq r_2.$$ 

(22)

Suppose further that $M_1$ and $M_2$ satisfy the inequality

$$M_1 \| P_v \| + M_2 \| P_w \| \leq 1.$$ 

(23)

Then $f_{\alpha}(V) \cap g_{\alpha}(W) \neq \emptyset$.

PROOF. Let us first remark that, as was shown by Granas, the conditions of Corollary 5 imply the existence of a constant $r > 0$ such that

$$\| G(- P_w x) - F(P_v x) \| \leq \| x \| \quad \text{for every } x \in X \text{ with } \| x \| \geq r;$$ 

(24)
i.e., the operator \( A(x) = G(-P_Wx) - F(P_Wx) \) is quasibounded. Assuming, without loss of generality, that there are no elements \( v \) in \( V \) and \( w \) in \( W \) with \( \| v - w \| = r \) such that \( f_1(v) = g_1(w) \), it is easy to see that whenever 
\[
Fv + \alpha w = Gw + \alpha w
\]
for some \( v \) in \( V \) and \( w \) in \( W \) with \( \| v - w \| = r \), then (24) implies that \( \alpha < 1 \). Hence condition (II) of Theorem 5 holds for \( \mu = 1 \). Furthermore, since \( G \) is completely continuous, (15) is clearly satisfied and, by Theorem 3 in [2], \( G(-P_W) \) is \( P \)-compact. Consequently, Corollary 5 follows from Theorem 5.

**Remark 7.** For the sake of completeness let us show that the conditions of Corollary 5 imply the validity of (24) for some \( r > 0 \). First let 
\[
r_0 = \max \{ r_1, r_2 \}
\]
and let \( c > 0 \) be a constant such that
\[
\| Fv \| \leq c \quad \text{for } v \text{ in } V \text{ with } \| v \| \leq r_0
\]
and
\[
\| Gw \| \leq c \quad \text{for } w \text{ in } W \text{ with } \| w \| \leq r_0.
\]
Taking
\[
r = \max \left\{ 2r_0, \frac{c}{1 - M_0} \right\},
\]
where
\[
M_0 = \max \{ M_1 \| P_v \|, M_2 \| P_W \| \} < 1,
\]
we obtain (24). Indeed, (24) follows trivially from (21), (22), and (23) if for \( x = v - w \) with \( \| x \| \geq r \) we have \( \| v \| \geq r_0 \) and \( \| w \| \geq r_0 \). On the other hand, if for \( \| x \| \geq r \) one of the conditions \( \| v \| \geq r_0 \) or \( \| w \| \geq r_0 \) is not satisfied (e.g. \( \| w \| \geq r_0 \)), then by our definition of \( c \) and \( M_0 \) we get the desired inequality
\[
\| G(-P_Wx) - F(P_Wx) \| \leq \| G(w) \| + \| F(v) \| \leq c + M_0 \| x \|
\]
\[
\leq (1 - M_0) \| x \| + M_0 \| x \| = \| x \| .
\]

**Remark 8.** Let us observe finally that the results obtained in Section 5 in [2] concerning the applicability of the projection method to the approximate solution of either the equation (3) or (6) remain also valid for unbounded \( P \)-compact operators provided, of course, that they satisfy the corresponding conditions assumed in [2].
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REFERENCES


