Galois objects for algebraic quantum groups

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A R T I C L E   I N F O

A B S T R A C T

The basic elements of Galois theory for algebraic quantum groups were given in the paper 'Galois Theory for Multiplier Hopf Algebras with Integrals' by Van Daele and Zhang. In this paper, we supplement their results in the special case of Galois objects: algebras equipped with a Galois coaction by an algebraic quantum group, such that only the scalars are coinvariants. We show how the structure of these objects is as rich as the one of the quantum groups themselves: there are two distinguished weak K.M.S. functionals, related by a modular element, and there is an analogue of the antipode squared. We show how to reflect the quantum group across a Galois object to obtain a (possibly) new algebraic quantum group. We end by considering an example.

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Introduction

The theory of *Hopf–Galois extensions* centers around special kinds of extensions of unital k-algebras $F \subseteq X$ over, say, a field k. They are of the following form: There should exist a Hopf algebra (A, Δ) over k and a coaction $\alpha : X \to X \otimes_k A$ of (A, Δ) on X, such that F is the set of coinvariant elements for α, and such that the map

$$X \underset{F}{\otimes} X \to X \underset{k}{\otimes} A : x \underset{F}{\otimes} y \to (x \otimes 1)\alpha(y)$$

is a bijection. Coactions satisfying this last condition are called *Galois coactions*. In special situations, one can associate a canonical (A, Δ) and α to such an extension [14], but most of the time one takes some (A, Δ) and α as part of the data. In case X and A are commutative, one may regard X as

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the function space of a bundle over the spectrum of F, with the spectrum of A acting freely and transitively on every fiber.

An interesting situation arises when F reduces to the scalar field k. In this case (X, α) is called a *Galois object* for (A, Δ). The most famous instance of this is the case where X is a finite field extension of k and A is the function algebra (w.r.t. k) of a finite group G: then a coaction α of (A, Δ) on X will make (X, α) into a Galois object for (A, Δ) if and only if $k \subseteq X$ is a Galois extension with G as its Galois group (by a result of Chase, Harrison and Rosenberg).

An important aspect of Hopf–Galois extensions is that they provide equivalences of certain categories, see e.g. [24] for an overview. As for Galois objects, their isomorphism classes are in one-to-one correspondence with the isomorphism classes of monoidal equivalences from the category of comodules of (A, Δ) to those monoidal categories which are the category of comodules for some Hopf algebra (see Theorem 3.2.2 of [24] for the more precise statement).

In the operator algebra framework, the same objects appear under a different name. In [1] a certain class of ergodic coactions of compact quantum groups on C*-algebras was created, for which there exist, in certain cases, a spectral subspace containing the corresponding irreducible corepresentation of the quantum group with a multiplicity *strictly greater* than the (classical) dimension of this corepresentation (which is impossible for an ordinary compact group, or even a compact quantum group of Kac type). These constructed coactions were named 'of full quantum multiplicity', which is precisely the condition that the coaction comes from an associated Galois object (when restricting all C*-algebras to natural dense *-subalgebras). However, the authors made use of the particular nature of the dual of a compact quantum group, which consists of a direct sum of matrix algebras, and also of some analytic machinery, which is somewhat in contrast with the techniques from the Hopf algebra approach.

Motivated by this, we try to develop in this paper a theory of Galois objects which comprises both the compact quantum group theory, and the Hopf-algebraic theory in case the Hopf algebra has integrals. Namely, we will consider the structure of Galois objects for algebraic quantum groups. These latter objects were developed by Van Daele in [26], motivated in turn by finding the right infinite-dimensional generalization of a finite-dimensional Hopf algebra which still allows for a dual object of the same kind, and in providing a purely algebraic framework for the study of some of the aspects of locally compact quantum groups. The main differences with a general ordinary Hopf algebra are the possible lack of a unit in the algebra, and the existence of a non-zero left invariant functional.

A first study of the general Hopf–Galois theory for algebraic quantum groups appeared in [27], whose main result is a Morita context between F and the smash product $X \# \widehat{A}$ for an (A, Δ)-Galois extension $F \subseteq X$ by an algebraic quantum group (A, Δ). As already said, our Galois objects will specialize this to the situation in which $F = k$, the ground field.

We come to the specific content of this paper. *In the first part*, we study the further algebraic structure of a Galois object (X, α) over an algebraic quantum group (A, Δ). The main results are the following. There are two distinguished functionals φ_X and ψ_X on X, with ψ_X invariant with respect to the action α. They are related by an invertible element δ_X inside the multiplier algebra $M(X)$ of X, namely $\varphi_X(\cdot \, \delta_X) = \psi_X$. They also satisfy the weak K.M.S. condition: if ω is a functional on X, we say that it satisfies this condition if there exists an algebra automorphism σ_ω of X such that $\omega(y\sigma_\omega(x)) = \omega(xy)$ for all $x, y \in X$. We call a functional satisfying this condition a *modular* functional. It is interesting to see this structure, much used in the theory of von Neumann algebras, appear in a natural way in a purely algebraic (non-finite-dimensional) setting. There also is a distinguished automorphism θ_X on X, which plays the rôle of the antipode squared. Finally, we associate a scaling constant to X, and show that it equals the scaling constant of A. Thus (X, α) is almost as rich in structure as (A, Δ). We also show that when working with a well-behaving *-structure on X and A, all these maps become simultaneously diagonalizable. We end this part by considering what happens when (A, Δ) is of a special type, namely compact or discrete.

In the second part, we construct a new algebraic quantum group (C, Δ_C), starting from a Galois object (X, α) for an (A, Δ). This (C, Δ_C) also comes with a left coaction γ on X, commuting with α and making (X, γ) into a (left) Galois object. Then (X, α, γ) can be called an A-C-bi-Galois object. Note that when (A, Δ) is of compact type (in particular, a Hopf algebra), there is also a Hopf algebraic method to obtain a new Hopf algebra from a Galois object (see [22]). This new Hopf algebra will then

coincide with the algebraic quantum group (C, Δ_C) we construct (by uniqueness), but we want to emphasize that in our situation, the Hopf algebraic construction has to be supplemented with the construction of an invariant functional on the new Hopf algebra. After this, we briefly show how A, X and C can also be grouped together inside an even bigger structure, which can be seen as a (very special kind of) 'algebraic quantum groupoid'. Next, we show the connection between bi-Galois objects and monoidal equivalence of module categories, spending some more time on the *-algebraic case. (We have not investigated if this monoidal equivalence of module categories actually *characterizes* bi-Galois objects.) In our calculations, we make essential use of a natural subspace of the dual of X.

In the third part, we examine a concrete example, which will allow us to construct *new* examples of algebraic quantum groups of compact type. It illustrates how duality can be used to find in a fairly easy way the structure of a reflected quantum group. Although we do not leave the Hopf algebraic framework in this example, we do feel that our approach has some novelty to it.

In Appendix A, we repeat some notions concerning multiplier algebras and algebraic quantum groups.

We now set down conventions and notations. We work over a fixed field k, i.e. all algebras are k-algebras. Moreover, all algebras appearing are non-trivial ($0 \neq 1$) and non-degenerate (which means that sending an element to the map 'left (resp. right) multiplication with it' is injective). We denote the multiplier algebra of an algebra X as $M(X)$, and identify X with its image inside $M(X)$ whenever convenient. We also identify k with its image $k \cdot 1$ in $M(X)$ when convenient. We denote the tensor product over k by $\otimes$. Whenever it is harmless, we will use the elementary tensor notation $x \otimes y$ for a general element of a tensor product (in particular, we use this convention together with the Sweedler notation). We mostly work with a fixed algebraic quantum group (A, Δ) over k (note that the theory in [26] is developed in the case $k = \mathbb{C}$, but everything works just as well in the general case). We denote its antipode with S and its counit with ε. We denote by φ a non-zero left invariant functional, positive if A is a *-algebraic quantum group. As a right invariant functional we choose $\psi = \varphi \circ S$. We denote the modular element by δ, and the modular automorphisms of φ and ψ by respectively σ and σ'. We take the algebraic convention for the coproduct $\Delta_{\widehat{A}}$ on the dual $\widehat{A}$ of A: it is determined by $\Delta_{\widehat{A}}(\omega_1)(x \otimes y) = \omega_1(xy)$ for all $x, y \in A$, $\omega_1 \in \widehat{A}$ (when properly interpreted). The left integral $\varphi_{\widehat{A}}$ of $\widehat{A}$ is given by $\varphi_{\widehat{A}}(\psi(a \cdot)) = \varepsilon(a)$. For the further theory of algebraic quantum groups, we refer the reader to [26] and Appendix A.

1. The structure of Galois objects

1.1. Definitions

We begin by recalling some definitions concerning coactions for algebraic quantum groups.

Let X be an algebra. Let α be a *right coaction* of our fixed algebraic quantum group (A, Δ) on X: it is an injective homomorphism $X \to M(X \otimes A)$ satisfying

(i) $\alpha(X)(1 \otimes A) = X \otimes A$,
(ii) $(1 \otimes A)\alpha(X) = X \otimes A$, and
(iii) $(\alpha \otimes \iota)\alpha = (\iota \otimes \Delta)\alpha$,

where the third property can be made sense of for example by using the unique extensions of $(\alpha \otimes \iota)$ and $(\iota \otimes \Delta)$ to homomorphisms $M(X \otimes A) \to M(X \otimes A \otimes A)$ (which exist by the first two conditions). The maps $X \otimes A \to X \otimes A$ given by

$$T_1 : x \otimes a \to \alpha(x)(1 \otimes a),$$

$$T_2 : x \otimes a \to (1 \otimes a)\alpha(x)$$

are then well-defined bijections, their inverses determined by

$$T_1^{-1} : y \otimes S(b) \to (\iota \otimes S)\big((1 \otimes b)\alpha(y)\big),$$

$$T_2^{-1} : y \otimes S^{-1}(b) \to \big(\iota \otimes S^{-1}\big)\big(\alpha(y)(1 \otimes b)\big).$$

The injectivity of α implies that $(\iota \otimes \varepsilon)(\alpha(x)) = x$ for all $x \in X$ (where a priori the left-hand side has to be treated as a multiplier).

The coaction is called *reduced* if in addition $(X \otimes 1)\alpha(X) \subseteq X \otimes A$. The other inclusion $\alpha(X)(X \otimes 1) \subseteq X \otimes A$ follows from this (see the remark after Proposition 2.5 in [27]).

We can extend α to a map $M(X) \to M(X \otimes A)$, using the bijections T_1 and T_2: if $z \in M(X)$, we can define $\alpha(z)$ by the identities $\alpha(z)(\alpha(y)(1 \otimes a)) = \alpha(zy)(1 \otimes a)$ and $((1 \otimes a)\alpha(y))\alpha(z) = (1 \otimes a)\alpha(yz)$ for $y \in X$ and $a \in A$. The *algebra of coinvariants* $F \subseteq M(X)$ is then defined as the set of elements f in $M(X)$ such that $\alpha(f) = f \otimes 1$. Remark that it is a unital algebra.

The following terminology comes from [27]: the coaction α is called *Galois*, if it is reduced and if the map

$$V : X \underset{F}{\otimes} X \to X \otimes A : x \underset{F}{\otimes} y \to (x \otimes 1)\alpha(y)$$

is bijective. In fact, the bijectivity already follows from the surjectivity of this map (see Theorem 4.4 in [27]). Also, the map

$$W : X \underset{F}{\otimes} X \to X \otimes A : x \underset{F}{\otimes} y \to \alpha(x)(y \otimes 1)$$

is then bijective, with the inverse map given by

$$W^{-1}(x \otimes a) = V^{-1}\big(\big(1 \otimes S^{-1}(a)\big)\alpha(x)\big).$$

We can now give a definition of our main object of study:

Definition 1.1. Let α be a right Galois coaction of an algebraic quantum group (A, Δ) on a non-degenerate algebra X. Then (X, α) is called a right *Galois object* for (A, Δ) if the algebra F of coinvariants is the scalar field k.

Sometimes, we also use the expression 'A-Galois object' instead of 'Galois object over (A, Δ),' although this is less precise.

Let (X, α) be an A-Galois object. For $a \in A$, we denote by $\beta(a)$ the element in $M(X \otimes X)$ which satisfies

$$(x \otimes 1)\beta(a) = V^{-1}(x \otimes a),$$

$$\beta(a)(1 \otimes y) = W^{-1}\big(y \otimes S(a)\big).$$

Using the formula for W^{-1} in terms of V^{-1}, it is not difficult to see that $\beta(a)$ is indeed a well-defined multiplier on $X \otimes X$ for each $a \in A$.

We will show later that also the maps

$$x \otimes a \to \beta(a)(x \otimes 1)$$

and

$$x \otimes a \to (1 \otimes x)\beta(a)$$

are bijections from $X \otimes A$ to $X \otimes X$ (see Corollary 1.14). This will allow us to regard β rather as a map $A \to M(X^{\mathrm{op}} \otimes X)$, which is really the better viewpoint.

For computations we will use the Sweedler notation, denoting $\alpha(x)$ as $x_{(0)} \otimes x_{(1)}$ and $\beta(a)$ as $a^{[1]} \otimes a^{[2]}$. Then by definition we have the identities

$$xa^{[1]}a^{[2]}{}_{(0)} \otimes a^{[2]}{}_{(1)} = x \otimes a,$$

$$a^{[1]}{}_{(0)}a^{[2]}x \otimes a^{[1]}{}_{(1)} = x \otimes S(a),$$

for all $x \in X, a \in A$. Applying $\iota \otimes \varepsilon$, we obtain the formula

$$xa^{[1]}a^{[2]} = \varepsilon(a)x.$$

We want to remark and warn however that the use of the Sweedler notation here is more delicate than for Hopf algebras. Indeed, when doing computations with Sweedler notation, it is crucial that all expressions are covered. We refer to [9] for a careful analysis of this technique, and to Appendix A for the more intuitive approach.

In the following subsections, we will work with a fixed Galois object (X, α) for the algebraic quantum group (A, Δ).

1.2. The existence of the invariant functionals

For any functional ω on X, we can interpret $(\omega \otimes \iota)(\alpha(x))$ in a natural way as a multiplier of A. By an invariant functional on X we mean a functional ω on X such that $(\omega \otimes \iota)(\alpha(x)) = \omega(x)1$ for all $x \in X$. By a δ-invariant functional we mean a functional ω on X such that $(\omega \otimes \iota)(\alpha(x)) = \omega(x)\delta$ for all $x \in X$ (where we recall that $\delta \in M(A)$ denotes the modular element of (A, Δ)). For the following proposition, recall also that φ denotes a left invariant functional on (A, Δ).

Theorem 1.2. *There exists a faithful δ-invariant functional φ_X on X such that*

$$(\iota \otimes \varphi)(\alpha(x)) = \varphi_X(x)1$$

for all $x \in X$.

(The notion of faithfulness is recalled in the second paragraph of Section 4.3.)

Proof. Take $x, y \in X$ and $a \in A$. Denote z for $(\iota \otimes \varphi)(\alpha(x)) \in M(X)$. Then we compute in detail, using the definition of the extension of α to $M(X)$, of $(\alpha \otimes \iota)$ to $M(A \otimes A)$, and the defining left invariance property of φ:

$$
\begin{aligned}
\alpha(z)\big(\alpha(y)(1 \otimes a)\big) &= \alpha(zy)(1 \otimes a) \\
&= \alpha\big((\iota \otimes \varphi)\big(\alpha(x)(y \otimes 1)\big)\big)(1 \otimes a) \\
&= (\iota \otimes \iota \otimes \varphi)\big((\alpha \otimes \iota)\big(\alpha(x)(y \otimes 1)\big)(1 \otimes a \otimes 1)\big) \\
&= (\iota \otimes \iota \otimes \varphi)\big((\alpha \otimes \iota)\big(\alpha(x)\big)\big(\alpha(y) \otimes 1\big)(1 \otimes a \otimes 1)\big) \\
&= (\iota \otimes \iota \otimes \varphi)\big((\iota \otimes \Delta)\big(\alpha(x)\big)(y_{(0)} \otimes y_{(1)}a \otimes 1)\big) \\
&= (\iota \otimes \iota \otimes \varphi)\big((\iota \otimes \Delta)\big(\alpha(x)(y_{(0)} \otimes 1)\big)(1 \otimes y_{(1)}a \otimes 1)\big) \\
&= x_{(0)}y_{(0)} \otimes (\iota \otimes \varphi)\big(\Delta(x_{(1)})(y_{(1)}a \otimes 1)\big)
\end{aligned}
$$

$$= x_{(0)} y_{(0)} \otimes \varphi(x_{(1)}) y_{(1)} a$$

$$= (z \otimes 1)\big(\alpha(y)(1 \otimes a)\big),$$

where the reader should make sure for himself that these expressions are all well-covered. By the non-degeneracy of the algebra $X \otimes A$, it follows that $z = (\iota \otimes \varphi)(\alpha(x))$ is coinvariant: $\alpha(z) = z \otimes 1$. So $z = \varphi_X(x)$ for some scalar $\varphi_X(x)$, since (X, α) is a Galois object. It is clear that φ_X then defines a linear functional on X.

We show now that this map φ_X is δ-invariant: for $x, z \in X$ and $a \in A$, we have

$$\varphi_X(x_{(0)}) z \otimes x_{(1)} a = x_{(0)} z \otimes \varphi(x_{(1)}) x_{(2)} a$$

$$= x_{(0)} \varphi(x_{(1)}) z \otimes \delta a$$

$$= \varphi_X(x) z \otimes \delta a,$$

where we used that $(\varphi \otimes \iota)(\Delta(b)) = \varphi(b)\delta$ for $b \in A$.

Finally, we prove faithfulness. Suppose $x \in X$ is such that $\varphi_X(xy) = 0$ for all $y \in X$. Then

$$\varphi(x_{(1)} y_{(1)}) x_{(0)} y_{(0)} z = 0 \quad \text{for all } y, z \in X.$$

Using the Galois property, it follows that

$$\varphi(x_{(1)} a) x_{(0)} y = 0 \quad \text{for all } y \in X \text{ and } a \in A.$$

The faithfulness of φ implies that $x_{(0)} y \otimes x_{(1)} = 0$ for all $y \in X$, hence $x = 0$. Likewise $\varphi_X(yx) = 0$ for all $y \in X$ implies $x = 0$. □

Corollary 1.3. *The algebra X has local units*: *for any finite set of $x_i \in X$, there exists $y \in X$ with $yx_i = x_i y = x_i$ for all i.*

Proof. The proof is the same as the one of Proposition 2.6 in [10], and will be omitted. □

Theorem 1.4. *There exists a non-zero invariant functional ψ_X on X.*

Proof. Choose $y \in X$ and put

$$\psi_X^y(x) = \varphi_X(x_{(0)} y) \psi(x_{(1)}).$$

It is easy to see, using the right invariance property of ψ w.r.t. Δ, that this functional is invariant. Suppose that ψ_X^y is zero for all $y \in X$. Then $\varphi_X(x)\psi(a) = 0$ for all $x \in X$ and $a \in A$, which is impossible. So we can choose as ψ_X some non-zero ψ_X^y. □

We prove a uniqueness result concerning the invariant functionals. We can follow the method of Lemma 3.5 and Theorem 3.7 of [26] verbatim.

Proposition 1.5. *If ψ_X^1 and ψ_X^2 are two invariant non-zero functionals on X, then there exists a scalar $c \in k$ such that $\psi_X^1 = c\psi_X^2$.*

Proof. First, we show that if ψ_X is a non-zero invariant functional on X, then

$$\{\varphi_X(\cdot\, x) \mid x \in X\} = \{\psi_X(\cdot\, x) \mid x \in X\}.$$

Choose x, y, z in X. Then

$$\alpha(xy)(z \otimes 1) = \sum_i \alpha(xw_i)(1 \otimes a_i)$$

for some $w_i \in X$, $a_i \in A$. Likewise starting with $x, w \in X$, $a \in A$, there exist $y_i, z_i \in X$ with

$$\sum_i \alpha(xy_i)(z_i \otimes 1) = \alpha(xw)(1 \otimes a).$$

If we apply $\psi_X \otimes \varphi$ to these expressions we obtain respectively the equalities

$$\varphi_X(xy)\psi_X(z) = \sum_i \psi_X(xw_i)\varphi(a_i),$$

$$\sum_i \varphi_X(xy_i)\psi_X(z_i) = \psi_X(xw)\varphi(a).$$

Choosing either z with $\psi_X(z) = 1$ or a with $\varphi(a) = 1$, we get respectively $\subseteq$ and $\supseteq$.

Suppose now that ψ_X^1 and ψ_X^2 are invariant. Choose $y, z_1 \in X$ with $\varphi_X(yz_1) = 1$ and take $z_2 \in X$ with $\psi_X^1(\cdot\, z_1) = \psi_X^2(\cdot\, z_2)$. Choosing $x \in X$, applying $\psi_X^1 \otimes \varphi$ to $(x \otimes 1)\alpha(yz_i)$ and writing this last expression as $\sum_j (1 \otimes a_j)\alpha(w_j z_i)$ for certain $w_j \in X$, $a_j \in A$, we see that $\psi_X^1(x) = \varphi_X(yz_2)\psi_X^2(x)$, proving that all invariant functionals are scalar multiples of each other. $\square$

1.3. The existence of the modular element

Let ψ_X be a non-zero invariant functional on X. We prove the existence of a modular element δ_X, relating the functionals φ_X and ψ_X. We first prove an important proposition, which is a kind of strong right invariance formula. (It is easily seen, looking at the proof, that this formula is valid for *any* right coaction that has an invariant functional.)

Proposition 1.6. *For all $x, y \in X$ we have*

$$S\big((\psi_X \otimes \iota)\big((x \otimes 1)\alpha(y)\big)\big) = (\psi_X \otimes \iota)\big(\alpha(x)(y \otimes 1)\big).$$

Proof. Choose $a \in A$ and $x, y \in X$. Pick $z_i \in X$ and $b_i \in A$ such that

$$(1 \otimes a)\alpha(y) = \sum_i z_i \otimes b_i.$$

Then by the formula for T_2^{-1} given in the beginning of this section, we have

$$y \otimes S(a) = \sum_i \alpha(z_i)\big(1 \otimes S(b_i)\big).$$

If we denote $w = S((\psi_X \otimes \iota)((x \otimes 1)\alpha(y)))$, then

$$wS(a) = \sum_i \psi_X(xz_i)S(b_i)$$

$$= \sum_i \psi_X(x_{(0)}z_{i(0)})x_{(1)}z_{i(1)}S(b_i)$$

$$= \psi_X(x_{(0)}y)x_{(1)}S(a).$$

Since a was arbitrary, the formula is proven. $\square$

Theorem 1.7. *There exists a unique invertible element $\delta_X \in M(X)$ such that $\varphi_X(x\delta_X) = \psi_X(x)$ for all $x \in X$.*

Proof. We show first that for all $x \in X$:

$$\psi_X(x) = 0 \quad \Rightarrow \quad \psi(x_{(1)})x_{(0)} = 0.$$

We know that $\psi_X(x) = 0$ implies $\psi_X^w(x) = 0$ for all $w \in X$, i.e. $\psi(x_{(1)})\varphi_X(x_{(0)}w) = 0$ for all $w \in X$. So $\psi(x_{(1)})x_{(0)} = 0$ by the faithfulness of φ_X.

This means that $\psi(x_{(1)})x_{(0)} = c_x\delta_X'$ for some multiplier $\delta_X' \in M(X)$ and some number $c_x \in k$. Now $x \to c_x$ is easily seen to be a non-zero invariant functional, and replacing ψ_X by this invariant functional, we obtain $\psi(x_{(1)})x_{(0)} = \psi_X(x)\delta_X'$.

Now we show that δ_X' has an inverse δ_X, and that $\varphi_X(x\delta_X) = \psi_X(x)$. Choose $y \in X$ with $\psi_X(y) = 1$, then

$$\psi_X(x\delta_X') = \psi_X(xy_{(0)})\varphi\big(S(y_{(1)})\big)$$

$$= \psi_X(x_{(0)}y)\varphi(x_{(1)})$$

$$= \varphi_X(x).$$

Since furthermore $\{\varphi_X(\cdot x)\} = \{\psi_X(\cdot x)\}$, we have that for any $x \in X$ there exists $y \in X$ with $y\delta_X' = x$. To show that also left multiplication is surjective, we use another argument. Take $x \in X$ and $a \in A$ with $\psi(a) = 1$. Write $x \otimes a$ as $\sum_i p_{i(0)}q_i \otimes p_{i(1)}$ for certain $p_i, q_i \in X$, and put $y = \sum_i \psi_X(p_i)q_i$. Then $\delta_X'y = \sum_i \psi(p_{i(1)})p_{i(0)}q_i = \psi(a)x = x$.

Hence we obtain the formula

$$\psi_X(x) = \varphi_X(x\delta_X) \quad \text{for all } x \in X.$$

By the faithfulness of φ_X, this uniquely determines δ_X. $\square$

The faithfulness of φ_X then also establishes the following corollary.

Corollary 1.8. *Any non-zero invariant functional is faithful.*

1.4. The modularity of the invariant functionals

We first prove some identities. The first one is yet again a variant on the notion of strong left invariance for an algebraic quantum group.

Proposition 1.9. *For all $x \in X$ and $a \in A$, we have*

(i) $\varphi(ax_{(1)})x_{(0)} = \varphi_X(a^{[2]}x)a^{[1]}$,
(ii) $\varphi(x_{(1)}S(a))x_{(0)} = \varphi_X(xa^{[1]})a^{[2]}$.

Proof. The first equation follows from

$$\varphi(ax_{(1)})zx_{(0)} = \varphi\big(a^{[2]}{}_{(1)}x_{(1)}\big)za^{[1]}a^{[2]}{}_{(0)}x_{(0)}$$
$$= \varphi_X\big(a^{[2]}x\big)za^{[1]},$$

for all $a \in A$ and $x, z \in X$. The second follows from

$$\varphi\big(x_{(1)}S(a)\big)x_{(0)}z = \varphi\big(x_{(1)}a^{[1]}{}_{(1)}\big)x_{(0)}a^{[1]}{}_{(0)}a^{[2]}z$$
$$= \varphi_X\big(xa^{[1]}\big)a^{[2]}z,$$

for all $a \in A$ and $x, z \in X$. $\square$

Lemma 1.10. *For all $y, p, q \in X$ and $a \in A$, we have*

$$\varphi_X\big(a^{[2]}y\big)\varphi_X\big(pa^{[1]}q\big) = \varphi_X\big(yb^{[1]}\big)\varphi_X\big(pb^{[2]}q\big), \quad \text{where } b = \big(S^{-1}\sigma\big)(a).$$

(Recall that σ denotes the modular automorphism for (A, Δ).)

Proof. Using the identities of the previous lemma, we get

$$\varphi_X\big(yb^{[1]}\big)\varphi_X\big(pb^{[2]}q\big) = \varphi\big(y_{(1)}\sigma(a)\big)\varphi_X\big(py_{(0)}q\big)$$
$$= \varphi(ay_{(1)})\varphi_X(py_{(0)}q)$$
$$= \varphi_X\big(a^{[2]}y\big)\varphi_X\big(pa^{[1]}q\big). \quad \square$$

We show now that φ_X is modular.

Theorem 1.11. *There exists an automorphism σ_X of X such that*

$$\varphi_X\big(y\sigma_X(x)\big) = \varphi_X(xy) \quad \text{for all } x, y \in X.$$

Furthermore, $\varphi_X \circ \sigma_X = \varphi_X$.

Proof. Choose $x \in X$, and write x as a sum of elements of the form $\varphi_X(pa^{[1]}q)a^{[2]}$ with $p, q \in X$ and $a \in A$. Define w as $\sum \varphi_X(pb^{[2]}q)b^{[1]}$ with $b = (S^{-1}\sigma)(a)$. Then the previous lemma shows that $\varphi_X(yw) = \varphi_X(xy)$ for all $y \in X$.

It is clear that w is uniquely determined by this property, so we can denote $w = \sigma_X(x)$. Standard arguments imply that σ_X is indeed an algebra automorphism. It will leave φ_X invariant because $X^2 = X$. $\square$

Remark. As for algebraic quantum groups, the concrete way in which σ_X is constructed is not so important. What *is* important is its modular property, which makes up for the fact that φ_X does not have to be tracial.

Corollary 1.12. *The functional ψ_X is modular with modular automorphism*

$$\sigma'_X(x) = \delta_X \sigma_X(x)\delta_X^{-1}.$$

1.5. Formulas

In this section and the next, we collect some formulas. They strongly resemble the formulas which hold in algebraic quantum groups, and also their proofs are mostly straightforward adaptations. Nevertheless, it is remarkable that Galois objects carry as rich a structure as the algebraic quantum groups themselves. These results, together with the ones of the following section, show that one can really consider these Galois objects as a kind of *hybrid* (algebraic) quantum groups.

Proposition 1.13. *For all $a \in A$, we have*

(i) $\alpha \circ \sigma'_X = (\sigma'_X \otimes S^{-2}) \circ \alpha$,
(ii) $((S^{-1}\sigma)(a))^{[1]} \otimes ((S^{-1}\sigma)(a))^{[2]} = \sigma_X(a^{[2]}) \otimes a^{[1]}$.

Proof. Choose $x, y \in X$ and $a \in A$. Then

$$(\psi_X \otimes \varphi)\big((y \otimes a)\alpha\big(\sigma'_X(x)\big)\big) = \psi_X\big(y_{(0)}\sigma'_X(x)\big)\varphi\big(aS^{-1}(y_{(1)})\big)$$
$$= \psi_X(xy_{(0)})\varphi\big(aS^{-1}(y_{(1)})\big)$$
$$= \psi_X(x_{(0)}y)\varphi\big(aS^{-2}(x_{(1)})\big)$$
$$= \psi_X\big(y\sigma'(x_{(0)})\big)\varphi\big(aS^{-2}(x_{(1)})\big),$$

applying Proposition 1.6 twice. As φ and ψ_X are faithful, the first identity follows. The second formula was essentially proven in Lemma 1.10. $\square$

Corollary 1.14. *The maps*

$$x \otimes a \to \beta(a)(x \otimes 1),$$
$$x \otimes a \to (1 \otimes x)\beta(a)$$

are bijections from $X \otimes A$ to $X \otimes X$.

Proof. This follows from the second formula. $\square$

Note that this fact is not at all clear at first sight. It allows us for example to construct the *Miyashita–Ulbrich* action in this context: A acting on the right of X by $x \cdot a = a^{[1]}xa^{[2]}$, making it a Yetter–Drinfel'd module together with α. It also allows us to regard β rather as a map $\tilde{\beta}: A \to M(X^{\mathrm{op}} \otimes X)$. More precisely: denote by $S_{X^{\mathrm{op}}}$ the canonical map $X^{\mathrm{op}} \to X$ sending x^{op} to x for $x \in X$ (where $^{\mathrm{op}}$ is really just some isomorphism of vector spaces from X to a vector space copy X^{op} of X, with X^{op} then endowed with the multiplication which makes $^{\mathrm{op}}$ an anti-isomorphism of algebras. Also, the reason for writing this as if it were an antipode will become clear later on). Then indeed $\tilde{\beta} = (S_{X^{\mathrm{op}}}^{-1} \otimes \iota)\beta$ will be a map with the range $M(X^{\mathrm{op}} \otimes X)$. If however we were ignorant of the previous corollary, we would only know that $\tilde{\beta}(a)$ is a left multiplier of $X^{\mathrm{op}} \otimes X$. Now as is the case for Galois objects over Hopf algebras, the map $\tilde{\beta}$ will be a homomorphism. The argument for this is simple: choose $x \in X$ and $a, b \in A$, and write $xb^{[1]} \otimes b^{[2]} = \sum_i p_i \otimes q_i$ for certain $p_i, q_i \in X$. Then $\sum_i (p_i \otimes a)\alpha(q_i) = x \otimes ab$. Applying V^{-1} (where V was introduced in the beginning of this section), we obtain $\sum_i p_i a^{[1]} \otimes a^{[2]}q_i = x(ab)^{[1]} \otimes (ab)^{[2]}$, so $xb^{[1]}a^{[1]} \otimes a^{[2]}b^{[2]} = x(ab)^{[1]} \otimes (ab)^{[2]}$. This proves that $\tilde{\beta}$ is a homomorphism.

Definition 1.15. We call $\tilde{\beta}: A \to M(X^{\mathrm{op}} \otimes X)$ the *external comultiplication on* A.

We will also use a Sweedler notation for this map now: $\tilde{\beta}(a) = a_{[1]} \otimes a_{[2]}$ for $a \in A$.
The following proposition collects some formulas concerning the modular elements.

Proposition 1.16. *The following identities hold*:

(iii) $\alpha(\delta_X) = \delta_X \otimes \delta$,
(iv) $\tilde{\beta}(\delta) = \delta_{X^{\mathrm{op}}} \otimes \delta_X$ (*with* $\delta_{X^{\mathrm{op}}} = (\delta_X^{-1})^{\mathrm{op}}$),
(v) $\sigma_X(\delta_X) = \tau^{-1}\delta_X$,

where in the last formula τ denotes the scaling constant of A.

Proof. First note that for any $x, y \in X$ we have

$$\varphi_X(xy_{(0)})y_{(1)} = \varphi_X(x_{(0)}y)S^{-1}(x_{(1)})\delta,$$

which is proven in the same way as Proposition 1.6, using the δ-invariance of φ_X. Then if $x, y \in X$, we have

$$\varphi_X\big(x(y\delta_X)_{(0)}\big)(y\delta_X)_{(1)} = \varphi_X(x_{(0)}y\delta_X)S^{-1}(x_{(1)})\delta$$

$$= \psi_X(x_{(0)}y)S^{-1}(x_{(1)})\delta$$

$$= \psi_X(xy_{(0)})y_{(1)}\delta$$

$$= \varphi_X(xy_{(0)}\delta_X)y_{(1)}\delta.$$

By faithfulness of φ_X we have $\alpha(y\delta_X) = \alpha(y)(\delta_X \otimes \delta)$, hence $\alpha(\delta_X) = \delta_X \otimes \delta$ by definition of α on $M(X)$.

For the second formula, we have to prove that $x(a\delta)^{[1]} \otimes (a\delta)^{[2]} = x\delta_X^{-1}a^{[1]} \otimes a^{[2]}\delta_X$ for all $a \in A$ and $x \in X$. This follows immediately by applying V (which was introduced in the beginning of this section) and using the previous formula.

As for the final formula, we have for any $x \in X$ that

$$\varphi_X(\delta_X x) = \varphi(\delta x_{(1)})\delta_X x_{(0)}$$

$$= \tau^{-1}\varphi(x_{(1)}\delta)\delta_X(x_{(0)}\delta_X)\delta_X^{-1}$$

$$= \tau^{-1}\varphi\big((x\delta_X)_{(1)}\big)\delta_X(x\delta_X)_{(0)}\delta_X^{-1}$$

$$= \tau^{-1}\varphi_X(x\delta_X),$$

which means exactly that $\sigma_X(\delta_X) = \tau^{-1}\delta_X$. $\square$

Corollary 1.17. *If φ'_X is another δ-invariant functional, then there exists $c \in k$ with $\varphi'_X = c\varphi_X$.*

Proof. This follows immediately by the uniqueness of an invariant functional and the fact that $\varphi'_X(\cdot\,\delta_X)$ is invariant. $\square$

1.6. The square of no antipode?

On X there is a natural unital left $\widehat{A}$-module algebra structure defined by $\omega_1 \cdot x = (\iota \otimes \omega_1)\alpha(x)$ for $x \in X$ and $\omega_1 \in \widehat{A}$. The unitality means that $\widehat{A} \cdot X = X$. It allows us to extend the action to a left action of $M(\widehat{A})$ on X.

Consider then the map

$$\theta_X : X \to X : x \to \sigma_X(\delta_{\widehat{A}} \cdot x),$$

where $\delta_{\widehat{A}}$ is the modular element of the dual $(\widehat{A}, \Delta_{\widehat{A}})$. It is a bijective homomorphism, which can be shown using the module algebra structure and the fact that $\delta_{\widehat{A}}$ is grouplike. This θ_X plays the rôle of 'the square of the antipode' for X. Indeed: in case $X = A$ and $\alpha = \Delta$, then θ_X is exactly S^2. We can use θ_X to complete our set of formulas.

Proposition 1.18. *The following identities hold*:

(vi) $\alpha \circ \sigma_X = (\theta_X \otimes \sigma) \circ \alpha$,
(vii) $\alpha \circ \theta_X = (\theta_X \otimes S^2) \circ \alpha$,
(viii) $\alpha \circ \theta_X = (\sigma_X \otimes \sigma'^{-1}) \circ \alpha$,
(ix) $\sigma_X \circ \theta_X = \theta_X \circ \sigma_X$,
(x) $\theta_X(\delta_X) = \delta_X$,
(xi) $\varphi_X \circ \theta_X = \varphi_X(\delta_X^{-1} \cdot \delta_X) = \tau \varphi_X$.

Proof. We first make some remarks. The vector space $M(\widehat{A})$ can still be seen as a subspace of the vector space dual A^* of A. As such the element $\delta_{\widehat{A}}$ corresponds with $\varepsilon \circ \sigma^{-1} = \varepsilon \circ \sigma'^{-1}$ (see Proposition 5.14 in [16]). Thus we can also write $\theta_X(x) = \varepsilon(\sigma^{-1}(x_{(1)}))\sigma_X(x_{(0)})$. Also, the above formulas are known to hold in case $X = A$ and $\alpha = \Delta$. We will use them in the course of the proof.

Take $x, y \in X$ and $a \in A$. Then

$$
\begin{aligned}
\varphi_X\big(y\theta_X(x_{(0)})\big)\varphi(a\sigma(x_{(1)})) &= \varphi_X\big((\delta_{\widehat{A}} \cdot x_{(0)})y\big)\varphi(x_{(1)}a) \\
&= \varphi_X(x_{(0)}y)\varepsilon\big(\sigma'^{-1}(x_{(1)})\big)\varphi(x_{(2)}a) \\
&= \varphi_X(x_{(0)}y)\varphi\big(\sigma'^{-1}\big(S^{-2}(x_{(1)})\big)a\big) \\
&= \varphi_X(xy_{(0)})\psi\big(aS^{-1}(y_{(1)})\big) \\
&= \varphi_X\big(y_{(0)}\sigma_X(x)\big)\psi\big(aS^{-1}(y_{(1)})\big) \\
&= \varphi_X\big(y\sigma_X(x)_{(0)}\big)\varphi\big(a\sigma_X(x)_{(1)}\big),
\end{aligned}
$$

which proves the equality in (vi). The equality in (vii) then follows by the previous one, and the fact that $\theta_A = S^2$ in case $X = A$ and $\alpha = \Delta$.

Further,

$$
\begin{aligned}
\varphi_X\big(y\theta_X(x_{(0)})\big)\psi\big(S^2(x_{(1)})a\big) &= \varphi_X\big(y\sigma_X(x_{(0)})\big)\delta_{\widehat{A}}(x_{(1)})\psi\big(S^2(x_{(2)})a\big) \\
&= \varphi_X\big(y\sigma_X(x_{(0)})\big)\varepsilon\big(\sigma'^{-1}(x_{(1)})\big)\psi\big(S^2(x_{(2)})a\big) \\
&= \varphi_X\big(y\sigma_X(x_{(0)})\big)\psi\big(\sigma'^{-1}(x_{(1)})a\big),
\end{aligned}
$$

which together with (vi) proves (viii).

The commutation in (ix) is clear. As for (x) we have $\theta_X(\delta_X) = \sigma_X(\delta_X)\varepsilon(\sigma^{-1}(\delta))$, which equals δ_X by the formula (v). The same formula (v) also shows immediately the validity of (xi). This concludes the proof. $\square$

Recall that we already constructed a map $S_{X^{op}} : X^{op} \to X$, which was just a given formal isomorphism between the underlying vector spaces.

Definition 1.19. We call the map

$$S_{X^{op}} : X^{op} \to X : x^{op} \to x$$

the *antipode on* X^{op}. We call the map

$$S_X : X \to X^{op} : x \to \theta_X(x)^{op}$$

the *antipode on* X.

Then indeed, $S_{X^{op}} \circ S_X = \theta_X$, so that θ_X can be considered to be 'the square of the antipode'!... If the reader feels cheated at this point, we urge him to read on.

For example, the following formulas should give a more direct connection with the defining property of an antipode. We will also write $\theta_{X^{op}}(x^{op}) = \theta_X(x)^{op}$ for $x \in X$.

Proposition 1.20. *For all* $x, y \in X$ *and* $a \in A$, *we have*

(xiii) $S(a)_{[1]} \otimes S(a)_{[2]} = S_X(a_{[2]}) \otimes S_{X^{op}}(a_{[1]})$,

(xiv) $y^{op} S_X(x_{(0)}) x_{(1)[1]} \otimes x_{(1)[2]} = y^{op} \otimes x$,

(xv) $y^{op} a_{[1]} S_X(a_{[2]}) = \varepsilon(a) y^{op}$,

(xvi) $\tilde{\beta} \circ S^2 = (\theta_{X^{op}} \otimes \theta_X) \circ \tilde{\beta}$,

(xvii) $\tilde{\beta} \circ \sigma = (\theta_{X^{op}} \otimes \sigma_X) \circ \tilde{\beta}$.

We use here the Sweedler notation for the external comultiplication $\tilde{\beta}$, introduced in the previous subsection.

Proof. Applying $(\iota \otimes \varphi_X(\cdot\, x))$ to $\theta_X(a^{[2]}) \otimes a^{[1]}$ and using formula (vi), we get

$$\begin{aligned}
\varphi_X(a^{[1]}x)\theta_X(a^{[2]}) &= \varphi_X(\sigma_X^{-1}(x)a^{[1]})\theta_X(a^{[2]}) \\
&= \varphi(\sigma_X^{-1}(x)_{(1)} S(a))\theta_X(\sigma_X^{-1}(x)_{(0)}) \\
&= \varphi(\sigma^{-1}(x_{(1)}) S(a)) x_{(0)} \\
&= \varphi(S(a)x_{(1)}) x_{(0)} \\
&= \varphi_X(S(a)^{[2]}x) S(a)^{[1]},
\end{aligned}$$

so that $S(a)^{[1]} \otimes S(a)^{[2]} = \theta_X(a^{[2]}) \otimes a^{[1]}$. This is easily seen to be equivalent with the first formula.

As for the second formula, we have to show that for all $z \in X$ we have

$$\varphi_X(x_{(1)}{}^{[2]}z)x_{(1)}{}^{[1]}\theta_X(x_{(0)})y = \varphi_X(xz)y.$$

This reduces, by Proposition 1.9(i), to proving that

$$\varphi(x_{(1)}z_{(1)})y_{(0)}\theta_X(x_{(0)})y = \varphi_X(xz)y.$$

This follows again by formula (vi) and the defining property of φ_X.

The last formulae are a direct consequence of the first (using Proposition 1.13(ii) for the last one). $\square$

Note that the second identity in the last proposition shows that

$$X^{\mathrm{op}} \otimes X \to X^{\mathrm{op}} \otimes A : y^{\mathrm{op}} \otimes x \to y^{\mathrm{op}}S_X(x_{(0)}) \otimes x_{(1)}$$

is the inverse of the map

$$X^{\mathrm{op}} \otimes A \to X^{\mathrm{op}} \otimes X : y^{\mathrm{op}} \otimes a \to y^{\mathrm{op}}a_{[1]} \otimes a_{[2]},$$

which correspond to the exact same formula for a (multiplier) Hopf algebra if we replace X^{op} and X by A, S_X by S and $\tilde{\beta}$ by the comultiplication map. More directly, we also have that $S_{X^{\mathrm{op}}}(a_{[1]})a_{[2]} = \varepsilon(a)1 = a_{[1]}S_X(a_{[2]})$ (where the unit in the middle is really in different algebras for the left and right expression). If the reader is still not convinced at this point that S_X and $S_{X^{\mathrm{op}}}$ are to be treated as antipodes, we point him to the third subsection of the third section.

However, we want to give a little warning at this point, as the situation could get a bit confusing when we consider $(X, \alpha) = (A, \Delta)$ (which is evidently a Galois object for (A, Δ)). For then we have an antipode S for the *algebraic quantum group* (A, Δ), which will be an anti-isomorphism $A \to A$, but we also have an antipode S_X for the *Galois object* (A, Δ), which will be an anti-isomorphism $A \to A^{\mathrm{op}}$. In some sense, for an algebraic quantum group the antipode contains extra information, which is not present in its square. But for a Galois object, the antipode is really just a formal construction *using* its antipode squared.

We want to remark that the notion of an 'antipode squared' on a Galois object for a Hopf algebra was considered more or less in [11], but in a different set-up. Also, the antipode squared there was a part of the axiom system. The connection with Galois objects and the redundancy of having this 'antipode squared' in the axiom system, was established in [23]. The notion of an antipode for a Galois object was considered explicitly first in [2] (although it differs somewhat from our construction). As a final remark, note that we can easily get into Grünspans framework of quantum torsors, by means of the quantum torsor map

$$(\iota \otimes \tilde{\beta})\alpha : X \to M(X \otimes X^{\mathrm{op}} \otimes X).$$

However, we have not developed an independent theory for such 'algebraic quantum torsors' (which seems very plausible to exist).

1.7. Concerning *-structures

We now look at the case $k = \mathbb{C}$.

Definition 1.21. Let X be a *-algebra and (A, Δ) a *-algebraic quantum group. If $\alpha : X \to M(X \otimes A)$ is a coaction making (X, α) into a Galois object for (A, Δ) (when neglecting the *-structure), we call (X, α) a *-*Galois object* if α is *-preserving, and if X satisfies the following non-degeneracy condition: if $\sum x_i^* x_i = 0$ for certain $x_i \in X$, then $x_i = 0$ for all i.

By saying that X is a $*$-algebra, we mean that X is equipped with an involutive anti-linear anti-multiplicative map $*$. The fact that α is $*$-preserving means that $x^*_{(0)} \otimes x^*_{(1)} = (x^*)_{(0)} \otimes (x^*)_{(1)}$ for all $x \in X$ (where we note that the $*$-operation can be easily extended to the multiplier algebra level). If we want to say anything interesting about $*$-Galois objects, it seems that we really have to put the extra non-degeneracy condition on the $*$-algebra (although I do not know of any examples where it is not satisfied automatically), for example to be able to arrive at the following statement:

Proposition 1.22. *Let (X, α) be a $*$-Galois object for a $*$-algebraic quantum groups (A, Δ). Then the functional $\varphi_X = (\iota \otimes \varphi)\alpha$ is positive.*

Proof. We have to show that $\varphi_X(x^*x) \geqslant 0$ for all $x \in X$.

Take non-zero $x, y \in X$, and write $\alpha(x)(y \otimes 1) = \sum p_i \otimes q_i$ for certain $p_i \in X$ and $q_i \in A$. Then

$$
\begin{aligned}
\varphi_X(x^*x)y^*y &= (\iota \otimes \varphi)\big(\big(\alpha(x)(y \otimes 1) \big)^* \big(\alpha(x)(y \otimes 1) \big) \big) \\
&= \sum_{i,j} \varphi\big(q_j^* q_i \big) p_j^* p_i.
\end{aligned}
$$

By positivity of φ, the matrix $(\varphi(q_j^* q_i))_{i,j}$ will be positive-definite, so that we can write $\varphi_X(x^*x)y^*y = \sum_i z_i^* z_i$ for certain $z_i \in X$. Then $\varphi_X(x^*x)$ must necessarily be positive, or else we would violate the non-degeneracy property of X. $\quad\square$

Remark. We could also have started with a weaker non-degeneracy condition on X, namely the existence of a non-zero positive functional ω on X. Then choosing y in the above proof such that $\omega(y^*y) = 1$, it is easy to conclude that we still have that φ_X is positive. Now by using a local unit argument for X, one also has that $\varphi_X(x^*) = \overline{\varphi_X(x)}$ for $x \in X$. This means that $\langle x, y \rangle = \varphi_X(y^*x)$ defines a pre-Hilbert space structure on X. Then by the Cauchy–Schwartz inequality and the faithfulness of φ_X, we see that $\varphi_X(x^*x) = 0$ implies $x = 0$. Hence, automatically the stronger non-degeneracy condition on X is satisfied.

We show that ψ_X can be chosen to be positive. As for $*$-algebraic quantum groups, this is a non-trivial statement. In that case, the first proof of this statement consisted of establishing an analytic structure on the $*$-algebraic quantum group (see [17]). In [6], we found an easier way to arrive at this, in the meantime showing something more about the structure of $*$-algebraic quantum groups. Namely, almost all structure maps on A are diagonalizable, i.e. there exists a basis of simultaneous eigenvectors for σ, S^2 and left and right multiplication with δ, which implies for example that the scaling constant is 1. We prove now that also all structure maps on X are diagonalizable.

For instance, take $x \in X$ and choose $w \in X$ with $\varphi_X(w) = 1$. Write $x \otimes w$ as a sum of $ya^{[1]} \otimes a^{[2]}$ for certain $y \in X$ and $a \in A$. Write $a = \sum a_i$ with the a_i eigenvectors for left multiplication with δ. Then

$$
\begin{aligned}
x \delta_X^n &= \sum \varphi_X(a^{[2]}) ya^{[1]} \delta_X^n \\
&= \sum \varphi_X\big(\delta_X^n (\delta^{-n}a)^{[2]} \big) y(\delta^{-n}a)^{[1]} \\
&\in \operatorname{Span}\big\{ \omega(a_i^{[2]}) ya_i^{[1]} \mid \omega \in X^* \big\},
\end{aligned}
$$

showing that $\operatorname{Span}\{x\delta_X^n \mid n \in \mathbb{Z}\}$ is finite-dimensional. The same technique shows that $\operatorname{Span}\{\delta_X^n x \mid n \in \mathbb{Z}\}$ and $\operatorname{Span}\{\theta_X^n(x) \mid n \in \mathbb{Z}\}$ are finite-dimensional. Since the left action of $\delta_{\widehat{A}}$ is diagonalizable, we also have that $\operatorname{Span}\{\sigma_X^n(x) \mid n \in \mathbb{Z}\}$ is finite-dimensional. As in [6], all these operations can be shown to be self-adjoint with respect to the scalar product $\langle x, y \rangle = \varphi_X(y^*x)$ on X (possibly after multiplying the invariant functional, and hence the modular element with a scalar), and since they commute, we obtain:

Theorem 1.23. *There exists a basis $\{x_i\}$ of X such that all x_i are eigenvectors for σ_X, θ_X and multiplication with δ_X to the left and to the right.*

It follows also by the same methods as in [6] that then all these maps have positive eigenvalues (again, possibly after multiplying the invariant functional with a scalar). In particular, this shows that δ_X is of the form $(\delta_X^{1/2})^2$ for some self-adjoint invertible element $\delta_X^{1/2} \in M(X)$. If we then choose $z \in X$ with $\varphi_X((\delta_X^{-1/2}z)^*\delta_X^{-1/2}z) = 1$, we have for any $x \in X$ that

$$
\begin{aligned}
\psi_X(x^*x) &= \psi_X(x^*x)\varphi_X(z^*\delta_X^{-1}z) \\
&= \varphi_X(z^*x_{(0)}^*x_{(0)'}z)\psi(x_{(1)}^*x_{(1)'}) \\
&\geqslant 0,
\end{aligned}
$$

showing

Corollary 1.24. *The functional ψ_X is positive.*

We also have a nice formula relating $\tilde{\beta}$ and $*$. However, we have to choose the good $*$-operation on X^{op} for this: $(x^{\mathrm{op}})^* = \theta_X(x^*)^{\mathrm{op}}$. It is not so hard to see that this is again a $*$-algebra, using that φ_X is faithful and θ_X is diagonalizable with positive eigenvalues. Now $\theta_X(x)^* = \theta_X^{-1}(x^*)$: this follows from the definition of θ_X, using that $\sigma_X(x)^* = \sigma_X^{-1}(x^*)$ for $x \in X$ (and likewise for σ and $x \in A$), which follows easily from the fact that φ_X is faithful and $\varphi_X(\cdot^*) = \overline{\varphi_X(\cdot)}$, and using that $\delta_{\widehat{A}} = \varepsilon\sigma^{-1}$ with also $\varepsilon(\cdot^*) = \overline{\varepsilon(\cdot)}$. From this, it follows that $S_{X^{\mathrm{op}}}((x^{\mathrm{op}})^*) = (S_X^{-1}(x^{\mathrm{op}}))^*$, and $S_X(x^*) = (S_{X^{\mathrm{op}}}^{-1}(x))^*$.

Proposition 1.25. *For all $a \in A$, we have*

$$
\tilde{\beta}(a)^* = \tilde{\beta}(a^*).
$$

Proof. For any $x \in X$, $a \in A$, we have

$$
\varphi(ax_{(1)})x_{(0)} = \varphi_X(a^{[2]}x)a^{[1]},
$$

by Proposition 1.9(i). Applying $*$, we see that

$$
\varphi(x_{(1)}^*S(S(a)^*))x_{(0)}^* = \varphi_X(x^*a^{[2]*})a^{[1]*}.
$$

Since the left-hand side equals $\varphi_X(x^*(S(a)^*)^{[1]})(S(a)^*)^{[2]}$ by Proposition 1.9(ii), we get that $(a^{[1]})^* \otimes (a^{[2]})^* = (S(a)^*)^{[2]} \otimes (S(a)^*)^{[1]}$ by the faithfulness of φ_X. Using the identities just before the statement of the proposition, this becomes $S_X^{-1}((a_{[1]})^*) \otimes (a_{[2]})^* = (S(a)^*)_{[2]} \otimes S_{X^{\mathrm{op}}}((S(a)^*)_{[1]})$. Applying S_X to the first leg and using the identity (xiii) in Proposition 1.20, we arrive at the identity stated in the proposition. $\square$

We want to end this subsection with a remark. As we had said before, X^{op} was really just some copy of X as a vector space. Still working in the $*$-algebra setting, let $\theta_X^{1/2}$ be the square root of θ_X, seen as an operator on the pre-Hilbert space X: since θ_X is diagonalizable with positive eigenvalues, this makes sense. Then $\theta_X^{1/2}$ will again be an automorphism on X. Now take $X^{\mathrm{op}} = X$ as a vector space, and define $x^{\mathrm{op}} = \theta_X^{-1/2}(x)$. We get that the $*$-operation on X^{op} becomes to the original $*$-operation on X. Also, the formulas for the antipodes become $S_X = \theta_X^{1/2} = S_{X^{\mathrm{op}}}$, which are a bit more symmetric, and which really make θ_X the square of the antipode now! The fact that S_X coincides with $\theta_X^{1/2}$ may seem strange, but we refer again to the discussion in the penultimate paragraph of the previous subsection.

1.8. Concerning compactness and discreteness

Definition 1.26. A non-degenerate algebra X is called *of compact type* if X has a unit. It is called *of discrete type* if every subspace of the form xX or Xx, with $x \in X$, is finite-dimensional.

Remark. This terminology is not standard, and we use it solely in this subsection.

Theorem 1.27. *Let (X, α) be a Galois object for an algebraic quantum group (A, Δ). Then the algebra X is of compact type iff A is an algebraic quantum group of compact type. The algebra X is of discrete type iff A is an algebraic quantum group of discrete type.*

Proof. If A is compact, then $\alpha(x) \in X \otimes A$ for any $x \in X$. Choosing $x \in X$ with $\varphi_X(x) = 1$, we have that $(\iota \otimes \varphi)\alpha(x) \in X$ is a unit of X.

If X is compact, then $(x \otimes 1)\alpha(1) \in X \otimes A$ for all $x \in A$. Choosing x with $\psi_X(x) = 1$, it is again a small exercise to check that $(\psi_X \otimes \iota)((x \otimes 1)\alpha(1)) \in A$ is a unit in A.

Now suppose that A is an algebraic quantum group of discrete type. Choose a non-zero left cointegral $h \in A$, so $ah = \varepsilon(a)h$ for all $a \in A$. We can scale h so that $\varphi(h) = 1$. Then for all $x, y \in X$, we have $\varphi_X(x(S^{-1}(h))^{[1]})(S^{-1}(h))^{[2]}y = \varphi(x_{(1)}h)x_{(0)}y = xy$ by Proposition 1.9(ii). This shows that Xy is finite-dimensional. Also yX is finite-dimensional, by a similar reasoning.

Conversely, suppose that X is an algebra of discrete type. Take $a \in A$ and $x \neq 0$ fixed in X. Write $xa^{[1]} \otimes a^{[2]}$ as $\sum_i p_i \otimes q_i$, and choose $y \in X$ such that $p_i y = p_i$ for all i (Corollary 1.3). Then

$$\dim(Aa) = \dim\{x \otimes ba \mid b \in A\}$$

$$= \dim\left\{\sum_i p_i y b^{[1]} \otimes b^{[2]} q_i \mid b \in A\right\}$$

$$\leqslant \dim \operatorname{span}\left\{\sum_i p_i w \otimes z q_i \mid w, z \in X\right\}$$

$$< \infty.$$

We show that this is sufficient to conclude that (A, Δ) is an algebraic quantum group of discrete type.

First, applying S, we see that also all aA are finite-dimensional. Choose $a \in A$ with $\varepsilon(a) = 1$. Write $I = AaA$, which is a finite-dimensional ideal. Because φ is faithful, we can choose some $\omega = \varphi(\cdot b) \in \hat{A}$ such that $\omega_{|I} = \varepsilon_{|I}$. Take $e \in A$ with $ae = a$. Then for all $x \in A$, we have

$$\varphi(xab) = \varphi(xaeb)$$

$$= \omega_{|I}(xae)$$

$$= \varepsilon(x).$$

Hence $\varepsilon \in \hat{A}$, and A is an algebraic quantum group of discrete type. $\square$

Note that the proof above shows that the terminology we used is consistent: an algebraic quantum group is of discrete type (in the sense of [26]) iff its underlying algebra is of discrete type (as defined in Definition 1.26). Also note that if $k = \mathbb{C}$ and X is a *-algebra, the condition 'X is of discrete type' is equivalent with X being a direct sum of finite-dimensional matrix algebras.

Proposition 1.28. *If (A, Δ) is an algebraic quantum group of discrete type, and (X, α) a Galois object for (X, α), then X is a Frobenius algebra in the sense of [29]: there exists a left X-module isomorphism $L : XX^* \to X$, where X^* is the dual space of X.*

Here XX^* denotes functionals of the form $x \cdot \omega = \omega(\cdot\, x)$ for $\omega \in X^*$ and $x \in X$.

Proof. Let p be the right cointegral of A, so that $pa = \varepsilon(a)p$ for all $a \in A$. We assume p normalized, so that $\varphi(p) = 1$. We show then that

$$(x \otimes 1)\beta(p) = \beta(p)(1 \otimes x)$$

for all $x \in X$.

Take $x, y \in X$ and apply $(\iota \otimes \varphi_X(\cdot\, y))$ to $\beta(p)(1 \otimes x)$. Then we find

$$
\begin{aligned}
(\iota \otimes \varphi_X)\big(\beta(p)(1 \otimes xy)\big) &= \varphi_X\big(p^{[2]}xy\big)p^{[1]} \\
&= \varphi(px_{(1)}y_{(1)})x_{(0)}y_{(0)} \\
&= \varepsilon(x_{(1)}y_{(1)})x_{(0)}y_{(0)} \\
&= xy \\
&= \varphi(py_{(1)})xy_{(0)} \\
&= \varphi_X\big(p^{[2]}y\big)xp^{[1]} \\
&= (\iota \otimes \varphi_X)\big((x \otimes 1)\beta(p)(1 \otimes y)\big).
\end{aligned}
$$

As φ_X is faithful, this implies $(x \otimes 1)\beta(p) = \beta(p)(1 \otimes x)$ for all $x \in X$.

Consider then

$$L : XX^* \to X : \omega \to (\iota \otimes \omega)\big(\beta(p)\big),$$

$$K : X \to XX^* : x \to \varphi_X(\cdot\, x).$$

Then L and K are seen to be X-module morphisms, using the above identity. Moreover, they are each others inverse: choose $x \in X$ and $\omega \in XX^*$, then

$$
\begin{aligned}
\varphi_X\big(y \cdot (\iota \otimes \omega)\big(\beta(p)\big)\big) &= \varphi_X\big(yp^{[1]}\big)\omega\big(p^{[2]}\big) \\
&= \varphi\big(y_{(1)}S(p)\big)\omega(y_{(0)}) \\
&= \varphi\big(S(p)\big)\omega(y) \\
&= \varphi(p\delta)\omega(y) \\
&= \omega(y),
\end{aligned}
$$

showing that KL is the identity. The fact that LK is the identity follows from $\varphi_X(p^{[2]}x)p^{[1]} = x$ for all $x \in X$. $\quad\square$

2. Reflecting an algebraic quantum group across a Galois object

It is known that if (A, Δ) is an algebraic quantum group of compact type (or more generally a Hopf algebra) and (X, α) a Galois object for it, then a second Hopf algebra (C, Δ_C) can be constructed from (A, Δ) and (X, α) (see [22]). Moreover, this (C, Δ_C) has a left coaction γ on X, and (X, α, γ) will be what is termed a A-C-bi-Galois object: it is at the same time a left C-Galois object and right A-Galois object, with the two coactions commuting. We show that the same holds in our setting. Our approach is based on duality: we first construct the dual $(\widehat{C}, \Delta_{\widehat{C}})$, which seems to be more natural here.

A remark concerning notation: we will always denote elements in the dual X^* of X by $\omega, \omega', \omega'', \ldots$ and the elements in the dual A^* of A by $\omega_1, \omega_2, \omega_3, \ldots$ When elements of X^* are indexed by some set I, we will put the index in superscript, so then these elements take the form ω^i.

2.1. The dual of a Galois object

Definition 2.1. Let (X, α) be a Galois object for an algebraic quantum group (A, Δ). The *restricted dual* of X is the vector space $\widehat{X} = \{\varphi_X(\,\cdot\, x) \mid x \in X\}$ inside the dual X^* of X.

We have shown that then

$$\widehat{X} = \{\varphi_X(x\,\cdot\,) \mid x \in X\}$$
$$= \{\psi_X(\,\cdot\, x) \mid x \in X\}$$
$$= \{\psi_X(x\,\cdot\,) \mid x \in X\}$$

in Theorems 1.7 and 1.11.

Let (X, α) be a Galois object for (A, Δ). Let $(\widehat{A}, \Delta_{\widehat{A}})$ be the dual of (A, Δ) (where we still identify $\widehat{A}$ with a space of functionals on A). As already mentioned at some point, we have a left $\widehat{A}$-module structure on X, induced by α, by putting $\omega_1 \cdot x = (\iota \otimes \omega_1)(\alpha(x))$ for $x \in X$ and $\omega_1 \in \widehat{A}$. This leads to a *right* $\widehat{A}$-module structure on X^*, by putting $(\omega \cdot \omega_1)(x) = \omega(\omega_1 \cdot x)$ for $\omega \in X^*$, $\omega_1 \in \widehat{A}$ and $x \in X$. This will restrict to a natural right $\widehat{A}$-module structure on $\widehat{X}$. More concretely, we have

$$\psi_X(\,\cdot\, x) \cdot \omega_1 = \omega_1\big(S(x_{(1)})\big)\psi_X(\,\cdot\, x_{(0)}), \quad \text{for all } x \in X, \ \omega_1 \in \widehat{A},$$

by using Proposition 1.6.

We can dualize the multiplication on X to a map

$$\Delta_{\widehat{X}} : \widehat{X} \to (X \otimes X)^*.$$

We denote the image of ω by $\omega^{(1)} \otimes \omega^{(2)}$. While we cannot say that this element is 'in $M(\widehat{X} \otimes \widehat{X})$', since $\widehat{X}$ has no multiplication, we do have that expressions such as $\omega^{(1)} \otimes (\omega^{(2)} \cdot \omega_1)$ with $\omega \in \widehat{X}$ and $\omega_1 \in \widehat{A}$ define elements of $\widehat{X} \otimes \widehat{X}$, and that this provides bijections between $\widehat{X} \otimes \widehat{A}$ and $\widehat{X} \otimes \widehat{X}$. For example, the map

$$V^t : \widehat{X} \otimes \widehat{A} \to \widehat{X} \otimes \widehat{X} : \omega \otimes \omega_1 \to \omega^{(1)} \otimes \big(\omega^{(2)} \cdot \omega_1\big)$$

is just the dual of the map V, introduced at the beginning of the previous section:

$$\big(V^t(\omega \otimes \omega_1)\big)(x \otimes y) = (\omega \otimes \omega_1)\big(V(x \otimes y)\big).$$

Again in more concrete terms, we have $V^t((\varphi_X(\,\cdot\, x) \otimes \varphi(a\,\cdot\,)) = \varphi_X(\,\cdot\, a^{[1]}x) \otimes \varphi_X(a^{[2]}\,\cdot\,)$ for $x \in X$, $a \in A$, by using Proposition 1.9(i).

The space $\widehat{X}$ also carries a *natural* $\widehat{A}$-valued k-bilinear form, determined by

$$[\omega, \omega']_{\widehat{A}}(a) = (\omega \otimes \omega')\big(\beta(a)\big), \quad \omega, \omega' \in \widehat{X}, \ a \in A,$$

where β is still the map introduced just after Definition 1.1. If we then let $\widehat{A}$ act on the left of $\widehat{X}$ by $\omega_1 \cdot \omega := \omega \cdot S_{\widehat{A}}^{-1}(\omega_1)$ for $\omega \in \widehat{X}$ and $\omega_1 \in \widehat{A}$, we have

Proposition 2.2. *The form $[\cdot,\cdot]_{\widehat{A}}$ is $\widehat{A}$-bilinear, i.e. for all $\omega,\omega' \in \widehat{X}$ and $\omega_1 \in \widehat{A}$,*

$$[\omega,\omega' \cdot \omega_1]_{\widehat{A}} = [\omega,\omega']_{\widehat{A}} \cdot \omega_1,$$

$$[\omega_1 \cdot \omega,\omega']_{\widehat{A}} = \omega_1 \cdot [\omega,\omega']_{\widehat{A}}.$$

Proof. This is just a reformulation of the formulas 2.1.2 and 2.1.3 of Lemma 2.7 in [24]. To proof the linearity on the right for example, note that if $\omega = \varphi_X(x \cdot)$, then

$$[\omega,\omega']_{\widehat{A}}(a) = \varphi\big(x_{(1)} S(a)\big) \omega'(x_{(0)}),$$

using the second formula of Proposition 1.9. Then

$$
\begin{aligned}
\big([\omega,\omega']_{\widehat{A}} \cdot \omega_1\big)(a) &= \varphi\big(x_{(1)} S(a_{(1)})\big) \omega'(x_{(0)}) \omega_1(a_{(2)}) \\
&= \varphi\big(x_{(2)} S(a)\big) \omega'(x_{(0)}) \omega_1(x_{(1)}) \\
&= [\omega,\omega' \cdot \omega_1]_{\widehat{A}}(a).
\end{aligned}
$$

The other identity can be proven in the same way. $\quad\square$

The following formula shows how the bracket behaves with respect to the left action:

Lemma 2.3. *For $\omega,\omega',\omega'' \in \widehat{X}$, we have*

$$[\omega,\omega']_{\widehat{A}} \cdot \omega'' = \omega'' \cdot \big[\theta_{\widehat{X}}^{-1}(\omega'),\omega\big]_{\widehat{A}},$$

with $\theta_{\widehat{X}}^{-1}(\omega') = \omega' \circ \theta_X^{-1}$.

Proof. This follows from formula (xiii) of Proposition 1.20. $\quad\square$

2.2. Construction of the reflected algebraic quantum group

In this subsection, we construct a new algebraic quantum group $(\widehat{C},\Delta_{\widehat{C}})$, given a Galois object (X,α). (We use the $\widehat{}$-notation since $\widehat{C}$ plays the same role as $\widehat{A}$.)

Consider the vector space $\widehat{C}$ spanned by the linear maps $[\omega,\omega']_{\widehat{C}}$ on $\widehat{X}$, where $\omega,\omega' \in \widehat{X}$, defined by the identity

$$[\omega,\omega']_{\widehat{C}} \cdot \omega'' = \omega \cdot [\omega',\omega'']_{\widehat{A}}, \quad \omega,\omega',\omega'' \in \widehat{X}.$$

This will be an algebra: because of the right linearity of $[\cdot,\cdot]_{\widehat{A}}$ we have

$$b \cdot [\omega,\omega']_{\widehat{C}} = [b \cdot \omega,\omega']_{\widehat{C}} \quad \text{for } b \in \widehat{C}.$$

This also makes the maps of $\widehat{C}$ on $\widehat{X}$ commute with the action of $\widehat{A}$. Moreover, if $\omega_1 \in \widehat{A}$ then

$$[\omega \cdot \omega_1,\omega']_{\widehat{C}} = [\omega,\omega_1 \cdot \omega']_{\widehat{C}},$$

which follows from the left $\widehat{A}$-linearity of $[\cdot,\cdot]_{\widehat{A}}$. This provides a canonical map π from $\widehat{X} \otimes_{\widehat{A}} \widehat{X}$ to $\widehat{C}$.

Lemma 2.4. *The map $S_{\widehat{C}} : \widehat{C} \to \widehat{C} : [\omega,\omega']_{\widehat{C}} \to [\theta_{\widehat{X}}(\omega'),\omega]_{\widehat{C}}$ is a well-defined bijection.*

Proof. Choose arbitrary $x, y \in X$, and suppose $[\omega, \omega']_{\widehat{C}} = 0$. Then

$$
\begin{aligned}
\left(\left[\theta_{\widehat{X}}(\omega'), \omega \right]_{\widehat{C}} \cdot \varphi_X(\cdot \, x) \right)(y) &= \omega' \left(\theta_X(y_{(0)}) \right) \omega(x_{(0)}) \varphi(y_{(1)} x_{(1)}) \\
&= \omega(x_{(0)}) \omega' \left(\sigma_X(y)_{(0)} \right) \varphi \left(x_{(1)} \left(\sigma_X(y) \right)_{(1)} \right) \\
&= \left([\omega, \omega']_{\widehat{C}} \cdot \varphi_X(y \, \cdot) \right)(x) \\
&= 0,
\end{aligned}
$$

hence $[\theta_{\widehat{X}}(\omega'), \omega]_{\widehat{C}} = 0$. □

This allows us to define a right action of $\widehat{C}$ on $\widehat{X}$ by setting $\omega \cdot b = S_{\widehat{C}}^{-1}(b) \cdot \omega$ for $\omega \in B$, since by Lemma 2.3 we have $\omega'' \cdot [\omega, \omega']_{\widehat{C}} = [\omega'', \omega]_{\widehat{A}} \cdot \omega'$.

Corollary 2.5. *The maps* $[\cdot, \cdot]_{\widehat{C}}$ *and* $[\cdot, \cdot]_{\widehat{A}}$ *make* $\widehat{X}$ *into a strict Morita context between* $\widehat{C}$ *and* $\widehat{A}$.

Remark that this can be used to show that $\widehat{C}$ is a non-degenerate algebra.

Lemma 2.6. *For any finite collection* $\omega^i \in \widehat{X}$ *there exists* $b \in \widehat{C}$ *with* $b \cdot \omega^i = \omega^i$.

Proof. Put $\omega^i = \varphi_X(\cdot \, y_i)$. Then we have to prove that there exist ω'^j and $\omega''^j \in \widehat{X}$ such that for any $x \in X$

$$
\sum_j \omega'^j(x_{(0)}) \omega''^j(y_{i(0)}) \varphi(x_{(1)} y_{i(1)}) = \varphi_X(xy_i).
$$

Choose $z \in X$ with $\varphi_X(z) = 1$. Put $\omega = \varphi_X(\cdot \, z)$ and choose $\omega_1 \in \widehat{A}$ such that $\omega_1(y_{i(1)}) y_{i(0)} z \otimes y_{i(2)} = y_{i(0)} z \otimes y_{i(1)}$ for all i. Put $\sum_j \omega'^j \otimes \omega''^j = \omega^{(1)} \otimes (\omega^{(2)} \cdot \omega_1)$. Then we have

$$
\begin{aligned}
\sum_j \omega'^j(x_{(0)}) \omega''^j(y_{i(0)}) \varphi(x_{(1)} y_{i(1)}) &= \omega(x_{(0)} y_{i(0)}) \varphi(x_{(1)} y_{i(2)}) \omega_1(y_{i(1)}) \\
&= \omega(x_{(0)} y_{i(0)}) \varphi(x_{(1)} y_{i(1)}) \\
&= \varphi_X(xy_i). \quad \square
\end{aligned}
$$

Proposition 2.7. *The projection* $\pi : \widehat{X} \otimes_{\widehat{A}} \widehat{X} \to \widehat{C}$ *is bijective.*

Proof. Suppose $\pi(\sum_i \omega^i \otimes_{\widehat{A}} \omega'^i) = 0$. Let $b = \sum_j [\omega''^j, \omega'''^j]$ be a local unit for the ω^i. Then

$$
\begin{aligned}
\sum_i \omega^i \underset{\widehat{A}}{\otimes} \omega'^i &= \sum_i b \cdot \omega^i \underset{\widehat{A}}{\otimes} \omega'^i \\
&= \sum_{i,j} \left(\omega''^j \cdot [\omega'''^j, \omega^i]_{\widehat{A}} \right) \underset{\widehat{A}}{\otimes} \omega'^i \\
&= \sum_{i,j} \omega''^j \underset{\widehat{A}}{\otimes} \left([\omega'''^j, \omega^i]_{\widehat{A}} \cdot \omega'^i \right) \\
&= \sum_{i,j} \omega''^j \underset{\widehat{A}}{\otimes} \left(\omega'''^j \cdot [\omega^i, \omega'^i]_{\widehat{C}} \right) \\
&= 0. \quad \square
\end{aligned}
$$

In the following, we will identify $\widehat{X} \otimes_{\widehat{A}} \widehat{X}$ with $\widehat{C}$.

We can define a right action of $\widehat{C}$ on X by

$$x \cdot [\omega, \omega']_{\widehat{C}} = \omega(x_{(0)})\omega'\big(x_{(1)}^{[1]}\big)x_{(1)}^{[2]},$$

and then $(b \cdot \omega)(x) = \omega(x \cdot b)$ for all $b \in \widehat{C}$. We can then construct a map

$$V_{\widehat{C}}^t : \widehat{C} \otimes \widehat{X} \to \widehat{X} \otimes \widehat{X} : b \otimes \omega \to b \cdot \omega^{(1)} \otimes \omega^{(2)},$$

and it will be a bijection. Its inverse is given by

$$\big(V_{\widehat{C}}^t\big)^{-1}(\omega \otimes \omega') = [\omega, \omega'^{(1)}]_{\widehat{C}} \otimes \omega'^{(2)},$$

where the first factor is covered by $\widehat{A}$-balancedness and a local unit argument. Note that a same kind of expression can be used for the inverse of V^t (which was introduced in the paragraph before Proposition 2.2), we have

$$\big(V^t\big)^{-1}(\omega \otimes \omega') = \omega^{(1)} \otimes [\omega^{(2)}, \omega']_{\widehat{A}}.$$

Now we define a comultiplication on $\widehat{C}$. We first provide some formal intuition. Note that on $\widehat{X}$, we have

$$\Delta_{\widehat{X}}(\omega \cdot \omega_1) = \Delta_{\widehat{X}}(\omega) \cdot \Delta_{\widehat{A}}(\omega_1).$$

Also, using $(ab)^{[1]} \otimes (ab)^{[2]} = b^{[1]}a^{[1]} \otimes a^{[2]}b^{[2]}$ for $a, b \in A$, we have

$$\Delta_{\widehat{A}}\big([\omega, \omega']_{\widehat{A}}\big) = [\omega^{(2)}, \omega'^{(1)}]_{\widehat{A}} \otimes [\omega^{(1)}, \omega'^{(2)}]_{\widehat{A}}.$$

If we then want $\Delta_{\widehat{C}}(b) \cdot \Delta_{\widehat{X}}(\omega) = \Delta_{\widehat{X}}(b \cdot \omega)$, we are led to

$$\Delta_{\widehat{C}}\big([\omega, \omega']_{\widehat{C}}\big) = [\omega^{(1)}, \omega'^{(2)}]_{\widehat{C}} \otimes [\omega^{(2)}, \omega'^{(1)}]_{\widehat{C}}.$$

We now show that this is a well-defined comultiplication.

Remark that $m = [\omega^{(1)}, \omega'^{(2)}]_{\widehat{C}} \otimes [\omega^{(2)}, \omega'^{(1)}]_{\widehat{C}}$ automatically has a well-defined meaning as a multiplier, using $\widehat{C}$-linearity of $[\cdot, \cdot]_{\widehat{C}}$. In fact, for $c \in \widehat{C}$ any of the expressions $(1 \otimes c)m$, $(c \otimes 1)m$, $m(1 \otimes c)$ and $m(c \otimes 1)$ are elements of $\widehat{C} \otimes \widehat{C}$. Since the action of $\widehat{C}$ is unital (i.e. $\widehat{C} \cdot \widehat{X} = \widehat{X}$), we can let m act on the left of $\widehat{X} \otimes \widehat{X}$.

Lemma 2.8. *For all* $\omega'', \omega''' \in \widehat{X}$, *we have*

$$m \cdot (\omega'' \otimes \omega''') = V^t(b \otimes 1)\big(V^t\big)^{-1}(\omega'' \otimes \omega'''),$$

where $b = [\omega, \omega']_{\widehat{C}}$.

Proof. We have to show that for all $\omega'' \in \widehat{X}$ and $\omega_1 \in \widehat{A}$, we have

$$V^t(b \cdot \omega'' \otimes \omega_1) = m \cdot \big(\omega''^{(1)} \otimes \omega''^{(2)} \cdot \omega_1\big).$$

Choose $c \in \widehat{C}$. Then

$$\big((c \otimes 1)m\big) \cdot \big(\omega''^{(1)} \otimes \omega''^{(2)} \cdot \omega_1\big) = \big[c \cdot \omega^{(1)}, \omega'^{(2)}\big]_{\widehat{C}} \cdot \omega''^{(1)} \otimes \big[\omega^{(2)}, \omega'^{(1)}\big]_{\widehat{C}} \cdot \big(\omega''^{(2)} \cdot \omega_1\big)$$

$$= \big(c \cdot \omega^{(1)}\big) \cdot \big[\omega'^{(2)}, \omega''^{(1)}\big]_{\widehat{A}} \otimes \omega^{(2)} \cdot \big[\omega'^{(1)}, \omega''^{(2)} \cdot \omega_1\big]_{\widehat{A}}$$

$$= \big(c \cdot \omega^{(1)} \otimes \omega^{(2)}\big) \cdot \big(\Delta_{\widehat{A}}\big(\big[\omega', \omega''\big]_{\widehat{A}}\big)(1 \otimes \omega_1)\big)$$

$$= (c \otimes 1) \cdot \big(\Delta_{\widehat{X}}(\omega \cdot \big[\omega', \omega''\big]_{\widehat{A}}) \cdot (1 \otimes \omega_1)\big)$$

$$= (c \otimes 1) \cdot \big(V^t(b \cdot \omega'' \otimes \omega_1)\big).$$

As c was arbitrary, the lemma is proved. □

It is then immediate that if we define

$$\Delta_{\widehat{C}}\big(\big[\omega, \omega'\big]_{\widehat{C}}\big) = \big[\omega^{(1)}, \omega'^{(2)}\big]_{\widehat{C}} \otimes \big[\omega^{(2)}, \omega'^{(1)}\big]_{\widehat{C}},$$

then $\Delta_{\widehat{C}}$ is a well-defined multiplicative comultiplication $\widehat{C} \to M(\widehat{C} \otimes \widehat{C})$.

Now we show that $(\widehat{C}, \Delta_{\widehat{C}})$ is an algebraic quantum group. We will do this by explicitly constructing its counit, its antipode and its left invariant functional. This is sufficient by Proposition 2.9 in [26]. Define by $\varepsilon_{\widehat{C}}$ the map

$$\varepsilon_{\widehat{C}} : \widehat{C} \to k : [\omega, \omega']_{\widehat{C}} \to \omega(1)\omega'(1).$$

Lemma 2.9. *The functional $\varepsilon_{\widehat{C}}$ is well defined, and satisfies the counit property with respect to $\Delta_{\widehat{C}}$.*

Proof. The well-definedness is immediate. Choose $c \in \widehat{C}$, $\omega, \omega' \in \widehat{X}$. Then

$$(\iota \otimes \varepsilon_{\widehat{C}})\big((c \otimes 1)\Delta_{\widehat{C}}\big(\big[\omega, \omega'\big]_{\widehat{C}}\big)\big) = \omega^{(2)}(1)\omega'^{(1)}(1)\big[c \cdot \omega^{(1)}, \omega'^{(2)}\big]_{\widehat{C}}$$

$$= c \cdot [\omega, \omega']_{\widehat{C}}.$$

The other half of the counit property is proven similarly. □

Lemma 2.10. *The map $S_{\widehat{C}}$ defined in Lemma 2.4 satisfies the antipode property.*

Proof. For one half of it, we have to prove the following identity: for any $\omega, \omega' \in \widehat{X}$ and $c \in \widehat{C}$,

$$\big[\theta_{\widehat{X}}(\omega'^{(2)}), \omega^{(1)}\big]_{\widehat{C}} \cdot \big[\omega^{(2)}, \omega'^{(1)} \cdot c\big]_{\widehat{C}} = \omega(1)\omega'(1)c.$$

But the left hand expression equals $[[\theta_{\widehat{X}}(\omega'^{(2)}), \omega^{(1)}]_{\widehat{C}} \cdot \omega^{(2)}, \omega'^{(1)} \cdot c]_{\widehat{C}}$. Now for $\omega'' \in \widehat{X}$ we have

$$\big[\omega'', \omega^{(1)}\big]_{\widehat{C}} \cdot \omega^{(2)} = \omega(1)\omega'',$$

by using Lemma 2.3 and formula (xiv) of Proposition 1.20, so the first expression reduces to $\omega(1)[\theta_{\widehat{X}}(\omega'^{(2)}), \omega'^{(1)} \cdot c]_{\widehat{C}}$. Using formula (xv) of Proposition 1.20, we find that for $\omega'' \in \widehat{X}$

$$\big[\theta_X(\omega'^{(2)}), \omega'^{(1)}\big]_{\widehat{C}} \cdot \omega'' = \omega'(1)\omega'',$$

proving the identity. The other half of the antipode property follows similarly. □

Now we show that $\widehat{C}$ has a non-zero left invariant functional. Denote by $\widehat{}$ the map $\widehat{X} \to X : \psi_X(\cdot x) \to x$.

Lemma 2.11. *The functional*

$$\varphi_{\widehat{C}} : \widehat{C} \to k : [\omega', \omega]_{\widehat{C}} \to \omega'(\widehat{\omega})$$

is well defined and left invariant.

Proof. If $x \in X$ and $\omega_1 \in \widehat{A}$, then $\psi_X(\cdot x) \cdot \omega_1 = \omega_1(S(x_{(1)}))\psi_X(\cdot x_{(0)})$. Hence if $\omega = \psi_X(\cdot x)$, then $\widehat{\omega_1 \cdot \omega} = \omega_1 \cdot \widehat{\omega}$. If also $\omega' \in \widehat{X}$, then $(\omega' \cdot \omega_1)(\widehat{\omega}) = \omega'(\widehat{\omega_1 \cdot \omega})$. This proves that $\varphi_{\widehat{C}}$ is well defined on $\widehat{C} = \widehat{X} \otimes_{\widehat{A}} \widehat{X}$.

We prove that $\varphi_{\widehat{C}}$ is left invariant. First note that the expression

$$(\iota \otimes \varphi_{\widehat{C}}) \Delta_{\widehat{C}} \big([\omega', \omega]_{\widehat{C}} \big)$$

makes sense as a multiplier of $\widehat{C}$. As the action of $\widehat{C}$ on X is unital, also the expression

$$\big(\big((\iota \otimes \varphi_{\widehat{C}}) \Delta_{\widehat{C}} \big([\omega', \omega]_{\widehat{C}} \big) \big) \cdot \omega'' \big)(x)$$

is meaningful for $\omega'' \in \widehat{X}$ and $x \in X$. Since the action of $M(\widehat{C})$ is faithful, it is enough to prove that this expression equals $\omega'(\widehat{\omega})\omega''(x)$. Define

$$\xi_{\omega'',x} : \widehat{C} \to k : b \to (b \cdot \omega'')(x).$$

By unitality we have that $(\xi_{\omega'',x} \otimes \iota) \Delta_{\widehat{C}}(b) \in \widehat{C}$ for any $b \in \widehat{C}$, and a small calculation yields that for $b = [\omega', \omega]_{\widehat{C}}$ we have

$$(\xi_{\omega'',x} \otimes \iota) \Delta_{\widehat{C}}(b) = \omega'' \big(x_{(1)}{}^{[2]} \big) \big[\omega' \big(x_{(0)} \cdot \big), \omega \big(\cdot x_{(1)}{}^{[1]} \big) \big]_{\widehat{C}}.$$

If we then apply $\varphi_{\widehat{C}}$ and write $\omega = \psi_X(\cdot y)$, we obtain $\omega''(x_{(1)}{}^{[2]})\omega'(x_{(0)}x_{(1)}{}^{[1]}y)$, which reduces to $\omega'(\widehat{\omega})\omega''(x)$. $\quad\square$

We have proven

Theorem 2.12. *Together with the map $\Delta_{\widehat{C}}$ the algebra $\widehat{C}$ is an algebraic quantum group.*

2.3. X as a bi-Galois object

We still assume that (X, α) is a Galois object for an algebraic quantum group (A, Δ). We use notation as in the previous subsection.

Denote by C the vector space $\widehat{X} \otimes_{\widehat{A}} X$. Then

$$\widehat{C} \times C \to k : (b, \omega \otimes x) \to (b \cdot \omega)(x)$$

is a well-defined pairing. Now since the map $\widehat{X} \to X : \psi_X(\cdot x) \to x$ is left $\widehat{A}$-linear, we can define a natural bijection $\widehat{C} \to C$ by sending $[\omega, \psi_X(\cdot x)]_{\widehat{C}}$ to $\omega \otimes x$. We also have a canonical bijection $(\widehat{C})\widehat{\ } \to \widehat{C} : \varphi_{\widehat{C}}(\cdot b) \to b$ for $b \in \widehat{C}$, where $(\widehat{C})\widehat{\ }$ momentarily denotes the dual of $(\widehat{C}, \Delta_{\widehat{C}})$. We can use this to identify C with $(\widehat{C})\widehat{\ }$, so that there is no conflict of notation. In the following we will denote $\omega \otimes_{\widehat{A}} x$ by $[\omega, x]_C$. Then multiplication in C is essentially defined by

$$\big[\omega^{(1)}, x \big]_C \cdot \big[\omega^{(2)}, y \big]_C = [\omega, xy]_C.$$

Also its counit, antipode and right invariant functional are easily seen to be defined, using the results on the structure of the dual of an algebraic quantum group, by respectively

$$\varepsilon_C\big([\omega, x]_C\big) = \omega(x)$$

$$S_C\big([\varphi_X(\cdot\, w), x]_C\big) = [\varphi_X(x\,\cdot\,), w]_C,$$

$$\psi_C\big([\omega, x]_C\big) = \omega(1)\psi_X(x).$$

We have not found a particularly nice form for the comultiplication on C.

It is immediate that the right action of $\widehat{C}$ on X makes X a unital right $\widehat{C}$-module algebra. Then we know from [27] that there is a left coaction of C on X. If we denote it by γ, then we have the formula

$$(\iota \otimes \omega)\gamma(x) = [\omega, x]_C.$$

It is also clear that this coaction makes X a left C-Galois object, since the adjoint of the Galois map is exactly $V_{\widehat{C}}^t$. As X is a $\widehat{A}$-$\widehat{C}$-bi-module, it is clear that the coactions of C and A commute. This shows that we are in the situation of the following definition:

Definition 2.13. If (A, Δ_A) and (C, Δ_C) are two algebraic quantum groups, then an *A-C-bi-Galois object* consists of a triple (X, α, γ), where (X, α) is a right A-Galois object, (X, γ) is a left C-Galois object, and α and γ commute.

Theorem 2.14. *If (C, Δ_C) is the algebraic quantum group, and $\gamma : X \to M(C \otimes X)$ is the left coaction constructed from a Galois object (X, α) for an algebraic quantum group (A, Δ), then (X, α, γ) is an A-C-bi-Galois object. The δ-invariant functional φ_X for α will be invariant for γ, while the invariant functional ψ_X for α will be δ_C^{-1}-invariant for γ.*

Proof. We have already shown the validity of the first statement. As for the invariance of φ_X, we have to show that $\varphi_X(x \cdot b) = \varphi_X(x)\varepsilon_{\widehat{C}}(b)$ for $x \in X$ and $b \in \widehat{C}$. Choosing $\omega \in \widehat{X}$, $y \in X$, we have for all $x \in X$ that

$$\varphi_X\big(x \cdot [\omega, \varphi_X(y \,\cdot\,)]_{\widehat{C}}\big) = \omega(x_{(0)})\varphi\big(y_{(1)}S(x_{(1)})\big)\varphi_X(y_{(0)})$$

$$= \varphi_X(y)\varphi_X(x)\omega(1)$$

$$= \varphi_X(x)\varepsilon\big([\omega, \varphi_X(y \,\cdot\,)]_{\widehat{C}}\big).$$

As for the δ_C^{-1}-invariance of ψ_X, this follows from the fact that

$$\psi_X(x)1 = (\psi_C \otimes \iota)\big(\gamma(x)\big).$$

This shows that ψ_X bears the same relation to ψ_C as φ_X did to φ, and reasoning by duality the claim follows. $\quad\square$

We give another characterization of the algebra $\widehat{C}$. When $x \in X$, we will denote by R_x the map 'right multiplication with x'. By $B_0(X)$ we denote $X \otimes \widehat{X}$ seen as finite rank operators on X in the natural way.

Proposition 2.15. *The algebra $\widehat{C}$ consists exactly of those maps $F : X \to X$ which commute with the left action of $\widehat{A}$ and such that $\{R_x F \mid x \in X\} \subseteq B_0(X)$*

Proof. It is not difficult to show that every element of $\widehat{C}$ satisfies this condition. For the other way, we first show that F can be seen as an element of $M(\widehat{C})$. The commutation of F with the left action of $\widehat{A}$ lets identify F with a functional ω_F on $C = \widehat{X} \otimes_{\widehat{A}} X$ by sending $\omega \otimes x$ to $\omega(F(x))$. As such, for any $c \in C$, we have $(\omega_F \otimes \iota)(\Delta_C(c)) \in C$ and $(\iota \otimes \omega_F)(\Delta_C(c)) \in C$, for if $c = \langle \varphi_X(\,\cdot\,y), x\rangle_C$, then

$$(\omega_F \otimes \iota)\big(\Delta_C(c)\big) = \omega_F(x_{(-2)})\varphi_X(x_{(0)}y)x_{(-1)}$$
$$= \omega_F(x_{(-1)})\varphi_X(x_{(0)}y_{(0)})S_C^{-1}(y_{(-1)})$$
$$= \varphi_X\big(F(x)y_{(0)}\big)S_C^{-1}(y_{(-1)}) \in C,$$

while

$$(\iota \otimes \omega_F)\big(\Delta_C(c)\big) = \varphi_X(x_{(0)}y)\omega_F(x_{(-1)})x_{(-2)}$$
$$= \varphi_X\big((R_y F)(x_{(0)})\big)x_{(-1)} \in C.$$

The remark before Proposition 4.3 of [26] lets conclude that $\omega_F \in M(\widehat{C})$, and hence F is the right action by ω_F. As the map sending $m \otimes x \in M(\widehat{C}) \otimes X$ to $R_x m$ is seen to be injective, and as $\widehat{C} \otimes X \to B_0(X) : b \otimes x \to R_x b$ is seen to be a bijection, we conclude that $\omega_F \in \widehat{C}$. $\quad\square$

Corollary 2.16. *If $(C', \Delta_{C'})$ is another algebraic quantum group making X a A-C'-bi-Galois object, then (C, Δ_C) and $(C', \Delta_{C'})$ are isomorphic as algebraic quantum groups.*

Proof. It is enough to prove that the dual $(\widehat{C}', \Delta_{\widehat{C}'})$ of C' is isomorphic with $\widehat{C}$. But the previous proposition implies that $\pi(\widehat{C}') \subseteq \widehat{C}$, with π the associated faithful representation of $\widehat{C}'$ as operators on $\widehat{X}$. Since the Galois property forces the natural map $\widehat{C}' \otimes \widehat{X} \to \widehat{X} \otimes \widehat{X}$ to be an isomorphism, we have $\pi(\widehat{C}') = \widehat{C}$. The comultiplications necessarily coincide, since $\Delta_{\widehat{C}'}(c)(\Delta_{\widehat{X}}(\omega)(1 \otimes \omega_1)) = \Delta_{\widehat{C}}(c)(\Delta_{\widehat{X}}(\omega)(1 \otimes \omega_1))$ for all $c \in C$, $\omega \in \widehat{X}$ and $\omega_1 \in \widehat{A}$ (having already identified $\widehat{C}'$ with $\widehat{C}$ as an algebra). $\quad\square$

In fact, it follows easily by the foregoing that the Galois coaction γ' of C' on X will then also be isomorphic with γ (i.e. the isomorphism $\phi : C' \to C$ of algebraic quantum groups satisfies $(\phi \otimes \iota)\gamma' = \gamma$), since the comodule structure is determined by the module structure for its dual. Thus, as in the Hopf algebra situation, there is a one-to-one correspondence between Galois objects and bi-Galois objects.

We end this subsection with another definition:

Definition 2.17. If (A, Δ_A) and (C, Δ_C) are two algebraic quantum groups, we call them *monoidally equivalent* if there exists a (non-trivial) A-C-bi-Galois object (N, α, γ).

2.4. Linking quantum groupoids and their duals

We want to warn that this subsection is mainly for motivation, and details are not provided. A more systematic and independent treatment of the following notions will be given in [7] (in the multiplier Hopf algebra case) and [8] (in the algebraic quantum group case).

We want to show that there is a nice way in which $\widehat{C}, \widehat{X}, \widehat{A}$ and their comultiplications can be put together inside a groupoid-like object. To make notation a bit more systematic, we now denote $\widehat{Q}_{11} = \widehat{C}$, $\widehat{Q}_{12} = \widehat{X}$ and $\widehat{Q}_{22} = \widehat{A}$. Denote by $\widehat{Q}_{21}$ or $\widehat{X^{op}}$ (which is for now just a formal notation!) the vector space consisting of linear maps $\widehat{X} \to \widehat{A}$ of the form $\omega \to [\omega', \omega]_{\widehat{A}}$. Then we can form a 2 by 2 matrix algebra of the form $\widehat{Q} = \begin{pmatrix} \widehat{Q}_{11} & \widehat{Q}_{12} \\ \widehat{Q}_{21} & \widehat{Q}_{22} \end{pmatrix}$. As a vector space, this is just the direct sum of its entries. But the matrix notation is convenient to see how the algebra structure works: one just checks that

if we apply the ordinary matrix multiplication rules, we always obtain a coupling between objects of which we know a composition. For example, $\begin{pmatrix} 0 & \omega \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & \omega_1 \end{pmatrix} = \begin{pmatrix} 0 & \omega \cdot \omega_1 \\ 0 & 0 \end{pmatrix}$, while $\begin{pmatrix} 0 & \omega \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ [\omega', \cdot]_{\widehat{A}} & 0 \end{pmatrix} = \begin{pmatrix} [\omega, \omega']_{\widehat{C}} & 0 \\ 0 & 0 \end{pmatrix}$ since $(\omega \cdot) \circ ([\omega', \cdot]_{\widehat{A}}) = [\omega, \omega']_{\widehat{C}}$ on $\widehat{X}$.

There is an easy comultiplication structure available on $\widehat{Q}$. First, we introduce a comultiplication on $\widehat{X^{\mathrm{op}}}$ by putting $\Delta_{\widehat{X^{\mathrm{op}}}}([\omega, \cdot]_{\widehat{A}}) = [\omega^{(2)}, \cdot]_{\widehat{A}} \otimes [\omega^{(1)}, \cdot]_{\widehat{A}}$, where the range has again to be interpreted in some generalized multiplier space $M(\widehat{X^{\mathrm{op}}} \otimes \widehat{X^{\mathrm{op}}})$, using that $\widehat{X^{\mathrm{op}}}$ is a $\widehat{A}$-$\widehat{C}$-bimodule in a natural way. Then the comultiplication on $\widehat{Q}$ is simply $\Delta_{\widehat{Q}}((\omega_{ij})_{ij}) = (\widehat{\Delta}_{ij}(\omega_{ij}))_{ij}$, where the entries on the right are to be seen as elements of $M(\widehat{Q} \otimes \widehat{Q})$, again simply by using matrix multiplication rules, and where the $\widehat{\Delta}_{ij}$ are just the natural comultiplications which we had introduced on the $\widehat{Q}_{ij}$ (for example, $\widehat{\Delta}_{ij} = \Delta_{\widehat{X}}$). Note however, that while $\Delta_{\widehat{Q}}$ is a well-behaved coassociative homomorphism, it is not unit-preserving (i.e., is not non-degenerate)!

The non-unitality of the comultiplication is the reason why $(\widehat{Q}, \Delta_{\widehat{Q}})$ should really be treated as an *algebraic quantum groupoid*, although such an object has not yet been defined in an axiomatic way. Hence, we only show that $\widehat{Q}$ comes equipped with all the structure one would *expect* from an algebraic quantum groupoid, drawing inspiration from [4] (where a theory of weak Hopf algebras is developed, which are very near to quantum groupoids).

Define $\varepsilon_{\widehat{X}}(\omega) = \omega(1)$ for $\omega \in \widehat{X}$, and $\varepsilon_{\widehat{X^{\mathrm{op}}}}([\omega, \cdot]_{\widehat{A}}) = \omega(1)$. These, and the counits for $\widehat{A}$ and $\widehat{C}$, we will then also write with the index notation as we did for the comultiplications on the $\widehat{Q}_{ij}$. Defining $\varepsilon_{\widehat{Q}} : \widehat{Q} \to k$ by $(\omega_{ij})_{ij} \to \sum_{i,j} \omega_{ij}(1)$, we get a counit on $\widehat{Q}$, satisfying the weak multiplicativity condition in Definition 2.1 of [4]. Defining also $S_{\widehat{X}} : \widehat{X} \to \widehat{X^{\mathrm{op}}} : \omega \to [\omega, \cdot]_{\widehat{A}}$, $S_{\widehat{X^{\mathrm{op}}}} : \widehat{X^{\mathrm{op}}} \to \widehat{X} : [\omega, \cdot]_{\widehat{A}} \to \theta_{\widehat{X}}(\omega)$, and $S_{\widehat{Q}} : \widehat{Q} \to (\omega_{ij})_{ij} \to (\widehat{S}_{ji}(\omega_{ji}))_{ij}$, we get that this last map $S_{\widehat{Q}}$ will satisfy the antipode conditions in Definition 2.1 of [4].

Since the unit in the multiplier algebra of $\widehat{Q}$ obviously satisfies the weak comultiplicativity statement of that Definition, it is clear why we can call $(\widehat{Q}, \Delta_{\widehat{Q}})$ a weak multiplier Hopf algebra. But since we are working with algebraic quantum groups, we also expect some integral structure. Now $(\widehat{Q}, \Delta_{\widehat{Q}})$ has to be pictured as a quantum groupoid with an underlying classical set consisting of 2 objects: this corresponds to the 2-dimensional algebra spanned by $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $e_2 = 1 - e_1 \in M(\widehat{Q})$. Then $\widehat{Q}$ can be pictured as being the *groupoid algebra* of some (quantum) space of arrows between these two objects, with $(\widehat{A}, \Delta_{\widehat{A}})$ and $(\widehat{C}, \Delta_{\widehat{C}})$ playing the role of the endomorphism groups of the points. Since in the finite-dimensional classical case, we expect invariant functionals to be given by a balanced sum of Dirac functionals at the unit endomorphisms of the two objects (in analogy with the group algebra case), it should seem quite natural to take $\varphi_{\widehat{Q}} : \widehat{Q} \to k : (\omega_{ij})_{ij} \to \sum_i \widehat{\varphi}_i(\omega_{ii})$ as a *left invariant functional*, where $\widehat{\varphi}_1 = \varphi_{\widehat{C}}$ and $\widehat{\varphi}_2 = \varphi_{\widehat{A}}$. We omit however here the real justification for this terminology, and refer again to [8] for more details.

Next, we sketch what the *dual* of this linking algebraic quantum groupoid looks like (for since finite-dimensional weak Hopf algebras are self-dual, one should expect a duality theory for algebraic quantum groupoids). This dual should of course be connected with X, A and C. Since now the comultiplication is the dual of some kind of matrix multiplication, we expect to see expressions roughly of the form $x_{ij} \to \sum_k y_{ik} \otimes z_{ki}$. The concrete picture is as follows.

Put $Q = C \oplus X^{\mathrm{op}} \oplus X \oplus A$, with the direct sum algebra structure. We write it also as $Q = Q_{11} \oplus Q_{21} \oplus Q_{12} \oplus Q_{11}$. We then have a natural pairing between Q and $\widehat{Q}$: for $x_{ij} \in Q_{ij}$ and $\omega_{kl} \in \widehat{Q}_{kl}$, we put $\langle \omega_{kl}, x_{ij} \rangle = \delta_{k,i} \delta_{l,j} \omega_{ij}(x_{ij})$: the only expression for which this is not yet defined then, is the one between X^{op} and $\widehat{X^{\mathrm{op}}}$. This is determined by $\langle [\omega, \cdot]_{\widehat{A}}, x^{\mathrm{op}} \rangle = \omega(\theta_X(x))$.

This pairing then leads to *eight* different comultiplications on the parts of Q: We define $\Delta_{ik}^j(x_{ij}) \in M(Q_{ij} \otimes Q_{jk})$ for $x_{ij} \in Q_{ij}$ by the property that $(\omega_{ij} \otimes \omega'_{jk})(\Delta_{ik}^j(x_{ij})) = (\omega_{ij} \cdot \omega'_{jk})(x_{ij})$ for all $\omega_{ij} \in \widehat{Q}_{ij}$ and $\omega'_{jk} \in \widehat{Q}_{jk}$. These fit perfectly with the maps we already had constructed: it is easy to check that $\Delta_{11}^1 = \Delta_C$, $\Delta_{12}^1 = \gamma$, $\Delta_{12}^2 = \alpha$, $\Delta_{22}^2 = \Delta_A$ and $\Delta_{22}^1 = \tilde{\beta}$. The map Δ_{21}^1 for example is given by $\Delta_{21}^1(x^{\mathrm{op}}) = (x_{(0)})^{\mathrm{op}} \otimes S_C^{-1}(x_{(-1)})$, which gives a right coaction on X^{op} (which makes it even into a right C-Galois object).

We can group these comultiplications together in one comultiplication for Q, which is then given as $\Delta_Q(x_{ik}) = \Delta^1_{ik}(x_{ik}) + \Delta^2_{ik}(x_{ik})$ for $x_{ik} \in Q_{ik}$. A counit is defined on Q by the formula $\varepsilon_Q(x_{ij}) = \delta_{i,j}\varepsilon_i(x_{ij})$, while an antipode is determined by putting $S_Q(x) = S_X(x)$ and $S_Q(x^{op}) = S_{X^{op}}(x^{op})$ for $x \in X$ (our final argument for our claim that S_X and $S_{X^{op}}$ are antipodes!), and using just the ordinary antipode maps on the C and A-parts. Then some calculations should reveal that again, this gives us the structure of a 'weak multiplier Hopf algebra' on Q. Moreover, we have a 'left invariant functional' φ_Q, which is now given as just the algebraic left integrals on the A and C-parts, by φ_X on X, and by $\psi_X \circ S_X^{-1}$ on X^{op}. Then (Q, Δ_Q) will be like the *function algebra* on a 'quantum' groupoid, with φ_X the operation of integrating out along a Haar system, and then applying the counting measure on the (finite) basis.

We want to end by remarking that this quantum groupoid viewpoint of Galois objects for Hopf algebras appeared already in [3] (see also [12]), but in a somewhat non-symmetric way, and without considering the dual picture (it also seems to have been implicit in earlier work by Schauenburg [22]). It is an important ingredient in generalizing the theory to the locally compact quantum group case (see [5]).

2.5. Concerning $*$-structures

Suppose now again that (A, Δ) is a $*$-algebraic quantum group, and (X, α) a $*$-Galois object. Then we know that φ_X is a positive functional on X. We can introduce on $\widehat{X}$ the $*$-operation

$$\omega^*(x) = \overline{\omega(x^*)}.$$

Lemma 2.18. *For all $\omega \in \widehat{X}$ and $\omega_1 \in \widehat{A}$, we have*

$$(\omega \cdot \omega_1)^* = \omega_1^* \cdot \omega^*.$$

Proof. Choose $x \in X$, then

$$\begin{aligned}
(\omega \cdot \omega_1)^*(x) &= \overline{(\omega \cdot \omega_1)(x^*)} \\
&= \overline{\omega(x^*_{(0)})\omega_1(x^*_{(1)})} \\
&= \omega^*(x_{(0)})\big(S_{\widehat{A}}^{-1}(\omega_1^*)\big)(x_{(1)}) \\
&= (\omega_1^* \cdot \omega^*)(x). \quad \square
\end{aligned}$$

Now consider the following $*$-operation on the dual C of $\widehat{C}$:

$$[\omega, x]_C^* := [\omega^*, x^*]_C.$$

By the previous lemma, it is seen to be a well-defined involution. Writing elements of C in the form $(\iota \otimes \omega)(\gamma(x))$, and using the formula

$$(\omega^*)^{(1)} \otimes (\omega^*)^{(2)} = (\omega^{(2)})^* \otimes (\omega^{(1)})^*,$$

we find that $*$ is anti-multiplicative on C. Finally, Δ_C is $*$-preserving, which is again easily seen by writing an element of C in the aforementioned form and using that γ is $*$-preserving.

Now we show that ψ_C is positive. Take $c \in C$ and $z \in X$ with $\varphi_X(z^*z) = 1$. Write $c \otimes z$ as $\sum_i \gamma(x_i)(1 \otimes y_i)$, then $c^*c = \sum_{i,j}(\iota \otimes \varphi_X((y_i)^* \cdot y_j))(\gamma((x_i)^*x_j))$, so

$$c^*c = \sum_{i,j}\big\langle\varphi_X\big((y_i)^* \cdot y_j\big), (x_i)^*x_j\big\rangle_C.$$

Applying ψ_C, we get

$$\psi_C(c^*c) = \sum_{i,j} \varphi_X\big((y_i)^* y_j\big) \psi_X\big((x_i)^* x_j\big)$$
$$\geqslant 0,$$

since the matrices $(a_{i,j}) = (\varphi_X((y_i)^* y_j))$ and $(b_{i,j}) = (\psi_X((x_i)^* x_j))$ are both positive definite. Hence

Theorem 2.19. *If (X, α) is a $*$-Galois object for a $*$-algebraic quantum group (A, Δ), then the reflected algebraic quantum group (C, Δ_C) also has the structure of a $*$-algebraic quantum group.*

We can also consider the corresponding $*$-operation on $\widehat{C}$. Then we have a formula corresponding to the one of the above lemma, namely for $\omega \in \widehat{X}$ and $b \in \widehat{C}$, we have

$$(b \cdot \omega)^* = \omega^* \cdot b^*.$$

Remark. In some sense, this $*$-operation is not the right one to consider on a conceptual level: one should rather use the (external) $*$-operation $\widehat{X} \to \widehat{X^{\mathrm{op}}}$ (with notation as in the previous subsection), with $\omega^*(x^{\mathrm{op}}) = \overline{\omega(x^*)}$ (considering $\widehat{X^{\mathrm{op}}}$ as functionals on X^{op}). Then we see that, with this definition, actually $(S_{\widehat{X}}(\omega))^*(x) = \overline{\omega(x^*)}$ for $x \in X$, which is more in line with the definition for the $*$-structure on the dual of a $*$-algebraic quantum group. See again [7] and [8] for further discussion.

We end again by giving a definition:

Definition 2.20. Let (A, Δ_A) and (C, Δ_C) be two $*$-algebraic quantum groups. We call them *monoidally $*$-equivalent* if there exists a (non-trivial) A-C-$*$-bi-Galois object (N, α, γ).

Here a $*$-bi-Galois object (N, α, γ) is just a bi-Galois object such that (N, α) is a right A-$*$-Galois object and (N, γ) is a left C-$*$-Galois object. One can easily show that (C, Δ_C) (as a $*$-algebraic quantum group) is again uniquely determined by (N, α).

2.6. Monoidal equivalence

Definition 2.21. If (D, Δ) is an algebraic quantum group, we denote by D-Rep the monoidal category of unital left D-modules, where a morphism between two objects V and W is a linear map $V \to W$ which intertwines the module structure.

Note that this category is indeed monoidal by the usual tensor product of two modules, because of the unitality assumption. (We recall that the unitality of a left D-module M means that $D \cdot M = M$.)

Theorem 2.22. *If (A, Δ_A) and (C, Δ_C) are monoidally equivalent algebraic quantum groups, then the categories of unital left modules for their duals are monoidally equivalent.*

Proof. We construct a monoidal equivalence between $\widehat{A}$-Rep and $\widehat{C}$-Rep. This equivalence is given by the functor $F = \widehat{X} \otimes_{\widehat{A}} -$. This functor is monoidal with respect to the natural isomorphism

$$n_\otimes : \omega \underset{\widehat{A}}{\otimes} (v \otimes w) \to \big(\omega^{(1)} \underset{\widehat{A}}{\otimes} v\big) \otimes \big(\omega^{(2)} \underset{\widehat{A}}{\otimes} w\big),$$

where it is clear how to interpret this and how to show bijectivity (using $\widehat{A} \otimes_{\widehat{A}} V = V$).

We briefly argue how to construct the inverse, without entering into details. Using the symmetric quantum groupoid picture from Section 2.4, we know that X^{op} a right C-Galois object (by the 'comultiplication part' Δ_{21}^1), and we can apply on $\widehat{C}$-Rep the functor $\widehat{X^{op}} \otimes_{\widehat{C}} -$. Then $\widehat{X^{op}} \otimes_{\widehat{C}} (\widehat{X} \otimes_{\widehat{A}} -)$ is seen to be naturally equivalent with the identity, using first the isomorphism $\widehat{X^{op}} \otimes_{\widehat{C}} \widehat{X} \to \widehat{A} : [\omega, \cdot]_{\widehat{A}} \otimes \omega' \to [\omega, \omega']_{\widehat{A}}$, then by using that $\widehat{A} \otimes_{\widehat{A}} -$ is simply the identity. $\quad\square$

Suppose now again that A is a $*$-Galois object. We can equip the category of unital left $\widehat{A}$-modules with an anti-linear conjugation functor $Conj_{\widehat{A}}$ by sending (V, π) to $(\overline{V}, \overline{\pi})$, where $\overline{V}$ is the conjugate vector space of V and with

$$\overline{\pi}(\omega_1) \cdot \overline{v} = \overline{S_{\widehat{A}}(\omega_1^*) \cdot v}.$$

There is a natural isomorphism

$$n_{Conj} : \big(Conj_{\widehat{C}} \circ \big(\widehat{X} \underset{\widehat{A}}{\otimes} -\big) \circ Conj_{\widehat{A}}\big) \to \widehat{X} \underset{\widehat{A}}{\otimes} -,$$

given by

$$\overline{\overline{\omega} \underset{\widehat{A}}{\otimes} \overline{v}} \to \theta_{\widehat{X}}(\omega^*) \underset{\widehat{A}}{\otimes} v,$$

where we use the $*$-structure on $\widehat{X}$ introduced in the beginning of the previous subsection. To see that this is well defined, we have to prove that

$$\theta_{\widehat{X}}\big((\omega \cdot \omega_1)^*\big) \underset{\widehat{A}}{\otimes} v = \theta_{\widehat{X}}(\omega^*) \underset{\widehat{A}}{\otimes} S_{\widehat{A}}(\omega_1^*) \cdot v,$$

for all $\omega \in \widehat{X}, \omega_1 \in \widehat{A}$ and $v \in V$. Now $(\omega \cdot \omega_1)^* = \omega^* \cdot S_{\widehat{A}}^{-1}(\omega_1^*)$ by Lemma 2.18, and $\theta_{\widehat{X}}(\omega \cdot \omega_1) = \theta_{\widehat{X}}(\omega) \cdot S_{\widehat{A}}^2(\omega_1)$ by (vii) of Proposition 1.20, which proves the identity. To prove that it is a natural transformation, we have to show that

$$\theta_{\widehat{X}}\big(\big(S_{\widehat{C}}(b^*) \cdot \omega\big)^*\big) = b \cdot \theta_{\widehat{X}}(\omega^*)$$

for $b \in \widehat{C}$ and $\omega \in \widehat{X}$. But $\theta_{\widehat{X}}((S_{\widehat{C}}(b^*) \cdot \omega)^*) = \theta_{\widehat{X}}(S_{\widehat{C}}^{-2}(b) \cdot \omega^*)$, so the identity follows by the 'mirror version' of the identity (vii) in Proposition 1.20.

Now look at the category $\widehat{A}$-Rep* of unital left $\widehat{A}$-modules which have a pre-Hilbert space structure such that the resulting representation of $\widehat{A}$ is $*$-preserving. The morphisms in the category are now required to have an adjoint. If V is an object in this category, we have a canonical morphism Sc : $V \otimes \overline{V} \to \varepsilon$, sending $v \otimes \overline{w}$ to $\langle v, w \rangle$. Note that we take the convention where the scalar product is antilinear on its second argument. Using the natural isomorphisms for tensoring and conjugating, the map Sc is sent to a map $(\widehat{X} \otimes_{\widehat{A}} V) \otimes \overline{(\widehat{X} \otimes_{\widehat{A}} V)} \to \varepsilon$. This provides a sesquilinear form on $\widehat{X} \otimes_{\widehat{A}} V$. We will show in Proposition 2.23 that it equips $\widehat{X} \otimes_{\widehat{A}} V$ with a pre-Hilbert space structure. This is the *categorical approach* to arrive at the induced Hilbert space structure.

We can also use a specific $\widehat{A}$-valued inner product on $\widehat{X}$ to perform the induction, given by

$$\langle \omega, \omega' \rangle_{\widehat{A}} = [\omega^*, \omega']_{\widehat{A}},$$

again using the $*$-structure on $\widehat{X}$ introduced in the beginning of the previous subsection. The fact that $\langle \cdot, \cdot \rangle_{\widehat{A}}$ is a $\widehat{A}$-valued inner product means that it is a sesquilinear map with values in $\widehat{A}$, antilinear in the *first* argument, such that

1. $\langle \omega, \omega' \rangle_{\widehat{A}}^* = \langle \omega', \omega \rangle_{\widehat{A}}$,
2. $\langle \omega, \omega \rangle_{\widehat{A}} \geqslant 0$ with equality iff $\omega = 0$, and
3. $\langle \omega, \omega' \cdot \omega_1 \rangle_{\widehat{A}} = \langle \omega, \omega' \rangle_{\widehat{A}} \cdot \omega_1$ for $\omega_1 \in \widehat{A}$.

Moreover, we have

4. $\langle b \cdot \omega, \omega' \rangle_{\widehat{A}} = \langle \omega, b^* \cdot \omega' \rangle_{\widehat{A}}$, $b \in \widehat{C}$.

We will prove this below. We can then make an inner product on $\widehat{X} \otimes_{\widehat{A}} V$ by setting

$$\langle \omega \otimes v, \omega' \otimes w \rangle = \langle v, \langle \omega, \omega' \rangle_{\widehat{A}} \cdot w \rangle.$$

This is the *C*-algebraic* approach (cf. [21]).

Finally, there is a *von Neumann algebraic* approach. Namely, introduce in $\widehat{A}$ the inner product determined by $\langle \omega_1, \omega_2 \rangle = \psi_{\widehat{A}}(\omega_2^* \omega_1)$ (i.e. $\langle \varphi(\cdot a), \varphi(\cdot b) \rangle = \varphi(b^*a)$), and introduce in $\widehat{X}$ the inner product determined by

$$\langle \varphi_X(\cdot y), \varphi_X(\cdot x) \rangle = \varphi_X(x^*y).$$

For $\omega \in \widehat{X}$, denote by L_ω the map $\widehat{A} \to \widehat{X}$ which sends ω_1 to $\omega \cdot \omega_1$. Then $L_\omega^* L_{\omega'}$ will be left multiplication with some element of $\widehat{A}$, and identifying the operator with this element, we can define

$$\langle \omega \otimes v, \omega' \otimes w \rangle = \langle v, L_\omega^* L_{\omega'} \cdot w \rangle.$$

Remark however that with this scalar product on $\widehat{X}$, the right or left representation of $\widehat{A}$ on $\widehat{X}$ is in general *not* a *-representation. This is because the left action of $\widehat{A}$ on $\widehat{X}$ is not an analogue of the left multiplication of $\widehat{A}$ on $\widehat{A}$.

Proposition 2.23. *All three sesquilinear forms on $\widehat{X} \otimes_{\widehat{A}} V$ coincide, providing this space with a pre-Hilbert space structure such that the induced representation of C is *-preserving.*

Proof. We first show that $L_\omega^* L_{\omega'} = \langle \omega, \omega' \rangle_{\widehat{A}}$. This means that for all $\omega, \omega' \in \widehat{X}$ and $\omega_1, \omega_2 \in \widehat{A}$ we have

$$\langle \omega \cdot \omega_1, \omega' \cdot \omega_2 \rangle = \langle \omega_1, \langle \omega, \omega' \rangle_{\widehat{A}} \omega_2 \rangle.$$

Writing $\omega' = \varphi_X(\cdot x)$ and $\omega_2 = \varphi(\cdot a)$, we have

$$\begin{aligned}
\langle \omega \cdot \omega_1, \omega' \cdot \omega_2 \rangle &= \langle \omega \cdot \omega_1, \varphi_X(\cdot x_{(0)}) \varphi(\delta S(x_{(1)}) a) \rangle \\
&= (\omega \cdot \omega_1)(x_{(0)}^*) \varphi(a^* S(x_{(1)})^* \delta),
\end{aligned}$$

while

$$\begin{aligned}
\langle \omega_1, \langle \omega, \omega' \rangle_{\widehat{A}} \omega_2 \rangle &= \langle \omega_1, \omega^*(x_{(0)}) \varphi(\cdot x_{(1)}) \varphi(\delta S(x_{(2)}) a) \rangle \\
&= \omega(x_{(0)}^*) \varphi(a^* S(x_{(2)})^* \delta) \omega_1(x_{(1)}^*),
\end{aligned}$$

which shows the equality.

Now we show the equivalence of the categorical and the C*-algebraic approach. In the categorical approach, we have for $\omega \in \widehat{X}$ that

$$\left\langle \omega^{(1)} \cdot \omega_1 \otimes v, \left(\omega^{(2)} \circ \theta_X^{-1}\right)^* \otimes w \right\rangle = F(\mathrm{Sc})(\omega \otimes v \otimes w)$$
$$= \omega(1)\langle v, w \rangle,$$

where F still denotes the functor $\widehat{X} \otimes_{\widehat{A}} -$. Since

$$\left\langle \left(\theta_{\widehat{X}}^{-1}\left(\omega^{(2)}\right)\right)^*, \omega^{(1)} \right\rangle_{\widehat{A}} = \omega(1)\varepsilon,$$

we are done.

From the von Neumann algebraic picture, it follows that this inner product is indeed positive (using that the $*$-structure on $\widehat{A}$ is exactly the adjoint of left multiplication with respect to the stated scalar product on $\widehat{A}$). From the categorical picture, it follows quite immediately that the resulting representation of $\widehat{C}$ is $*$-preserving. Finally, also the non-degeneracy of the inner product follows from the categorical viewpoint: for the vectors of length zero in $F(V)$ form a subobject of V, which is sent to 0 by F^{-1}.

The fact that the $\widehat{A}$-valued map is indeed a $\widehat{A}$-hermitian product follows then in a straightforward manner from the categorical and von Neumann algebraic viewpoint. $\quad\square$

So in any case, $\widehat{X} \otimes_{\widehat{A}} -$ can be lifted to a functor between the $*$-representation categories.

Proposition 2.24. *The natural transformation $n_\otimes$ is unitary.*

Proof. This follows by using the C*-algebraic picture, and using the identity

$$\Delta_{\widehat{A}}\left(\langle \omega, \omega' \rangle_{\widehat{A}}\right) = \left\langle \omega^{(1)}, \omega'^{(1)} \right\rangle_{\widehat{A}} \otimes \left\langle \omega^{(2)}, \omega'^{(2)} \right\rangle_{\widehat{A}}. \qquad \square$$

The map n_{Conj} however will only be unitary in case $S^2 = \iota$, and moreover, if this is not the case, no unitary intertwiner between $\overline{\widehat{X} \otimes \overline{V}}$ and $\widehat{X} \otimes V$ can be constructed. We can however repair this situation by changing the definition of the conjugation operator: now we send (V, π) to $(\overline{V}, \overline{\pi})$ with

$$\overline{\pi}(\omega_1) \cdot \overline{v} = \overline{R_{\widehat{A}}(\omega_1)^* \cdot v},$$

where $R_{\widehat{A}}$ is the unitary antipode for $\widehat{A}$ (see [20]). As remarked already in the previous section, note that we can take the square $\theta_X^{1/2}$ of the positive diagonalizable operator θ_X, i.e. $\theta_X^{1/2} : X \to X$ is a diagonalizable map $X \to X$ with positive eigenvalues, and $\theta_X^{1/2}(\theta_X^{1/2}(x)) = \theta_X(x)$ for all $x \in X$. This $\theta_X^{1/2}$ will still be a multiplicative automorphism of X.

Proposition 2.25. *The map*

$$\overline{\omega \underset{\widehat{A}}{\otimes} \overline{v}} \to \theta_{\widehat{X}}^{1/2}(\omega^*) \underset{\widehat{A}}{\otimes} v$$

provides a well-defined unitary intertwiner between $\overline{\widehat{X} \otimes \overline{V}}$ and $\widehat{X} \otimes V$.

Proof. To see if this map is well defined, we have to check the identity

$$\theta_{\widehat{X}}^{1/2}\left((\omega \cdot \omega_1)^*\right) = \theta_{\widehat{X}}^{1/2}(\omega^*) \cdot R_{\widehat{A}}\left(\omega_1^*\right),$$

with $\omega \in \widehat{X}$, $\omega_1 \in \widehat{A}$. Using Lemma 2.18, this reduces to proving that

$$\theta_{\widehat{X}}^{1/2}(\omega \cdot \omega_1) = \theta_{\widehat{X}}^{1/2}(\omega) \cdot \theta_{\widehat{A}}^{1/2}(\omega_1),$$

where $\theta_{\widehat{A}} = S_{\widehat{A}}^2$ and $\theta_{\widehat{A}}^{1/2}$ is its positive root (so that $\theta_{\widehat{A}}^{1/2} \circ R_{\widehat{A}} = S_{\widehat{A}}$). This follows from $\theta_{\widehat{X}}(\omega \cdot \omega_1) = \theta_{\widehat{X}}(\omega) \cdot S_{\widehat{A}}^2(\omega_1)$, by taking ω an eigenvector for $\theta_{\widehat{X}}$ and ω_1 an eigenvector for $S_{\widehat{A}}^2$.

The way to prove that this is a natural transformation is very similar, and we omit it.

Finally, to prove unitarity we have to prove the identity

$$R_{\widehat{A}}\big([\omega', \omega]_{\widehat{A}}\big) = \big[\theta_{\widehat{X}}^{-1/2}(\omega), \theta_{\widehat{X}}^{1/2}(\omega')\big]_{\widehat{A}},$$

where $\omega, \omega' \in \widehat{X}$. Applying $\theta_{\widehat{A}}^{1/2}$ and using Lemma 2.18, this reduces to proving that

$$\big[\theta_{\widehat{X}}^{1/2}(\omega), \theta_{\widehat{X}}^{1/2}(\omega')\big]_{\widehat{A}} = \theta_{\widehat{A}}^{1/2}\big([\omega, \omega']_{\widehat{A}}\big).$$

This identity is true when $\theta_{\widehat{X}}^{1/2}$ is replaced by $\theta_{\widehat{X}}$ and $\theta_{\widehat{A}}^{1/2}$ is replaced by $S_{\widehat{A}}^2$, using formula (xvi) of Proposition 1.20. Again by using an eigenvector argument, it is also true as stated. $\square$

In any case, Proposition 2.24 was already sufficient to conclude:

Theorem 2.26. *Let (A, Δ_A) and (C, Δ_C) be two monoidally *-equivalent *-algebraic quantum groups. Then the monoidal *-categories of unital *-representations in Hilbert spaces of their duals are monoidally *-equivalent.*

For the notions of *-category and monoidal *-equivalence, we refer for example to [19].

3. An example

The following examples of infinite-dimensional Hopf algebras with a left invariant functional can be found in [26] and [28]. We slightly generalize the construction to fit them both in a family.

Definition 3.1. Let $n > 1$, $m \geqslant 1$ be natural numbers, and $\lambda \in k$ such that λ^m is a primitive nth root of unity. Let $A_\lambda^{n,m}$ be the unital algebra over k generated by elements a, a^{-1} and b, and with defining relations: a^{-1} is the inverse of a, $ab = \lambda ba$ and $b^n = 0$. Then we can define a comultiplication on $A_\lambda^{n,m}$ determined on the generators by

$$\Delta(a) = a \otimes a,$$

$$\Delta(b) = b \otimes a^m + 1 \otimes b.$$

This makes $(A_\lambda^{n,m}, \Delta)$ an algebraic quantum group of compact type.

To prove that this comultiplication is indeed well defined, we only have to use the well-known fact that $(s + t)^l = s^l + t^l$ when s, t are variables satisfying the commutation $st = qts$ with q a primitive lth root of unity (see e.g. [15]). Now $(A_\lambda^{n,1}, \Delta)$ is the example in [28], and with the further relation $a^n = 1$, this reduces to the two-generator Taft algebras. The Hopf algebra $(A_\lambda^{n,2}, \Delta)$ is isomorphic with the example constructed in [26].

The left invariant functional φ of $(A, \Delta) = (A_\lambda^{n,m}, \Delta)$ is defined by $\varphi(a^p b^q) = \delta_{p,0}\delta_{q,n-1}$, $p \in \mathbb{Z}$, $0 \leqslant q < n$. As A is infinite-dimensional, the dual $\widehat{A}$ is necessarily of discrete type and not compact, i.e. it is a genuine multiplier Hopf algebra. This is a difference with the Taft algebras, which are self-dual. Remark that there can still be defined a pairing between A and itself, but it will be degenerate.

In [18] the Galois objects for the Taft algebras were classified. It provides the motivation for the following construction. Fix $(A, \Delta) = (A_\lambda^{n,m}, \Delta)$ as above, and assume moreover that λ is a primitive nth root of unity and m and n are coprime. The condition 'λ^m is a primitive nth root of unity' follows from this assumption.

Definition 3.2. Take $\mu \in k$. Let $X = X_{\lambda,\mu}^{n,m}$ be the unital algebra generated by x, x^{-1} and y, with the defining relations: x^{-1} is the inverse of x, $xy = \lambda yx$ and $y^n = \mu x^{mn}$. A right coaction α of (A, Δ) on X is defined on the generators by

$$\alpha(x) = x \otimes a,$$

$$\alpha(y) = y \otimes a^m + 1 \otimes b.$$

It is again easy to show that this has a well-defined extension to the whole of X.

Proposition 3.3. (X, α) is a right A-Galois object.

Proof. First of all, we have to see if X is not trivial. We follow standard procedure. Let V be a vector space over k which has a basis of vectors of the form $e_{p,q}$ with $p \in \mathbb{Z}$ and $0 \leqslant q < n$. Define operators x' and y' by

$$x' \cdot e_{p,q} = e_{p+1,q} \qquad \text{for all } p \in \mathbb{Z},\ 0 \leqslant q < n,$$

$$y' \cdot e_{p,q} = \lambda^{-p} e_{p,q+1} \qquad \text{if } p \in \mathbb{Z},\ 0 \leqslant q < n-1,$$

$$y' \cdot e_{p,n-1} = \mu \lambda^{-p} e_{p+nm,0} \qquad \text{if } p \in \mathbb{Z}.$$

Then it is easy to see that x' is invertible and that $x'y' = \lambda y'x'$. Also:

$$
\begin{aligned}
y'^n \cdot e_{p,q} &= \lambda^{-p(n-1-q)} y'^{1+q} \cdot e_{p,n-1} \\
&= \mu \lambda^{-p(n-1-q)} \lambda^{-p} y'^q \cdot e_{p+nm,0} \\
&= \mu \lambda^{-p(n-1-q)} \lambda^{-p} \lambda^{-pq} e_{p+nm,q} \\
&= \mu \lambda^{-pn} e_{p+nm,q} \\
&= \mu x'^{mn} \cdot e_{p,q}.
\end{aligned}
$$

This gives us a non-trivial representation of X. Moreover, it is easy to see that this representation is faithful.

Define by $\tilde{\beta} : A \to X^{\mathrm{op}} \otimes X$ the homomorphism generated by

$$\tilde{\beta}(a) = (x^{-1})^{\mathrm{op}} \otimes x,$$

$$\tilde{\beta}(b) = -(yx^{-m})^{\mathrm{op}} \otimes x^m + 1 \otimes y.$$

This is well defined: for example, we have

$$
\begin{aligned}
\tilde{\beta}(b)^n &= ((-yx^{-m})^n)^{\mathrm{op}} \otimes x^{mn} + \mu(1 \otimes x^{mn}) \\
&= ((-1)^n \lambda^{mn(n-1)/2} + 1)\mu(1 \otimes x^{mn}) \\
&= 0,
\end{aligned}
$$

using that λ^m is a primitive root of unity. Denoting $\beta = (S_{X^{\mathrm{op}}} \otimes \iota)\tilde{\beta}$ with $S_{X^{\mathrm{op}}}$ the canonical map $X^{\mathrm{op}} \to X$, and writing $\beta(c) = c^{[1]} \otimes c^{[2]}$ for $c \in A$, it is easy to compute that

$$z_{(0)} z_{(1)}{}^{[1]} \otimes z_{(1)}{}^{[2]} = 1 \otimes z,$$

$$c^{[1]} c^{[2]}{}_{(0)} \otimes c^{[2]}{}_{(1)} = 1 \otimes c$$

for all $z \in \{x, y, x^{-1}\}$ and $c \in \{a, b, a^{-1}\}$, and hence for all $z \in X, c \in A$. This is enough to know that the action is Galois. $\quad\square$

The extension $k \subseteq X$ will be cleft (see e.g. Definition 2.2.3 in [24]), by the comodule isomorphism $\Psi_X : X \to A : x^p y^q \to a^p b^q$, $p \in \mathbb{Z}$ and $0 \leqslant q < n$. The associated scalar cocycle η is given by $\eta(a^p b^q \otimes a^r b^s) = 0$, except for $q = s = 0$, where it is 1, and when $q + s = n$, in which case it equals $\mu \lambda^{-rq}$. (We want to thank the referee for pointing out that this Hopf algebra is... pointed, so that any Galois object is automatically cleft (see [13]).)

We determine the extra structure occurring in this example. First note that we have shown that the elements of the form $x^p y^q$ with $p \in \mathbb{Z}$ and $0 \leqslant q < n$ form a basis. Then we have

$$\varphi_X(x^p y^q) = \delta_{q,n-1} \delta_{p,0} \quad \text{for } p \in \mathbb{Z}, \ 0 \leqslant q < n,$$

$$\psi_X(x^p y^q) = \delta_{q,n-1} \delta_{p,m(1-n)} \lambda^{-m} \quad \text{for } p \in \mathbb{Z}, \ 0 \leqslant q < n,$$

$$\delta_X = x^{(n-1)m},$$

$$\sigma_X(x) = \lambda^{-1} x, \qquad \theta_X(x) = x$$

$$\sigma_X(y) = y, \qquad \theta_X(y) = \lambda^m y,$$

by some easy computations (where we have used notation as in the first section). It is of course the nature of the example which makes the structure so similar to the one of A.

Now we determine the associated algebraic quantum group (C, Δ_C). Note that we could determine the structure with the help of the cocycle, but we wish to directly use the Galois object itself, since this is easier. In particular, we exploit the pairing between (C, Δ_C) and its dual $(\widehat{C}, \Delta_{\widehat{C}})$.

We first give a heuristic reasoning. We determine the algebra structure of $\widehat{C}$. We need a description of the dual $\widehat{A}$ of $A_\lambda^{n,m}$. It has a basis consisting of expressions $e_p d^q$ with $p \in \mathbb{Z}$ and $0 \leqslant q < n$, where $e_p \in \widehat{A}$ and $d \in M(\widehat{A})$, such that $e_p e_q = \delta_{p,q} e_p$, $d e_p = e_{p-m} d$ and $d^n = 0$. With $c = \sum_k \lambda^{-k} e_k \in M(\widehat{A})$, the comultiplication is determined by

$$\Delta(e_p) = \sum_t e_t \otimes e_{p-t},$$

$$\Delta(d) = d \otimes c + 1 \otimes d.$$

Now the left action of $\widehat{A}$ on X is given by

$$e_s \cdot x^p y^q = \delta_{p,s-mq} x^p y^q,$$

$$d \cdot x^p y^q = C_q x^p y^{q-1}, \quad 0 < q < n,$$

$$d \cdot x^p = 0,$$

where $C_q = \frac{(1-\lambda^{mq})}{(1-\lambda^m)} \lambda^{m(q-1)}$. Consider the operators g_s and h acting on the right of X by

$$x^p y^q \cdot g_s = \delta_{p,-s} x^p y^q,$$

$$x^p y^q \cdot h = C_q x^{p+m} y^{q-1}, \quad 0 < q < n - 1,$$

$$x^p \cdot h = 0.$$

Then it is easy to see that h and g_s commute with the left action of $\widehat{A}$. We see that $h \cdot g_s = g_{s+m} \cdot h$, that $g_s g_t = \delta_{s,t} g_s$ and that $h^n = 0$. The span of $g_s h^q$ will form our algebra $\widehat{C}$. Now denote by $u_{p,q}$ the elements in C such that $\langle u_{p,q}, e_r d^s \rangle = \delta_{p,r} \delta_{q,s}$, and denote $u = u_{-1,0}$, $v = u_{0,0}$ and $w = u_{0,1}$. Then we have $\gamma(x) = u \otimes x$ and $\gamma(y) = v \otimes y + w \otimes x^m$ by using the action of $\widehat{C}$. Since this has to commute with α, we find that $v = 1$. Using that $y^n = \mu x^{mn}$ we find that $\mu + w^n = \mu u^{mn}$, and using $xy = \lambda yx$, we get $uw = \lambda wu$. Furthermore, the fact that x is invertible gives that u is invertible. This then completely determines the structure of C. The coalgebra structure is determined by the usual

$$\Delta(u) = u \otimes u,$$

$$\Delta(w) = w \otimes u^m + 1 \otimes w.$$

We can now make things exact.

Proposition 3.4. *Let C be the unital algebra generated by three elements u, u^{-1} and w, with defining relations: u^{-1} is the inverse of u, $uw = \lambda wu$ and $\mu \cdot 1 + w^n = \mu u^{mn}$. Then C is not trivial. We can define a unital multiplicative comultiplication Δ_C on C, given on the generators by*

$$\Delta_C(u) = u \otimes u,$$

$$\Delta_C(w) = w \otimes u^m + 1 \otimes w,$$

making it an algebraic quantum group of compact type. It has a left coaction γ on X determined by

$$\gamma(x) = u \otimes x,$$

$$\gamma(y) = 1 \otimes y + w \otimes x^m,$$

making it a A-C-bi-Galois object.

Proof. It is easy to see that Δ_C and γ can be extended, that Δ_C is coassociative and γ a coaction, and that γ commutes with the right coaction of A. Since now C is already a bi-algebra, it follows from the general theory of Hopf–Galois extensions that if γ can be shown to make X a left C-Galois object, then automatically C will be a Hopf algebra, hence the reflected algebraic quantum group of A.

We can again show this by explicitly constructing a homomorphism $\tilde{\beta}_C : C \to X \otimes X^{\mathrm{op}}$, $\beta_C = (\iota \otimes S_{X^{\mathrm{op}}})\tilde{\beta}_C$, $\beta_C(c) = c^{[-2]} \otimes c^{[-1]}$. On generators it is given by $\beta_C(u) = x \otimes x^{-1}$ and $\beta_C(w) = y \otimes x^{-m} - 1 \otimes yx^{-m}$. Again the same chore shows that it has a well-defined extension to C, and that it provides the good inverse for the Galois map associated with γ. This concludes the proof. □

Remarks. 1. If the characteristic of k is zero, then C will not be isomorphic to A when $\mu \neq 0$. For in A, the only group-like elements are powers of a. Thus any isomorphism would send u to a power a^l of a. But then $\mu(a^{lmn} - 1)$ would have to be an nth power in A, hence, dividing out by b, also in $k[a, a^{-1}]$. This is impossible.

2. As we have remarked, this example is a cocycle (double) twist construction by a cocycle η. We have already given the 2-cocycle as a function on $A \otimes A$. But it is also natural to see it as a multiplier of $\widehat{A} \otimes \widehat{A}$. Then we have the expression

$$\eta = 1 \otimes 1 + \mu \sum_{q=0}^{n-1} \frac{1}{(\lambda^m; \lambda^m)_{q-1} \cdot (\lambda^m; \lambda^m)_{n-q-1}} d^q \otimes d^{n-q} c^q,$$

with the notation for the dual as before, and where $(a; z)_k$ denotes the z-shifted factorial [15]. Now consider the algebra generated by c and d as the fiber at λ^m of the field of algebras on $\mathbb{C}_0$ with the fiber in z generated by c_z, d_z with c_z invertible, $d_z^n = 0$ and $c_z d_z = z d_z c_z$, and with the extra relation $c_z^k = 1$ if z is a primitive kth root of unity. Then we can formally write

$$\eta = 1 \otimes 1 + \mu \cdot \lim_{z \to \lambda^m} \frac{1}{(z; z)_{n-1}} (d_z \otimes c_z + 1 \otimes d_z)^n,$$

where we take a limit over points which are *not* roots of unity. In this way, since c, d generate a finite-dimensional 2-generator Taft algebra, we find back a part of the cocycles of [18]. In fact, *any* of those cocycles should give a cocycle inside $M(\widehat{A} \otimes \widehat{A})$, hence a cocycle functional on $A \otimes A$. We have however not carried out the computations in this general case.

3. There does not seem to be any straightforward modification of the two-generator Taft algebra Galois objects that provides a Galois object for the *dual* of some $A_\lambda^{n,m}$. It would be interesting to see if such non-trivial Galois objects exist.

Appendix A

A.1. Multipliers

Let A be a non-degenerate algebra over a field k, with or without a unit. The non-degeneracy condition means that $ab = 0$ for all $b \in A$ implies $a = 0$, and $ab = 0$ for all $a \in A$ implies $b = 0$. As a set, the multiplier algebra $M(A)$ of A consists of couples (λ, ρ), where λ and ρ are linear maps $A \to A$, obeying the following law:

$$a\lambda(b) = \rho(a)b, \quad \text{for all } a, b \in A.$$

In practice, we write m for (λ, ρ), and denote $\lambda(a)$ by ma and $\rho(a)$ by am. Then the above law is simply an associativity condition. With the obvious multiplication by composition of maps, $M(A)$ becomes an algebra, called the multiplier algebra of A. Moreover, if $k = \mathbb{C}$ and A is a $*$-algebra, $M(A)$ also carries a $*$-operation: for $m \in M(A)$ and $a \in A$, we define m^* by $m^*a = (a^*m)^*$ and $am^* = (ma^*)^*$.

There is a natural map $A \to M(A)$, letting an element a correspond with left and right multiplication by it. Because of non-degeneracy, this algebra morphism will be an injection. In this way, non-degeneracy compensates the possible lack of a unit. Note that, when A is unital, $M(A)$ is equal to A.

Let B be another non-degenerate algebra, and f a non-degenerate algebra homomorphism $A \to M(B)$, where by the non-degeneracy we mean that $f(A)B = B = Bf(A)$. Then f can be extended to an algebra morphism from $M(A)$ to $M(B)$, by defining $f(m)(f(a)b) = f(ma)b$ and $(bf(a))f(m) = bf(am)$ for $m \in M(A), a \in A$ and $b \in B$.

A.2. Multiplier Hopf algebras

A *regular multiplier Hopf algebra* [25] consists of a couple (A, Δ), with A a non-degenerate algebra, and Δ, the *comultiplication*, a non-degenerate homomorphism $A \to M(A \otimes A)$. Moreover, (A, Δ) has to satisfy the following conditions:

- $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$ (coassociativity).

- The maps

$$T_{\Delta 2} : A \otimes A \to M(A \otimes A) : a \otimes b \to \Delta(a)(1 \otimes b),$$

$$T_{1\Delta} : A \otimes A \to M(A \otimes A) : a \otimes b \to (a \otimes 1)\Delta(b),$$

$$T_{\Delta 1} : A \otimes A \to M(A \otimes A) : a \otimes b \to \Delta(a)(b \otimes 1),$$

$$T_{2\Delta} : A \otimes A \to M(A \otimes A) : a \otimes b \to (1 \otimes a)\Delta(b)$$

all induce linear bijections $A \otimes A \to A \otimes A$.

The first condition can be made sense of by showing that $(\Delta \otimes \iota)$ and $(\iota \otimes \Delta)$ are non-degenerate maps $A \otimes A \to M(A \otimes A \otimes A)$, hence can be extended to $M(A \otimes A)$. The T-maps can be used to define a counit (which will be a homomorphism from A to k) and an antipode (which will be an anti-automorphism). Both counit and antipode will be unique, and will satisfy the corresponding equations of those defining them in the Hopf algebra case.

When A is a $*$-algebra over $\mathbb{C}$ and Δ is $*$-preserving, we call (A, Δ) a *regular multiplier Hopf $*$-algebra*. In this case ε is a $*$-homomorphism, while S satisfies $S(a^*) = (S^{-1}(a))^*$.

A.3. Algebraic quantum groups

An *algebraic quantum group* [26] is a regular multiplier Hopf algebra (A, Δ) for which there exists a non-zero functional φ on A such that

$$(\iota \otimes \varphi)\big(\Delta(a)(b \otimes 1)\big) = \varphi(a)b, \quad \text{for all } a, b \in A.$$

A $*$-*algebraic quantum group* is an algebraic quantum group which is at the same time a multiplier Hopf $*$-algebra, such that the functional φ is positive: for every $a \in A$, we have $\varphi(a^*a) \geqslant 0$. This extra condition is very restrictive.

For any algebraic quantum group, the functional φ will be unique up to multiplication with a scalar. It will be faithful in the following sense: if $\varphi(ab) = 0$ for all $b \in A$ or $\varphi(ba) = 0$ for all $b \in A$, then $a = 0$. The functional φ is called the *left invariant functional*. Then (A, Δ) will also have a functional ψ, such that

$$(\psi \otimes \iota)\big(\Delta(a)(1 \otimes b)\big) = \psi(a)b, \qquad \text{for all } a, b \in A.$$

This map ψ is called the *right invariant functional*. If A is a $*$-algebraic quantum group, then ψ can still be chosen so that it is positive.

For any algebraic quantum group, there exists a unique automorphism σ of the algebra A, satisfying $\varphi(ab) = \varphi(b\sigma(a))$ for all $a, b \in A$. It is called *the modular automorphism*. There also exists a unique invertible multiplier $\delta \in M(A)$ such that

$$(\varphi \otimes \iota)\big(\Delta(a)(1 \otimes b)\big) = \varphi(a)\delta b,$$

$$(\varphi \otimes \iota)\big((1 \otimes b)\Delta(a)\big) = \varphi(a)b\delta,$$

for all $a, b \in A$. It is called *the modular element*.

There is a particular number that can be associated with an algebraic quantum group. Since $\varphi \circ S^2$ is a left invariant functional, the uniqueness of φ implies there exists $\tau \in k$ such that $\varphi(S^2(a)) = \tau\varphi(a)$, for all $a \in k$. One can also show that $\sigma(\delta) = \tau^{-1}\delta$. If A is a $*$-algebraic quantum group then $\tau = 1$ [6].

Note that all formulas which are proven in the second section of this article were known to hold when $X = A$ and $\alpha = \Delta$ (in which case $\beta = (S \otimes \iota) \circ \Delta$), and we have used some of them in proving our statements.

To any algebraic quantum group (A, Δ), one can associate another algebraic quantum group $(\widehat{A}, \Delta_{\widehat{A}})$ which is called its dual. As a set it consists of functionals on A of the form $\varphi(\cdot\, a)$ with $a \in A$, where φ is the left invariant functional on A. Its multiplication and comultiplication are dual to respectively the comultiplication and multiplication on A. Intuitively, this means that

$$\Delta_{\widehat{A}}(\omega_1)(a \otimes b) = \omega_1(ab),$$

$$(\omega_1 \cdot \omega_2)(a) = (\omega_1 \otimes \omega_2)(\Delta(a)),$$

for $a, b \in A$ and $\omega_1, \omega_2 \in \widehat{A}$, but some care is needed in giving sense to these formulas.

The counit on $\widehat{A}$ is defined by evaluation in 1, while the antipode is the dual of the antipode of A: if $S_{\widehat{A}}$ denotes the antipode of $\widehat{A}$, then

$$S_{\widehat{A}}(\omega_1)(a) = \omega_1(S(a)),$$

for $\omega_1 \in \widehat{A}$ and $a \in A$. The left integral $\varphi_{\widehat{A}}$ of $\widehat{A}$ is determined by $\varphi_{\widehat{A}}(\psi(a\, \cdot\,)) = \varepsilon(a)$.

If A is a *-algebraic quantum group, then one can endow $\widehat{A}$ with the *-operation $\omega_1^*(a) = \overline{\omega_1(S(a)^*)}$ for $\omega_1 \in \widehat{A}$ and $a \in A$. Then $\varphi_{\widehat{A}}$ turns out to be a positive left invariant functional, so that also $(\widehat{A}, \Delta_{\widehat{A}})$ is a *-algebraic quantum group.

A.4. Covering issues

When working with multiplier Hopf algebras, it is advantageous to use the Sweedler notation to gain insight into certain formulas. However in this context this is not as straightforward as for Hopf algebras. The problem is that if (A, Δ) is a multiplier Hopf algebra, then $\Delta(a)$ is an element of $M(A \otimes A)$, and can in general not be written as a sum of elementary tensors. So if we denote $\Delta(a) = \sum a_{(1)} \otimes a_{(2)}$, then this is purely formal, as the right-hand side is no well-defined sum of finitely many elements. This gives problems if we want to apply a map to one of the legs of $\Delta(a)$. This is the situation in which we need coverings. For elements of the form $\Delta(a)(1 \otimes b)$ with $a, b \in A$ *are* finite sums of elementary tensors in $A \otimes A$, so if we denote this by $\sum a_{(1)} \otimes a_{(2)}b$, there is no trouble in applying a map to the first leg. We then say that 'the variable $a_{(2)}$ is covered on the right by b'. Although this seems simple, the situation can become quite complicated when multiple coverings are needed (see e.g. the examples in [9]). However, in our paper the situation is not so bad, probably because we are working with algebraic quantum groups in stead of general multiplier Hopf algebras: mostly it is seen at first sight if an expression is well covered or not. This is why we have opted not to emphasize the covering issues too much, since this would probably have obscured certain proofs and statements.

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