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A note on parallel and alternating time

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Dedicted to Henryk Wozniakowski on the occasion of his 60th birthday

Abstract

A long standing open question in complexity theory over the reals is the relationship between parallel time and quantifier alternation. It is known that alternating digital quantifiers is weaker than parallel time, which in turn is weaker than alternating unrestricted (real) quantifiers. In this note we consider some complexity classes defined through alternation of mixed digital and unrestricted quantifiers in different patterns. We show that the class of sets decided in parallel polynomial time is sandwiched between two such classes for different patterns.

Keywords: Complexity classes; Parallelism; Alternation

1. Introduction

In classical complexity theory (that is, in the theory built upon the Turing machine) it was early realized [3,7] that the three following resources:

1. parallel time,
2. alternating time, and
3. space

were equivalent under polynomial bounds. In other words, the complexity classes defined by parallel polynomial time, alternating polynomial time, and polynomial space are actually the same class.

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In contrast with the above, it was soon remarked that in the theory of complexity over the reals developed by Blum, Shub, and Smale [2], every decidable set can be decided using constant workspace [8]. This was at the expense of an exponential increase in the running time. Yet, the contrast with the classical situation prevailed. If we denote by $\mathrm{PSPACE}_{\mathbb{R}}$ the class of sets of real vectors decidable in exponential time and using polynomial space, by $\mathrm{PAT}_{\mathbb{R}}$ the class of sets decidable using polynomial alternating time, and by $\mathrm{PAR}_{\mathbb{R}}$ the one of those decidable using parallel polynomial time, one can still show [4] that

$$\mathrm{PAR}_{\mathbb{R}} \subset \mathrm{PSPACE}_{\mathbb{R}} \subseteq \mathrm{PAT}_{\mathbb{R}}. \tag{1}$$

Note that, in the classical setting, the requirement of exponential time is superfluous when polynomial space is ensured. Hence the (somehow abusive) notation $\mathrm{PSPACE}_{\mathbb{R}}$. Note also that the first inclusion above is strict.

Along with these inclusions, a major achievement in algorithmics over the reals was the inclusion

$$\mathrm{PH}_{\mathbb{R}} \subseteq \mathrm{PAR}_{\mathbb{R}}, \tag{2}$$

where $\mathrm{PH}_{\mathbb{R}}$ denotes the polynomial hierarchy over the reals (see, e.g., [9]). This is the class of sets which can be decided in polynomial alternating time with a constant (though not universally bounded) number of alternations. The inclusions in (1) and (2) together draw a critical boundary among the sets decidable with alternation: if the number of alternations between existential and universal guesses is constant (i.e., independent on the input size) then the set can be decided within parallel polynomial time. If no such constant bound exists then the problem may need exponential parallel time (and in some cases it actually does).

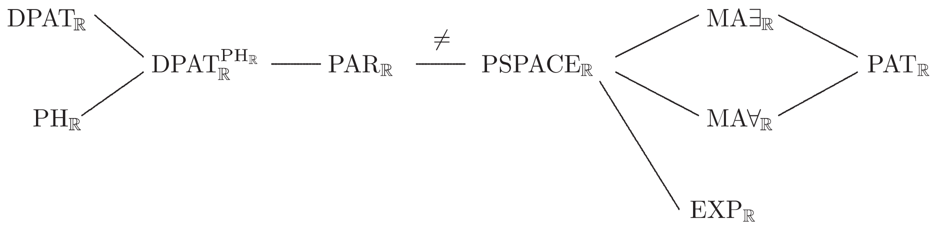
The goal of this paper is to further investigate this boundary by looking at quantifiers with a restricted expressive power and consider the classes they define.

The quantifiers we will look at are the *digital quantifiers* introduced in [5]. These are simply the usual quantifiers $\exists$ and $\forall$ but with variables ranging on the set $\{0, 1\}$. We will denote then by $\exists_B$ and $\forall_B$, respectively. Digital versions of $\mathrm{NP}_{\mathbb{R}}$ and $\mathrm{coNP}_{\mathbb{R}}$ are naturally defined and a number of problems are shown to belong to these classes (see [6] for a non-trivial example). Alternation is also naturally defined and with it, the class $\mathrm{DPAT}_{\mathbb{R}}$ of digital polynomial alternating time. Since any computation in this class makes only a polynomial number of guesses (digital, either universal or existential) one can simulate the computation in parallel polynomial time by independently computing its outcome for the exponential number of possible guesses and then checking whether the set of outcomes satisfy the prefix of quantifiers. Therefore, $\mathrm{DPAT}_{\mathbb{R}} \subseteq \mathrm{PAR}_{\mathbb{R}}$. Recall from (2), we also have $\mathrm{PH}_{\mathbb{R}} \subseteq \mathrm{PAR}_{\mathbb{R}}$. One of our main results is to extend both these inclusions by proving that $\mathrm{DPAT}_{\mathbb{R}}^{\mathrm{PH}_{\mathbb{R}}} \subseteq \mathrm{PAR}_{\mathbb{R}}$.

Our second main result involves classes defined by alternating digital and ordinary quantifiers. We define the class $\mathrm{MA}\exists_{\mathbb{R}}$ (*mixed alternation with real existentials*) containing all sets decidable alternating digital universal and real existential guesses in polynomial time. Similarly, one defines the class $\mathrm{MA}\forall_{\mathbb{R}}$ (*mixed alternation with real universals*) containing all sets decidable alternating digital existential and real universal guesses in polynomial time. Precise definitions are in Section 4. Then, we show that $\mathrm{PAR}_{\mathbb{R}} \subset \mathrm{MA}\exists_{\mathbb{R}}$ and $\mathrm{PAR}_{\mathbb{R}} \subset \mathrm{MA}\forall_{\mathbb{R}}$ (we will actually show that $\mathrm{PSPACE}_{\mathbb{R}}$ is included in both $\mathrm{MA}\exists_{\mathbb{R}}$ and $\mathrm{MA}\forall_{\mathbb{R}}$, hence the strict inclusions for $\mathrm{PAR}_{\mathbb{R}}$).

Together with our first result mentioned above this sharpens the relationship of $\mathrm{PAR}_{\mathbb{R}}$ with alternation. For, on the one hand, we characterize the class $\mathrm{DPAT}_{\mathbb{R}}^{\mathrm{PH}_{\mathbb{R}}}$ by a form of alternation where one first alternates a polynomial number of digital quantifiers and then a polynomial number of real quantifiers (but these ones with only a bounded number of alternations). And, on the other hand, the classes $\mathrm{MA}\exists_{\mathbb{R}}$ and $\mathrm{MA}\forall_{\mathbb{R}}$ allow real quantifiers to alternate with digital ones provided all the real quantifiers are of the same kind.

We can summarize the relationship between complexity classes which emerges from our results in the following diagram (where a line means inclusion of the left-hand side class in the right-hand side one and $\mathrm{EXP}_{\mathbb{R}}$ denotes the class of sets decidable in exponential time).



Probably the absence of a characterization of $\mathrm{PAR}_{\mathbb{R}}$ in terms of quantifier alternation was an obstacle in the search for complete problems in $\mathrm{PAR}_{\mathbb{R}}$ of which, to the best of our knowledge, there are no known natural examples. In Section 3 we provide one such problem (whose completeness is used later on in the paper).

2. Preliminaries

We denote by $\mathbb{R}^\infty$ the disjoint union of the Euclidean spaces $\mathbb{R}^n$, for $n \geqslant 1$. Given $x \in \mathbb{R}^\infty$ we denote by $|x|$ its *size*, i.e., the only $n \geqslant 1$ such that $x \in \mathbb{R}^n$.

Sequential machines over $\mathbb{R}$ were introduced in [2]. Roughly speaking, they take inputs from $\mathbb{R}^\infty$ and compute their output by performing arithmetic operations and comparisons. The class $\mathrm{P}_{\mathbb{R}}$ of subsets $S \subset \mathbb{R}^\infty$ decidable in polynomial time is then readily defined. Nondeterministic machines over $\mathbb{R}$ were also introduced in [2], together with the class $\mathrm{NP}_{\mathbb{R}}$ of subsets decidable in nondeterministic polynomial time. Alternating machines are defined similarly. See [1] for the latter as well as for details on the definition of the polynomial hierarchy $\mathrm{PH}_{\mathbb{R}}$ and its levels.

A parallel machine over the reals is defined in [1, Chapter 18]. It is a collection of processors, each with its own memory, and able to read other processors' memory. The class $\mathrm{PAR}_{\mathbb{R}}$ is the class of all subsets of $\mathbb{R}^\infty$ decidable by parallel machines with a single exponential number of processors, and in polynomial time. It is shown there that $\mathrm{PAR}_{\mathbb{R}}$ can also be defined as the class of subsets decidable by $\mathrm{P}_{\mathbb{R}}$-uniform families of decisional circuits with polynomial depth (and hence, exponential size). See [1, Chapter 18] for details.

3. A $\mathrm{PAR}_{\mathbb{R}}$-complete problem

While the nature of the class $\mathrm{PAR}_{\mathbb{R}}$ suggests it must have complete problems, to the best of our knowledge, no natural $\mathrm{PAR}_{\mathbb{R}}$-complete problem has been exhibited in the literature. In this section we provide such a completeness result. Consider the following decisional problem:

$\mathrm{SCE}_{\mathbb{R}}$ (*Succinct circuit evaluation*): Given a tuple $(M, x, 1^p, 1^t)$ decide whether

(i) M is a machine "describing" a circuit $\mathscr{C}$ in time at most p (i.e., with input i, M returns the encoding of the ith gate of $\mathscr{C}$—or NIL if the size of $\mathscr{C}$ is less than i—in time at most p),

(ii) $\mathscr{C}$ has depth at most t,
(iii) $\mathscr{C}$ has $\text{size}(x)$ input gates and one output gate, and
(iv) $\mathscr{C}(x) = 1$.

Proposition 3.1. *The problem* $\text{SCE}_{\mathbb{R}}$ *is* $\text{PAR}_{\mathbb{R}}$*-complete.*

Proof. Let $S \in \text{PAR}_{\mathbb{R}}$. Then, there exists a $\text{P}_{\mathbb{R}}$-uniform family of circuits $\{\mathscr{C}_n\}_{n\in\mathbb{N}}$ deciding S in polynomial depth. Let $p(n)$ and $t(n)$ be the polynomials bounding the running time of the machine M describing the circuits, and the depth of $\mathscr{C}_n$, respectively. On input (n, i), the machine M returns the ith gate of $\mathscr{C}_n$. For all $n \in \mathbb{N}$, let M'_n be the machine computing the circuit $\mathscr{C}_n$. Since M'_n computes the function

$$i \mapsto M(n, i)$$

the code of the machine M'_n can be computed in time polynomial in n (from the code of M). Then the map

$$x \mapsto (M'_{|x|}, x, 1^{p(|x|)}, 1^{t(|x|)})$$

gives a many-one reduction from S to $\text{SCE}_{\mathbb{R}}$. This proves the hardness.

For the membership, simply check that the following algorithm solves $\text{SCE}_{\mathbb{R}}$ in $\text{PAR}_{\mathbb{R}}$:

```
input (M, x, 1^p, 1^t)
% check conditions (i), (ii), and (iii) %
for i = 1, ..., 2^t in parallel do
    compute the output g_i of p steps of M with input i
    if g_i is not NIL or a gate encoding HALT and REJECT
end for
let C = {g_i}_{i ≤ 2^t}
if C not a circuit with |x| input gates
and 1 output gate HALT and REJECT
% check condition (iv) %
let m := size(C)
for i = 1, ..., m in parallel do μ_i := 0 end for
for j = 1, ..., t do
    for i = 1, ..., m in parallel do
        if the μ's corresponding to the parents of the ith
        gate are both 1 then evaluate g, set μ_i := 1 and
        set v_i to be the result of the evaluation
    end for
end for
if μ_m = 1 and v_m = 1 then HALT and ACCEPT
else REJECT.  □
```

4. An upper bound for $\text{PAR}_{\mathbb{R}}$: $\text{MA}\exists_{\mathbb{R}}$

In this section we sharpen the inclusion $\text{PAR}_{\mathbb{R}} \subset \text{PAT}_{\mathbb{R}}$ by showing that $\text{PAR}_{\mathbb{R}} \subset \text{MA}\exists_{\mathbb{R}} \subseteq \text{PAT}_{\mathbb{R}}$ (the second inclusion being trivial).

For a quantifier Q and a variable x, let us use the notation $Q_B x$ and $Q_{\mathbb{R}} x$ instead of $Qx \in \{0, 1\}$ and $Qx \in \mathbb{R}$, respectively.

Definition 4.1. We define $\mathrm{MA}\exists_{\mathbb{R}}$ to be the class of sets $S \subseteq \mathbb{R}^\infty$ such that there exists a set $B \subseteq \mathbb{R}^\infty$ in $\mathrm{P}_{\mathbb{R}}$ and a polynomial p such that, for $x \in \mathbb{R}^\infty$, x belongs to S if and only if

$$\forall_B\, y_1\, \exists_{\mathbb{R}} z_1 \ldots \forall_B\, y_{p(|x|)}\, \exists_{\mathbb{R}} z_{p(|x|)} \quad (x, y, z) \in B.$$

We define the class $\mathrm{MA}\forall_{\mathbb{R}}$ to be the class of sets whose complement is in $\mathrm{MA}\exists_{\mathbb{R}}$.

The main result of this section is the following.

Proposition 4.2. *The class* $\mathrm{PSPACE}_{\mathbb{R}}$ *is included in* $\mathrm{MA}\exists_{\mathbb{R}}$ *and in* $\mathrm{MA}\forall_{\mathbb{R}}$.

Proof. Since $\mathrm{PSPACE}_{\mathbb{R}}$ is closed by complementation, we just need to show that $\mathrm{PSPACE}_{\mathbb{R}}$ is included in $\mathrm{MA}\exists_{\mathbb{R}}$. To do so we will closely follow the main argument in the proof of the inclusion $\mathrm{PSPACE}_{\mathbb{R}} \subseteq \mathrm{PAT}_{\mathbb{R}}$ given in [4].

We define $\mathcal{M}$ to be the set of true formulas of the form

$$Q_1 X_1\, Q_2 X_2\, \ldots\, Q_n X_n, \quad \varphi(X_1, X_2, \ldots, X_n),$$

where the Q_i are either $\forall_B$ or $\exists_{\mathbb{R}}$, and the expression $\varphi(X_1, X_2, \ldots, X_n)$ denotes a semi-algebraic system.

Clearly, $\mathcal{M}$ is a $\mathrm{MA}\exists_{\mathbb{R}}$-complete problem. To prove our statement it is therefore enough to reduce any problem in $\mathrm{PSPACE}_{\mathbb{R}}$ to $\mathcal{M}$.

Let S be a language in $\mathrm{PSPACE}_{\mathbb{R}}$ and M a machine over $\mathbb{R}$ deciding S in exponential time and polynomial space. Let p be a polynomial bounding the space used by M and q one bounding the logarithm (base 2) of the time bound for M.

Fix $x \in \mathbb{R}^n$. It is shown in [4] that any configuration of the computation of M with input x may be represented by a real vector of size $p(n) + 3$ (which, roughly speaking, encodes the current instruction and the current contents of the memory of M). For $\alpha, \beta \in \mathbb{R}^{p(n)+3}$ we define the formulas

$$\mathsf{Next}(\alpha, \beta), \quad \mathsf{Equal}(\alpha, \beta), \quad \mathsf{Initial}(\alpha, x), \quad \text{and} \quad \mathsf{Accepts}(\alpha)$$

meaning, respectively, "β is the configuration resulting from α after one step of M", "α and β are the same configuration", "α is the initial configuration of M with input x" and "α is an accepting configuration".

These formulas may be constructed in time polynomial in n by a real machine (whose code uses that of M). Our next goal is to describe a formula $\mathsf{Access\_2}^m(\alpha, \beta)$, also constructible in polynomial time, expressing that the configuration β is reached from α after at most 2^m steps of M.

If $m = 0$ we take

$$\mathsf{Access\_2}^0(\alpha, \beta) = \mathsf{Equal}(\alpha, \beta) \vee \mathsf{Next}(\alpha, \beta).$$

For $m > 0$ we could define

$$\mathsf{Access\_2}^m := \exists\gamma\, \mathsf{Access\_2}^{m-1}(\alpha, \gamma) \wedge \mathsf{Access\_2}^{m-1}(\gamma, \beta),$$

but the length of this expression doubles at each iteration. To avoid the exponential growth of the expanded formula, we introduce a Boolean universal quantifier meant to describe the two calls to $\mathsf{Access\_2}^{m-1}$ above with only one such call. We define $\mathsf{Access\_2}^{m}(\alpha,\beta)$ as follows:

$$\exists_{\mathbb{R}}\gamma\,\forall_B\mathbf{b}\,\exists_{\mathbb{R}}\alpha'\,\exists_{\mathbb{R}}\beta'\,[(\mathsf{Equal}(\alpha',\alpha)\wedge\mathsf{Equal}(\beta',\gamma)\wedge\mathbf{b}=0)\\ \vee(\mathsf{Equal}(\alpha',\gamma)\wedge\mathsf{Equal}(\beta',\beta)\wedge\mathbf{b}=1)]\wedge\mathsf{Access\_2}^{m-1}(\alpha',\beta').$$

Let us denote by $\mathbf{z}_m$ the vector of the variables present in this step of the recursion, that is, $\mathbf{z}_m=(\alpha,\beta,\gamma,\mathbf{b},\alpha',\beta')$. We denote by ϕ the formula

$$\phi(\mathbf{z}_m)=[(\mathsf{Equal}(\alpha',\alpha)\wedge\mathsf{Equal}(\beta',\gamma)\wedge\mathbf{b}=0)\\ \vee(\mathsf{Equal}(\alpha',\gamma)\wedge\mathsf{Equal}(\beta',\beta)\wedge\mathbf{b}=1)].$$

With these notations,

$$\begin{aligned}\mathsf{Access\_2}^{m}(\alpha,\beta)&=\exists_{\mathbb{R}}\gamma_m\,\forall_B\mathbf{b}_m\,\exists_{\mathbb{R}}\alpha_m\,\exists_{\mathbb{R}}\beta_m\,\phi(\mathbf{z}_m)\wedge\mathsf{Access\_2}^{m-1}(\alpha_m,\beta_m)\\&=\exists_{\mathbb{R}}\gamma_m\,\ldots\,\exists_{\mathbb{R}}\beta_{m-1}\,\phi(\mathbf{z}_m)\wedge\phi(\mathbf{z}_{m-1})\wedge\mathsf{Access\_2}^{m-2}(\alpha_{m-1},\beta_{m-1})\\&\;\vdots\\&=\exists_{\mathbb{R}}\gamma_m\,\forall_B\mathbf{b}_m\,\exists_{\mathbb{R}}\alpha_m\,\exists_{\mathbb{R}}\beta_m\,\ldots\,\exists_{\mathbb{R}}\gamma_1\,\forall_B\mathbf{b}_1\,\exists_{\mathbb{R}}\alpha_1\,\exists_{\mathbb{R}}\beta_1\\&\quad\phi(\mathbf{z}_m)\wedge\cdots\wedge\phi(\mathbf{z}_1)\wedge\mathsf{Access\_2}^{0}(\alpha_1,\beta_1).\end{aligned}$$

Note that we let the inner quantifiers migrate in front since the corresponding variables are not used in the previous part of the formula.

Our reduction from S to $\mathcal{M}$ can be now simply described. To a point $x\in\mathbb{R}^\infty$ we associate the formula

$$\exists_{\mathbb{R}}\alpha\exists_{\mathbb{R}}\beta\,[\mathsf{Initial}(\alpha,x)\wedge\mathsf{Accepts}(\beta)\wedge\mathsf{Access\_2}^{q(|x|)}(\alpha,\beta))],$$

which is constructed in polynomial time in $|x|$, has the form required by $\mathcal{M}$, and belongs to $\mathcal{M}$ if and only if $x\in S$. □

Corollary 4.3. *The class* $\mathrm{PAR}_{\mathbb{R}}$ *is included in* $\mathrm{MA}\exists_{\mathbb{R}}$ *and in* $\mathrm{MA}\forall_{\mathbb{R}}$.

Proof. It follows from Proposition 4.2 and the inclusion $\mathrm{PAR}_{\mathbb{R}}\subset\mathrm{PSPACE}_{\mathbb{R}}$ shown in [4, Lemma 5.3]. □

5. A lower bound for $\mathrm{PAR}_{\mathbb{R}}$: $\mathrm{DPAT}_{\mathbb{R}}^{\mathrm{PH}_{\mathbb{R}}}$

Roughly speaking, oracle machines are theoretical computational devices which, during the computation, may query whether an intermediately computed value, say $z\in\mathbb{R}^\infty$, belongs to a fixed set $A\subseteq\mathbb{R}^\infty$ (called the *oracle*). The underlying computational device can be sequential, parallel, nondeterministic, alternating . . . . Formal definitions can be found, e.g., in [1].

Given a complexity class $\mathcal{C}$ (defined in terms of a class of resource-bounded machines) and a set A as the above one denotes by $\mathcal{C}^A$ the class of sets decidable by machines in $\mathcal{C}$ which query the oracle A. Also, given complexity classes $\mathcal{C}$ and $\mathcal{D}$ one defines

$$\mathcal{C}^{\mathcal{D}}=\bigcup_{A\in\mathcal{D}}\mathcal{C}^A.$$

Probably the best known example of classes defined this way are the levels of the polynomial hierarchy. Recall, for $k \geqslant 1$, one defines $\Sigma_{\mathbb{R}}^k$ to be the class of sets $S \subseteq \mathbb{R}^\infty$, for which there exists a set $B \in \mathrm{P}_{\mathbb{R}}$ and polynomials $p_1, \ldots, p_k$ such that, for all $x \in \mathbb{R}^\infty$, $x \in S$ if and only if

$$\exists y_1 \in \mathbb{R}^{p_1(|x|)} \, \forall y_2 \in \mathbb{R}^{p_2(|x|)} \ldots Q_k y_k \in \mathbb{R}^{p_k(|x|)} \quad (x, y_1, \ldots, y_k) \in B. \tag{3}$$

Here $Q_k = \exists$ if k is odd and $Q_k = \forall$ otherwise. A well-known result (cf. [1, Chapter 18]) shows that

$$\Sigma_{\mathbb{R}}^k = \mathrm{NP}_{\mathbb{R}}^{\Sigma_{\mathbb{R}}^{k-1}}.$$

Our next result provides a similar characterization for $\mathrm{DPAT}_{\mathbb{R}}^{\mathrm{PH}_{\mathbb{R}}}$.

Lemma 5.1. *For a set $S \subseteq \mathbb{R}^\infty$ the following are equivalent*:

(i) $S \in \mathrm{DPAT}_{\mathbb{R}}^{\mathrm{PH}_{\mathbb{R}}}$,

(ii) *there exists $B \in \mathrm{P}_{\mathbb{R}}$, $k \geqslant 0$, and polynomials $q, p_1, \ldots, p_k$ such that, for all $x \in \mathbb{R}^\infty$, $x \in S$ if and only if*

$$\exists_B b_1 \, \forall_B b_2 \ldots Q_B b_{q(|x|)} \, \exists y_1 \in \mathbb{R}^{p_1(|x|)} \, \forall y_2 \in \mathbb{R}^{p_2(|x|)} \ldots Q_k y_k \in \mathbb{R}^{p_k(|x|)}$$
$$(x, b_1, \ldots, b_{q(|x|)}, y_1, \ldots, y_k) \in B.$$

Here $Q_B = \exists_B$ if $q(|x|)$ is odd and $Q_B = \forall_B$ otherwise. Similarly for Q_k.

Proof. We begin with (i) $\Longrightarrow$ (ii). To do so, let $S \in \mathrm{DPAT}_{\mathbb{R}}^{\mathrm{PH}_{\mathbb{R}}}$. Then, there exist a digitally alternating machine M and a set $A \in \Sigma_{\mathbb{R}}^\ell$ (for some $\ell \geqslant 0$) such that M decides S with oracle A. The idea of the proof is that we can modify M so that all the oracle queries are performed after the alternation has been done. This is obtained by replacing a query "$z \in A$?" in the program of M by an existential binary guess (which replaces the answer to the query "$z \in A$"). Then, once all the alternation has been performed, the program adds the following instructions (here $d_1, \ldots, d_r$ are the binary guesses corresponding to the oracle queried values $z_1, \ldots, z_r$)

if for all $j = 1, \ldots, r$
$(z_j \in A$ and $d_j = 1)$ or $(z_j \notin A$ and $d_j = 0)$
then continue
else REJECT

Note that, once fixed all the binary values corresponding to the alternation (this includes $d_1, \ldots, d_r$) the computation in the instructions above is performed in $\mathrm{P}_{\mathbb{R}}^A$ which is known to be included in $\Sigma_{\mathbb{R}}^{\ell+1}$. And computations in $\Sigma_{\mathbb{R}}^{\ell+1}$ can be described by a quantifier prefix as that described in (3) with $k = \ell + 1$. This shows that S can be described as in (ii).

For the direction (ii) $\Longrightarrow$ (i) consider a set S as described in (ii). Define a set $A \subseteq \mathbb{R}^\infty$ consisting of the points $z \in \mathbb{R}^\infty$ satisfying that:

(1) z is of the form $(x, b_1, \ldots, b_{q(|x|)})$ with $x \in \mathbb{R}^\infty$ and $b_i \in \{0, 1\}$, for $i \leqslant q(|x|)$.

(2) $\exists y_1 \in \mathbb{R}^{p_1(|x|)} \, \forall y_2 \in \mathbb{R}^{p_2(|x|)} \ldots Q_k y_k \in \mathbb{R}^{p_k(|x|)} (z, y_1, \ldots, y_k) \in B$.

Since (1) is checked in $\mathrm{P}_{\mathbb{R}}$ and (2) in $\Sigma_{\mathbb{R}}^k$ we have $A \in \Sigma_{\mathbb{R}}^k$. To show that $S \in \mathrm{DPAT}_{\mathbb{R}}^{\mathrm{PH}_{\mathbb{R}}}$ one considers the machine that, given $x \in \mathbb{R}^\infty$, first guesses the elements $b_1, \ldots, b_{q(|x|)} \in \{0, 1\}$

(alternating existential and universal guesses) and then queries whether $z = (x, b_1, \ldots, b_{q(|x|)})$ is in A. This machine decides S in $\mathrm{DPAT}_{\mathbb{R}}^{A} \subseteq \mathrm{DPAT}_{\mathbb{R}}^{\Sigma_{\mathbb{R}}^k}$. □

Proposition 5.2. *We have* $\mathrm{DPAT}_{\mathbb{R}}^{\mathrm{PH}_{\mathbb{R}}} \subseteq \mathrm{PAR}_{\mathbb{R}}$.

Proof. Let $S \in \mathrm{DPAT}_{\mathbb{R}}^{\mathrm{PH}_{\mathbb{R}}}$. Then, S can be characterized as in Lemma 5.1(ii). Consider now a parallel machine which, with input $x \in \mathbb{R}^{\infty}$, independently generate the $2^{q(|x|)}$ elements in $\{0, 1\}^{q(|x|)}$ and for each one of them, say $\mathbf{b}$, checks whether

$$\exists y_1 \in \mathbb{R}^{p_1(|x|)}\, \forall y_2 \in \mathbb{R}^{p_2(|x|)} \ldots Q_k y_k \in \mathbb{R}^{p_k(|x|)} (x, \mathbf{b}, y_1, \ldots, y_k) \in B.$$

This checking can be done in $\mathrm{PAR}_{\mathbb{R}}$ (we saw in the proof of Lemma 5.1 that it can be done in $\Sigma_{\mathbb{R}}^k$ and now we use that $\Sigma_{\mathbb{R}}^k \subseteq \mathrm{PAR}_{\mathbb{R}}$). Therefore, we compute in $\mathrm{PAR}_{\mathbb{R}}$ the $2^{q(|x|)}$ bits corresponding to all the possible guesses $\mathbf{b}$. We now check that these bits satisfy the prefix of quantifiers corresponding to $\mathbf{b}$, which can also be done in $\mathrm{PAR}_{\mathbb{R}}$. □

Proposition 5.3. *If* $\mathrm{DPAT}_{\mathbb{R}}^{\mathrm{PH}_{\mathbb{R}}} = \mathrm{PAR}_{\mathbb{R}}$ *then there exists* $k \geqslant 0$ *such that* $\mathrm{DPAT}_{\mathbb{R}}^{\Sigma_{\mathbb{R}}^k} = \mathrm{PAR}_{\mathbb{R}}$.

Proof. Assume $\mathrm{DPAT}_{\mathbb{R}}^{\mathrm{PH}_{\mathbb{R}}} = \mathrm{PAR}_{\mathbb{R}}$. Then, $\mathrm{SCE}_{\mathbb{R}} \in \mathrm{DPAT}_{\mathbb{R}}^{\mathrm{PH}_{\mathbb{R}}}$ and therefore, there exists $k \geqslant 0$ such that $\mathrm{SCE}_{\mathbb{R}} \in \mathrm{DPAT}_{\mathbb{R}}^{\Sigma_{\mathbb{R}}^k}$. Since $\mathrm{SCE}_{\mathbb{R}}$ is complete in $\mathrm{PAR}_{\mathbb{R}}$ (Proposition 3.1) all problems in $\mathrm{PAR}_{\mathbb{R}}$ must be in $\mathrm{DPAT}_{\mathbb{R}}^{\Sigma_{\mathbb{R}}^k}$. □

Remark 5.4. (i) One can prove Proposition 5.2 differently. First, we claim that the inclusion $\mathrm{DPAT}_{\mathbb{R}} \subseteq \mathrm{PAR}_{\mathbb{R}}$ relativizes (i.e., that for every set $A \subseteq \mathbb{R}^{\infty}$, one has $\mathrm{DPAT}_{\mathbb{R}}^{A} \subseteq \mathrm{PAR}_{\mathbb{R}}^{A}$).

Indeed, any computation in $\mathrm{DPAT}_{\mathbb{R}}^{A}$ makes only a polynomial number of guesses (digital, either universal or existential). Therefore, one can simulate the computation in parallel polynomial time by independently computing its outcome for the exponential number of possible guesses (each of these computations being in $\mathrm{P}_{\mathbb{R}}^{A}$) and then checking whether the set of outcomes satisfy the prefix of quantifiers. This shows the claim.

Now Proposition 5.2 follows from this claim by taking $A = \mathrm{PH}_{\mathbb{R}} \subseteq \mathrm{PAR}_{\mathbb{R}}$ and noting that if $A \subseteq \mathrm{PAR}_{\mathbb{R}}$ then $\mathrm{PAR}_{\mathbb{R}}^{A} = \mathrm{PAR}_{\mathbb{R}}$.

(ii) A natural question arising from Proposition 5.2 is whether $\mathrm{PH}_{\mathbb{R}}^{\mathrm{DPAT}_{\mathbb{R}}} \subseteq \mathrm{PAR}_{\mathbb{R}}$. We do not have an answer for it. Actually we note that we do not have a result similar to Lemma 5.1 (that would characterize $\mathrm{PH}_{\mathbb{R}}^{\mathrm{DPAT}_{\mathbb{R}}}$ by alternating first real quantifiers—with a bounded number of alternations—and then digital ones) nor can we show that the inclusion $\mathrm{PH}_{\mathbb{R}} \subseteq \mathrm{PAR}_{\mathbb{R}}$ relativizes (the proofs known for this inclusion being too involved (e.g., [9])).

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