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The majorization theorem of connected graphs ☆

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ABSTRACT

Let $\pi=(d_1,d_2,\ldots,d_n)$ and $\pi'=(d_1',d_2',\ldots,d_n')$ be two non-increasing degree sequences. We say π is majorizated by π', denoted by $\pi \lhd \pi'$, if and only if $\pi \neq \pi'$, $\sum_{i=1}^{n} d_i=\sum_{i=1}^{n} d_i'$, and $\sum_{i=1}^{j} d_i \leqslant \sum_{i=1}^{j} d_i'$ for all $j=1,2,\ldots,n$. If the degree of vertex v is (resp. not) equal to 1, then we call v a pendant (resp. non-pendant) vertex of G. We use C_π to denote the class of connected graphs with degree sequence π. Suppose π and π' are two non-increasing c-cyclic degree sequences. Let G and G' be the graphs with greatest spectral radii in C_π and $C_{\pi'}$, respectively. In this paper, we shall prove that if $\pi \lhd \pi'$, G and G' have the same number of pendant vertices, and the degrees of all non-pendant vertices of G' are greater than c, then $\rho(G)<\rho(G')$.

1. Introduction

Throughout the paper, $G=(V,E)$ is a connected undirected simple graph with $V=\{v_1,v_2,\ldots,v_n\}$ and $E=\{e_1,e_2,\ldots,e_m\}$, i.e., $|V|=n$ and $|E|=m$. If $m=n+c-1$, then G is called a *c-cyclic graph*. Especially, if $c=1$, then G is called a *unicyclic graph*. Let uv be an edge, of which the end vertices are u and v. The symbol $N(v)$ denotes the neighbor set of vertex v, then $d(v)=|N(v)|$ is called the degree of v. If the degree of vertex v is (resp. not) equal to 1, then we call v a *pendant* (resp. *non-pendant*) *vertex* of G.

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Let $A(G)$ be the adjacency matrix of G. The spectral radius of G, denoted by $\rho(G)$, is the largest eigenvalue of $A(G)$. When G is connected, $A(G)$ is irreducible and by the Perron-Frobenius Theorem (see, e.g. [1]), $\rho(G)$ is simple and there is a unique positive unit eigenvector corresponding to $\rho(G)$. We refer to such an eigenvector f as the *Perron vector* of G.

If $d_i = d(v_i)$ for $i = 1, 2, \ldots, n$, then we call the sequence $\pi = (d_1, d_2, \ldots, d_n)$ the *degree sequence* of G. Throughout this paper, we enumerate the degrees in non-increasing order, i.e., $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_n$.

A non-increasing sequence $\pi = (d_1, d_2, \ldots, d_n)$ is called *graphic* if there exists a graph having π as its degree sequence. It is called a *c-cyclic degree sequence*, if it is the degree sequence of some connected c-cyclic graph. Specially, if there exists a connected unicyclic graph with π as its degree sequence, then π is called a *unicyclic degree sequence*.

We use C_π to denote the class of connected graphs with degree sequence π. If $G \in C_\pi$ and $\rho(G) \geqslant \rho(G')$ for any other $G' \in C_\pi$, then we call G has *greatest spectral radius* in C_π.

Suppose $\pi = (d_1, d_2, \ldots, d_n)$ and $\pi' = (d'_1, d'_2, \ldots, d'_n)$ are two non-increasing graphic degree sequences, we write $\pi \lhd \pi'$ if and only if $\pi \neq \pi'$, $\sum_{i=1}^{n} d_i = \sum_{i=1}^{n} d'_i$, and $\sum_{i=1}^{j} d_i \leqslant \sum_{i=1}^{j} d'_i$ for all $j = 1, 2, \ldots, n$. Such an ordering is sometimes called *majorization*.

The work on determining the graph which has greatest spectral radius among some class of graphs, can be traced back to 1985 when Brualdi and Hoffman [2] investigated the maximum spectral radius of the adjacency matrix of a (not necessarily connected) graph in the set of all graphs with a given number of vertices and edges. Their work was followed by other people, in the connected graph case as well as in the general case, and a number of literatures have been written. Recently, Bıyıkoğlu and Leydold had firstly considered the majorization theorem for the graphs, which have greatest spectral radii, between two degree sequences, and they once obtained.

Theorem A [3]. *Let π and π' be two different non-increasing graphic degree sequences with $\pi \lhd \pi'$. Let G and G' be the graphs with greatest spectral radii in C_π and $C_{\pi'}$, respectively. Then, $\rho(G) < \rho(G')$.*

Unfortunately, the following example shows that Theorem A is not correct.

Example 1.1. Let $\pi = (4, 3, 3, 3, 2, 2, 1)$ and $\pi' = (4, 4, 3, 2, 2, 2, 1)$. Let G_1 and G_2 be the graphs as shown in Fig. 1. It is easy to see that (see the data of the spectra of connected graphs with seven vertices [4, pp. 163–220]) G_1 and G_2 are the graphs with greatest spectral radii in C_π and $C_{\pi'}$, respectively. Clearly, $\pi \lhd \pi'$, but $\rho(G_1) = 3.09787 > 3.05401 = \rho(G_2)$.

Very recently, Bıyıkoğlu and Leydold had changed Theorem A from the general graphs to the class of trees, i.e.,

Theorem B [5]. *Let π and π' be two different non-increasing degree sequences of trees with $\pi \lhd \pi'$. Let T and T' be the trees with greatest spectral radii in C_π and $C_{\pi'}$, respectively. Then, $\rho(T) < \rho(T')$.*

In this note, we consider the similar problem to the general graphs with additional restrictions, and we shall prove that

Theorem 1.1. *Suppose $\pi = (d_1, d_2, \ldots, d_n)$ and $\pi' = (d'_1, d'_2, \ldots, d'_n)$ are two different non-increasing c-cyclic degree sequences. Let G and G' be the graphs with greatest spectral radii in C_π and $C_{\pi'}$, respectively.*

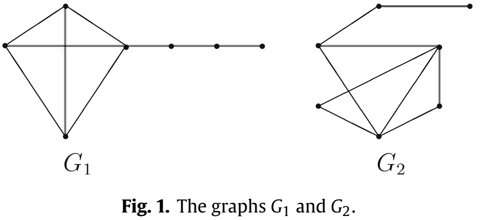

Fig. 1. The graphs G_1 and G_2.

If $\pi \lhd \pi'$, G and G' have the same number of pendant vertices, and the degrees of all non-pendant vertices of G' are greater than c, then $\rho(G) < \rho(G')$.

By Theorem 1.1, it is easy to follow that

Corollary 1.1. *Suppose $\pi = (d_1, d_2, \ldots, d_n)$ and $\pi' = (d'_1, d'_2, \ldots, d'_n)$ are two different non-increasing unicyclic degree sequences. Let G and G' be the unicyclic graphs with greatest spectral radii in C_π and $C_{\pi'}$, respectively. If $\pi \lhd \pi'$, G and G' have the same number of pendant vertices, then $\rho(G) < \rho(G')$.*

Remark. By Example 1.1, the condition of "the degrees of all non-pendant vertices of G' are greater than c" in Theorem 1.1 cannot be deleted.

2. The proof of Theorem 1.1

Suppose $uv \in E$, the notion $G - uv$ denotes the new graph yielded from G by deleting the edge uv. Similarly, if $uv \notin E$, then $G + uv$ denotes the new graph obtained from G by adding the edge uv.

Lemma 2.1 [6,7]. *Let u, v be two vertices of the connected graph G, and $w_1, w_2, \ldots, w_k$ $(1 \leqslant k \leqslant d(v))$ be some vertices of $N(v) \setminus N(u)$. Let $G' = G + w_1u + w_2u + \cdots + w_ku - w_1v - w_2v - \cdots - w_kv$. Suppose f is a Perron vector of G, if $f(u) \geqslant f(v)$, then $\rho(G') > \rho(G)$.*

Given a graphic degree sequence $\pi = (d_1, d_2, \ldots, d_n)$, let $\pi(1)$ denote the cardinality of 1 in π, and $d(\pi) = min\{d_i : d_i \neq 1$ and d_i is a component of $\pi\}$. We use $min(\pi)$ to denote the minimum component of π, i.e., $min(\pi) = d_n$.

By the Theorem 1 of [5], the next result follows immediately.

Lemma 2.2. *Suppose $\pi = (d_1, d_2, \ldots, d_n)$ is a non-increasing c-cyclic degree sequence. If G has greatest spectral radius in C_π with the Perron vector f, then there exists an ordering of $V(G) = \{v_1, v_2, \ldots, v_n\}$ such that $d(v_i) = d_i$ for $1 \leqslant i \leqslant n$, and $f(v_1) \geqslant f(v_2) \geqslant \cdots \geqslant f(v_n)$.*

Lemma 2.3. *If $\pi \lhd \pi'$, then $min(\pi) \geqslant min(\pi')$.*

Proof. Suppose $\pi = (d_1, d_2, \ldots, d_n)$ and $\pi' = (d'_1, d'_2, \ldots, d'_n)$. Assume that the contrary holds, i.e., $d_n < d'_n$. Since $\pi \lhd \pi'$, then $\sum_{i=1}^{n} d_i = \sum_{i=1}^{n} d'_i$. Combining with $d_n < d'_n$, we have $\sum_{i=1}^{n-1} d_i > \sum_{i=1}^{n-1} d'_i$, a contradiction to the definition of $\pi \lhd \pi'$. Thus, $d_n \geqslant d'_n$ follows. $\square$

Lemma 2.4. *Let $\pi = (d_1, d_2, \ldots, d_n)$ and $\pi' = (d'_1, d'_2, \ldots, d'_n)$ be two non-increasing degree sequences with $min(\pi') \geqslant 1$. If $\pi \lhd \pi'$ and only two components of π and π' are different from 1, then $\pi'(1) \geqslant \pi(1)$.*

Proof. Without loss of generality, we may assume that $d_i = d'_i$ for $i \neq p, q$, and $d_p + 1 = d'_p$, $d_q - 1 = d'_q$. Since $\pi \lhd \pi'$, then $1 \leqslant p < q \leqslant n$. Thus, $d_p \geqslant d_q = d'_q + 1 \geqslant min(\pi') + 1 \geqslant 2$. This implies that $\pi'(1) \geqslant \pi(1)$. $\square$

Lemma 2.5 [8]. *Let π and π' be two different non-increasing graphic degree sequences. If $\pi \lhd \pi'$, then there exists a series non-increasing graphic degree sequences $\pi_1, \ldots, \pi_k$ such that $(\pi =)\pi_0 \lhd \pi_1 \lhd \cdots \lhd \pi_k \lhd \pi_{k+1}(= \pi')$, and only two components of π_i and π_{i+1} are different from 1, where $0 \leqslant i \leqslant k$.*

Lemma 2.6. *Let π and π' be two different non-increasing c-cyclic degree sequences with $\pi(1) = \pi'(1)$ and $d(\pi') \geqslant c + 1$. If $\pi \lhd \pi'$, then there exists a series non-increasing c-cyclic degree sequences $\pi_1, \ldots, \pi_k$ such that $(\pi =)\pi_0 \lhd \pi_1 \lhd \cdots \lhd \pi_k \lhd \pi_{k+1}(= \pi')$ with $\pi(1) = \pi_1(1) = \cdots = \pi_k(1) = \pi'(1)$, $d(\pi) \geqslant d(\pi_1) \geqslant \cdots \geqslant d(\pi_k) \geqslant d(\pi')$, and only two components of π_i and π_{i+1} are different from 1, where $0 \leqslant i \leqslant k$.*

Proof. Since $\pi \lhd \pi'$, by Lemma 2.5 there exists a series non-increasing graphic degree sequences $\pi_1, \ldots, \pi_k$ such that $(\pi =)\pi_0 \lhd \pi_1 \lhd \cdots \lhd \pi_k \lhd \pi_{k+1}(=\pi')$, and only two components of π_i and π_{i+1} are different from 1, where $0 \leqslant i \leqslant k$. By Lemma 2.3, we can conclude that $min(\pi) \geqslant min(\pi_1) \geqslant \cdots \geqslant min(\pi_k) \geqslant min(\pi') \geqslant 1$. Thus, $\pi(1) \leqslant \pi_1(1) \leqslant \cdots \leqslant \pi_k(1) \leqslant \pi'(1)$ follows from Lemma 2.4. Moreover, since $\pi(1) = \pi'(1)$, then $\pi(1) = \pi_1(1) = \cdots = \pi_k(1) = \pi'(1)$ follows.

In the following, let $\pi_i = (d_1, d_2, \ldots, d_n)$ and $\pi_{i+1} = (d'_1, d'_2, \ldots, d'_n)$. Since only two components of π_i and π_{i+1} are different from 1, we may assume that $d_j = d'_j$ for $j \neq p, q$, and $d_p + 1 = d'_p$, $d_q - 1 = d'_q$. We only need to show the following facts:

Fact 1. $d(\pi) \geqslant d(\pi_1) \geqslant \cdots \geqslant d(\pi_k) \geqslant d(\pi')$.

Proof of Fact 1. It is sufficient to show that $d(\pi_i) \geqslant d(\pi_{i+1})$ for $0 \leqslant i \leqslant k$. Since $\pi_i \lhd \pi_{i+1}$, then $1 \leqslant p < q \leqslant n$. Thus, $d_p \geqslant d_q = d'_q + 1 \geqslant min(\pi_{i+1}) + 1 \geqslant 2$. Combining with $\pi_i(1) = \pi_{i+1}(1)$, then $d'_q \neq 1$ (Otherwise, $\pi_i(1) < \pi_{i+1}(1)$). Thus, $d(\pi_i) \geqslant d(\pi_{i+1})$.

Fact 2. Each π_i is a c-cyclic degree sequence for all $1 \leqslant i \leqslant k$.

Proof of Fact 2. It is sufficient to show that: For each $i \in \{0, 1, \ldots, k\}$, if there exists a connected c-cyclic graph G with π_i as its degree sequence, then there must exist a connected c-cyclic graph G' with π_{i+1} as its degree sequence. Once this is proved, we are done.

Since $\pi_i \lhd \pi_{i+1}$, then $d_p \geqslant d_q \geqslant 2$. Recall that $\pi_i(1) = \pi_{i+1}(1)$, and $d_j = d'_j$ for $j \neq p, q$, then $d'_q \neq 1$. By Fact 1, $d_q = d'_q + 1 \geqslant d(\pi') + 1 \geqslant c + 2$. Let $P_{v_p v_q}$ be a shortest path from v_p to v_q in G. Note that G is a connected c-cyclic graph and $d_q \geqslant c + 2$, then there must exist some $w \in N(v_q) \setminus N(v_p)$, but $w \notin P_{v_p v_q}$ (Otherwise, G is not a c-cyclic graph). Let $G' = G + v_p w - v_q w$, then G' is also a connected c-cyclic graph with π_{i+1} as its degree sequence.

This completes the proof of this lemma. $\square$

Lemma 2.7. *Let $\pi = (d_1, d_2, \ldots, d_n)$ and $\pi' = (d'_1, d'_2, \ldots, d'_n)$ be two non-increasing c-cyclic degree sequences with $\pi(1) = \pi'(1)$ and $d(\pi') \geqslant c + 1$. Let G_1 and G_2 be the connected c-cyclic graphs with greatest spectral radii in C_π and $C_{\pi'}$, respectively. If $\pi \lhd \pi'$ and only two components of π and π' are different from* 1, *then $\rho(G_1) < \rho(G_2)$.*

Proof. Notice that G_1 has greatest spectral radius in C_π, by Lemma 2.2 there exists an ordering of $V(G_1) = \{v_1, v_2, \ldots, v_n\}$ such that $d(v_i) = d_i$ for $1 \leqslant i \leqslant n$, and $f(v_1) \geqslant f(v_2) \geqslant \cdots \geqslant f(v_n)$. Recall that only two components of π and π' are different from 1, we may assume that $d_i = d'_i$ for $i \neq p, q$, and $d_p + 1 = d'_p$, $d_q - 1 = d'_q$. Since $\pi \lhd \pi'$, then $1 \leqslant p < q \leqslant n$. This implies that $d_p \geqslant d_q \geqslant 2$ and $f(v_p) \geqslant f(v_q)$.

Let $P_{v_p v_q}$ be a shortest path from v_p to v_q in G_1. Since $\pi(1) = \pi'(1)$, $d_p \geqslant d_q \geqslant 2$, then $d'_q \neq 1$. Thus, $d_q = d'_q + 1 \geqslant d(\pi') + 1 \geqslant c + 2$. This guarantees that there must exist some $w \in N(v_q) \setminus N(v_p)$, but $w \notin P_{v_p v_q}$. Let $G' = G_1 + v_p w - v_q w$, then $G' \in C_{\pi'}$. Moreover, since $f(v_p) \geqslant f(v_q)$, then $\rho(G_1) < \rho(G')$ by Lemma 2.1. This implies that $\rho(G_1) < \rho(G_2)$ because G_2 has greatest spectral radius in $C_{\pi'}$. $\square$

The Proof of Theorem 1.1. Since $G \in C_\pi$, $G' \in C_{\pi'}$, G and G' have the same number of pendant vertices, and the degrees of all non-pendant vertices of G' are greater than c, then $\pi(1) = \pi'(1)$ and $d(\pi') \geqslant c + 1$. Combining with $\pi \lhd \pi'$, by Lemma 2.6 there exists a series non-increasing c-cyclic degree sequences $\pi_1, \ldots, \pi_k$ such that $(\pi =)\pi_0 \lhd \pi_1 \lhd \cdots \lhd \pi_k \lhd \pi_{k+1}(=\pi')$ with $\pi(1) = \pi_1(1) = \cdots = \pi_k(1) = \pi'(1)$, $d(\pi) \geqslant d(\pi_1) \geqslant \cdots \geqslant d(\pi_k) \geqslant d(\pi') \geqslant c + 1$, and only two components of π_i and π_{i+1} are different from 1, where $0 \leqslant i \leqslant k$.

Let G_i be the connected c-cyclic graph with greatest spectral radius in C_{π_i} for $1 \leqslant i \leqslant k$. By Lemma 2.7, we can conclude that $\rho(G) < \rho(G_1) < \cdots < \rho(G_k) < \rho(G')$. Thus, Theorem 1.1 follows. $\square$

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