A SIMPLE PROOF OF SOME CONGRUENCES FOR COLORED GENERALIZED FROBENIUS PARTITIONS

Louis Worthy KOLITSCH
Department of Mathematics and Computer Science, University of Tennessee at Martin, Martin, Tennessee 38238, USA

Received 26 January 1988

In this paper we present a very simple analytic proof of some congruences for generalized Frobenius partitions with k colors. The proof highlights yet another combinatorial property of these objects.

In the author's doctoral thesis and in two recent papers [2, 3] the following congruences for colored generalized Frobenius partitions, colored F-partitions for short, were proven:

\[ \sum_{d \mid (k, n)} \mu(d) \phi_{k/d,s} \left( \frac{n}{d} \right) = 0 \pmod{k^2}, \quad \text{provided } (s + 1, k) = 1 \text{ if } s > 1, \]

where \( \phi_{h,s}(r) \) is the number of F-partitions of \( r \) using \( h \) colors with (at most) \( s \) repetitions where \( s \) can be any positive integer or \( \infty \) (to represent no restriction on repetitions).

The proofs of these congruences were based on some interesting congruence properties of compositions and were combinatorial in nature. Though the proofs were straightforward, they were somewhat lengthy and tedious. During some recent work with colored F-partitions the following analytic proof of the above congruences was discovered. This alternate proof highlights yet another combinatorial property of these colored F-partitions.

As was shown in each of the papers [2, 3] the sum,

\[ \sum_{d \mid (k, n)} \mu(d) \phi_{k/d,s} \left( \frac{n}{d} \right) \]

enumerates the number of F-partitions of \( n \) using \( k \) colors with \( s \) repetitions whose order is \( k \) under cyclic permutation of the \( k \) colors, denoted by \( c_{k,s}(n) \). This can be seen very easily by noting that \( c_{k/d,s}(n/d) \) is the number of F-partitions of \( n \) using \( k \) colors with \( s \) repetitions whose order under cyclic permutation of the \( k \) colors divides \( k/d \). In each F-partition of \( n/d \) enumerated by \( c_{k/d,s}(n/d) \) simply repeat each part \( d \) times and increment the color by \( k/d \) each time. A simple inclusion/exclusion argument completes the observation.

We will now separate the F-partitions enumerated by \( c_{k,s}(n) \) into \( k \) equal classes so that any \( k \) F-partitions equivalent under cyclic permutation of the \( k \)
colors will be in the same class. In doing this we will have \( k \) classes with the cardinality of each class divisible by \( k \). Hence \( k^2 \) divides \( c_{k,s}(n) \) which is the desired result.

The classes are described as follows. Each class, whose cardinality is denoted by \( c_{k,s}(j, k, n) \) for \( j = 0, 1, \ldots, k - 1 \), will contain those \( F \)-partitions enumerated by \( c_{k,s}(n) \) in which the difference in the colors on the top row and the bottom row is congruent to \( j \) modulo \( k \). In other words, an \( F \)-partition is counted by \( c_{k,s}(j, k, n) \) if \( d(1) + 2d(2) + \cdots + kd(k) = j (\text{mod } k) \) where \( d(i) \) is the number of appearances of color \( i \) on the top row minus the number of appearances of color \( i \) on the bottom row as defined in [1]. Clearly any two \( F \)-partitions equivalent under cyclic permutation of the \( k \) colors will be in the same class.

We now consider the generating functions for \( c_{k,s}(n) \) and include a parameter \( t \) that will keep track of the difference in the colors on the top and bottom rows. If we let \( c_{k,s}(m, n) \) denote the number of \( F \)-partitions of \( n \) using \( k \) colors with \( s \) repetitions whose order is \( k \) and whose difference in colors is \( m \), we have

\[
\sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} c_{k,s}(m, n) t^m q^n = \sum_{j=0}^{k-1} \sum_{m=j \pmod{k}}^{\infty} c_{k,s}(m, n) t^m q^n
\]

If we replace \( t \) by an \( r \)th root of unity, other than \( t = 1 \), with \( r \mid k \), we see that

\[
\sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} c_{k,s}(m, n) t^m q^n = \sum_{j=0}^{k-1} t^j \sum_{n=0}^{\infty} c_{k,s}(m, n) q^n = \sum_{j=0}^{k-1} t^j \sum_{n=j \pmod{k}}^{\infty} c_{k,s}(m, n) q^n
\]

\[
= \sum_{n=0}^{\infty} \left( \sum_{j=0}^{k-1} c_{k,s}(j, k, n) t^j \right) q^n.
\]

It is not difficult to see that each of the divisor sums is now identically zero provided in the second sum \((s + 1, k) = 1\).

Thus \( \Sigma_{j=0}^{k-1} c_{k,s}(j, k, n) t^j = 0 \) for all \( n \) and all \( k \)th roots of unity except \( t = 1 \).

Therefore

\[
c_{k,s}(0, k, n) = c_{k,s}(1, k, n) = \cdots = c_{k,s}(k - 1, k, n) = \frac{1}{k} c_{k,s}(k, n),
\]
References

[2] L.W. Kolitsch, A Congruence for $c\phi_{n,k}(n)$, to be published.