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Canonical representations on the two-sheeted hyperboloid

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ABSTRACT

A two-sheeted hyperboloid in $\mathbb{R}^3$ can be identified with the Riemann sphere $\overline{\mathbb{C}}$ without the equator. Canonical representations of the group $\mathrm{SU}(1,1)$ on $\overline{\mathbb{C}}$ are defined as the restriction of spherical representations of the "overgroup" $\mathrm{SL}(2,\mathbb{C})$. We decompose these canonical representations into irreducible constituents and decompose Berezin forms.

Canonical representations on Hermitian symmetric spaces G/K were introduced by Vershik, Gelfand and Graev [7] (for the Lobachevsky plane) and Berezin [1]. They are unitary with respect to some invariant non-local inner product (the Berezin form). We think that it is natural to consider canonical representations in a wider sense: to give up the condition of unitarity and let these representations act on sufficiently extensive spaces, in particular, on distributions. The notion of canonical representation (in this wide sense) can be extended to other classes of semisimple symmetric spaces G/H, in particular, to para-Hermitian symmetric spaces, see [5]. Moreover, sometimes it is natural to consider several spaces G/H_i together, possibly with different H_i, embedded as open G-orbits into a compact manifold Ω, where G acts, so that Ω is the closure of these orbits.

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One of sources of getting canonical representations consists of the following. We take some group $\widetilde{G}$ (overgroup) containing G, take a series of representations $\widetilde{R}$ of $\widetilde{G}$ induced by characters of some parabolic subgroup $\widetilde{P}$ associated with G/H, and restrict these representations to G.

In this paper we carry out this program for a crucial example: the two-sheeted hyperboloid $\mathcal{L}$ in $\mathbb{R}^3$. Here the group G is $\mathrm{SU}(1,1)$, the manifold Ω is the Riemann sphere $\overline{\mathbb{C}}$. The hyperboloid $\mathcal{L}$ can be identified with this sphere without the equator S: $z\bar{z}=1$. Both hemispheres can be identified with the Lobachevsky plane. The overgroup $\widetilde{G}$ is $\mathrm{SL}(2,\mathbb{C})$. We decompose canonical representations $R_{\lambda,\nu}$, $\lambda \in \mathbb{C}$, $\nu = 0, 1$, into irreducible constituents and decompose the Berezin form considered on corresponding spaces.

This paper is closely related to our paper [6], where the Lobachevsky plane was considered. For the two-sheeted hyperboloid, analysis of canonical representations turns out to be in some sense more natural. Essential differences from [6] are the following. Now the inverse of the Berezin transform $Q_{\lambda,\nu}$ can be easy written: it is the Berezin transform $Q_{-\lambda-2,\nu}$. It allows to write a decomposition (a "Plancherel formula") of the Berezin form $(f,h)_{\lambda,\nu}$ for all λ in a transparent form.

Let us introduce some notation and agreements.

We use the following notation for a character of the group $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$:

$$t^{\lambda,\nu} = |t|^{\lambda}(\operatorname{sgn} t)^{\nu},$$

where $t \in \mathbb{R}^*$, $\lambda \in \mathbb{C}$, $\nu = 0, 1$. By $\mathbb{N}$ we denote $\{0, 1, 2, \dots\}$. The sign "$\equiv$" means the congruence modulo 2. If M is a manifold, then $\mathcal{D}(M)$ denotes the Schwartz space of compactly supported infinitely differentiable complex valued functions on M, with the usual topology, and $\mathcal{D}'(M)$ denotes the space of distributions on M—of antilinear continuous functionals on $\mathcal{D}(M)$.

1. THE TWO-SHEETED HYPERBOLOID

Let us equip the space $\mathbb{R}^3$ with the bilinear form $[x,y] = -x_1y_1 + x_2y_2 + x_3y_3$. Let $G_1 = \mathrm{SO}_0(1,2)$, the connected component of the identity of the group of linear transformations of $\mathbb{R}^3$ preserving the form $[x,y]$. We consider that G_1 acts on $\mathbb{R}^3$ from the right. Let $\mathcal{L}$ denote the two-sheeted hyperboloid $[x,x] = -1$. It consists of two connected parts $\mathcal{L}^+$: $x_1 \geqslant 1$ and $\mathcal{L}^-$: $x_1 \leqslant -1$. Each of these parts is the Lobachevsky plane—the homogeneous space G_1/K_1 where $K_1 = \mathrm{SO}(2)$, the stabilizer of points $(\pm 1, 0, 0)$. A measure on $\mathcal{L}$ invariant with respect to G_1 is $dx = |x_1|^{-1}\,dx_2\,dx_3$.

Let us realize $\mathcal{L}$ on the Riemann sphere—the extended complex plane $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$: to a point $x \in \mathcal{L}$ we attach the point

$$z = \frac{x_2 + ix_3}{x_1 + 1}, \tag{1.1}$$

the inverse map carries a point $z = \xi + i\eta$ to the point

$$x = \left(\frac{1+z\bar{z}}{1-z\bar{z}}, \frac{2\xi}{1-z\bar{z}}, \frac{2\eta}{1-z\bar{z}}\right) \tag{1.2}$$

in $\mathcal{L}$. Under this map the sheet $\mathcal{L}^+$ goes onto the open disk D: $z\bar{z} < 1$, and the sheet $\mathcal{L}^-$ goes onto its exterior D': $z\bar{z} > 1$ including the point ∞ which corresponds to the point $(-1, 0, 0)$. The group G_1 is isomorphic to the group of fractional linear transformations

$$(1.3)\qquad z \mapsto z \cdot g = \frac{az+\bar{b}}{bz+\bar{a}}, \quad g = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}, \ a\bar{a} - b\bar{b} = 1.$$

These matrices g form the group $G = \mathrm{SU}(1,1)$. We obtain the homomorphism $G \to G_1$ with the kernel $\{\pm E\}$. The group K_1 is the image under this homomorphism of the group K consisting of diagonal matrices. The sphere $\overline{\mathbb{C}}$ splits into three G-orbits with respect to the action (1.3): D, D' and the circle S: $z\bar{z} = 1$. A G-invariant measure $d\mu(z)$ is given by

$$(1.4)\qquad d\mu(z) = p^{-2}\, dx\, dy,$$

where

$$p = 1 - z\bar{z}.$$

Let U be a representation of G on functions on D by translations:

$$\big(U(g)f\big)(z) = f(z \cdot g),$$

a quasiregular representation. In particular, the representation U on the space $L^2(D, d\mu)$ is unitary with respect to the inner product

$$(1.5)\qquad (f,h) = \int_D f(z)\overline{h(z)}\, d\mu(z).$$

These manifolds are convenient to consider together—as sections of the cone $\mathcal{C}$: $-x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0$, $x \neq 0$, in $\mathbb{R}^4$. Namely, the section $x_4 = 1$ is the hyperboloid $\mathcal{L}$, the section $x_1 + x_4 = 2$ can be identified with the complex plane $\mathbb{C}$: to a point $z = \xi + i\eta$ in $\mathbb{C}$ we attach the point $(1 + z\bar{z}, 2\xi, 2\eta, 1 - z\bar{z})$, so that we shall denote this section as $\mathbb{C}$ again, finally, the section Ω: $x_1 = 1$ is a two-dimensional sphere. These three manifolds are mapped to each other by the central projection from the origin. In particular, it gives (1.1) and (1.2). Under this correspondence we have $dx = 4\, d\mu(z)$ and, moreover, if points $x, y \in \mathcal{L}$ correspond to points $z, w \in \overline{\mathbb{C}}$ respectively, then

$$(1.6)\qquad \frac{1 - [x,y]}{2} = \frac{(1 - z\bar{w})(1 - w\bar{z})}{(1 - z\bar{z})(1 - w\bar{w})}.$$

The orbits D, D' and S are mapped to hemispheres $x_4 < 0$, $x_4 > 0$ and the equator $x_4 = 0$ respectively.

Let us realize the space $\mathbb{R}^4$ as the space of Hermitian matrices of order 2:

$$x = (x_1, x_2, x_3, x_4) \quad \longleftrightarrow \quad x = \frac{1}{2}\begin{pmatrix} x_1 - x_4 & x_2 - ix_3 \\ x_2 + ix_3 & x_1 + x_4 \end{pmatrix}.$$

The group $\widetilde{G} = \mathrm{SL}(2, \mathbb{C})$ acts on this space as follows: $x \mapsto \bar{g}' x g$, the prime denotes matrix transposition. This action preserves $\det x$. The cone C is characterized by $\det x = 0$, $x \neq 0$, it is invariant with respect to $\widetilde{G}$.

2. ELEMENTARY REPRESENTATIONS OF $G/\{\pm E\}$

In this section we recall a description of the principal non-unitary series of representations of the group G which are trivial on the center, see, for example, [8].

A representation T_σ, $\sigma \in \mathbb{C}$, of G acts on $\mathcal{D}(S)$ by:

$$\big(T_\sigma(g)\varphi\big)(s) = \varphi(s \cdot g)|bs + \bar{a}|^{2\sigma}.$$

Let ds denote the Euclidean measure on S: $ds = d\alpha$, if $s = \mathrm{e}^{i\alpha}$. The inner product from $L^2(S, ds)$:

$$\langle \psi, \varphi \rangle_S = \int_S \psi(s)\overline{\varphi(s)}\, ds \tag{2.1}$$

is invariant with respect to the pair $(T_\sigma, T_{-\bar{\sigma}-1})$. An operator A_σ on $\mathcal{D}(S)$ defined by

$$(A_\sigma \varphi)(s) = \int_S |1 - s\bar{u}|^{-2\sigma-2} \varphi(u)\, du$$

intertwines T_σ and $T_{-\sigma-1}$, i.e., $T_{-\sigma-1}(g)A_\sigma = A_\sigma T_\sigma(g)$. A sesqui-linear form $\langle A_\sigma \psi, \varphi\rangle_S$ is invariant with respect to the pair $(T_\sigma, T_{\bar{\sigma}})$. In particular, for $\sigma \in \mathbb{R}$, this form is an invariant Hermitian form for T_σ.

The composition $A_\sigma A_{-\sigma-1}$ is a scalar operator—the multiplication by $[2\pi\omega(\sigma)]^{-1}$ where

$$\omega(\sigma) = \frac{1}{2\pi^2}\left(\sigma + \frac{1}{2}\right)\cot\sigma\pi, \tag{2.2}$$

a "Plancherel measure", see Theorem 3.1. The function $\psi_0(s) = 1$ is an eigenfunction of A_σ with the eigenvalue

$$j(\sigma) = 2\pi\Gamma(-2\sigma - 1)/\Gamma^2(-\sigma),$$

so that

$$j(\sigma)j(-\sigma-1) = \big[2\pi\omega(\sigma)\big]^{-1}.$$

The operator A_σ as well as the factor $j(\sigma)$ is meromorphic in σ with simple poles at $\sigma \in -1/2 + \mathbb{N}$. Residues of A_σ are differential operators.

If $\sigma \notin \mathbb{Z}$, then T_σ is irreducible and T_σ is equivalent to $T_{-\sigma-1}$ by means of the operator A_σ or its residue.

There are four series of unitarizable irreducible representations: the *continuous* series: T_σ, $\sigma = -1/2 + i\rho$, $\rho \in \mathbb{R}$, an inner product is (2.1); the *complementary* series: T_σ, $-1 < \sigma < 1$, an inner product is the form $\langle A_\sigma \psi, \varphi\rangle_S$ with a suitable factor; the *holomorphic* and *antiholomorphic* series consisting of subfactors of T_σ, $\sigma \in \mathbb{Z}$.

3. HARMONIC ANALYSIS ON THE LOBACHEVSKY PLANE

In this section we recall some facts from [6] about harmonic analysis on the Lobachevsky plane realized as the disk D (of course, the Plancherel formula on D is a classical result).

The *Poisson transform* $P_\sigma : \mathcal{D}(S) \to C^\infty(D)$ and the *Fourier transform* $F_\sigma : \mathcal{D}(D) \to \mathcal{D}(S)$ are defined by

$$(P_\sigma \varphi)(z) = \int_S \big(T_\sigma(g^{-1})\psi_0\big)(s)\varphi(s)\,ds = p^{-\sigma} \int_S |1 - z\bar{s}|^{2\sigma} \varphi(s)\,ds,$$

$$\begin{aligned}(F_\sigma f)(s) &= \int_D f(z)\big(T_\sigma\big(g^{-1}\big)\psi_0\big)(s)\,d\mu(z)\\ &= \int_D f(z)p^{-\sigma-2}|1 - z\bar{s}|^{2\sigma}\,dx\,dy, \qquad z = x + iy,\end{aligned}$$

where $z = 0 \cdot g$. They intertwine $T_{-\sigma-1}$ with U and U with T_σ respectively. With A_σ they interact as follows:

$$P_\sigma A_\sigma = j(\sigma)P_{-\sigma-1}. \tag{3.1}$$

$$A_\sigma F_\sigma = j(\sigma)F_{-\sigma-1}. \tag{3.2}$$

For a function $f \in \mathcal{D}(D)$, we call $F_\sigma f \in \mathcal{D}(S)$ a *Fourier component* of f corresponding to T_σ.

Both transforms are entire functions of σ. They are conjugated to each other:

$$(P_\sigma \varphi, f) = \langle \varphi, F_{\bar{\sigma}} f\rangle_S$$

(in the left-hand side form (1.5) stands).

The Poisson transform $\Psi_\sigma = P_\sigma \psi_0$ of the K-invariant ψ_0 is the *spherical function* corresponding to T_σ. It belongs to $C^\infty(D)$ and is expressed in terms of Legendre functions: $\Psi_\sigma(z) = 2\pi P_\sigma(c)$ (here P_σ is the Legendre function!) with $c = x_1$ (see (1.2)).

Theorem 3.1. *The quasiregular representation U of G on $L^2(D, d\mu)$ decomposes into the integral of unitary irreducible representations of the continuous series with multiplicity one. Namely, let us assign to a function $f \in \mathcal{D}(D)$ the family $\{F_\sigma f\}$ of its Fourier components of the continuous series ($\sigma = -(1/2) + i\rho$). This correspondence is G-equivariant. There is an inversion formula*:

$$f = \int_{-\infty}^{\infty} \omega(\sigma)P_{-\sigma-1}F_\sigma f|_{\sigma=-(1/2)+i\rho}\,d\rho \tag{3.3}$$

(for $\omega(\sigma)$, see (2.2)) and a Plancherel formula:

$$(3.4)\qquad (f,h)=\int_{-\infty}^{\infty}\omega(\sigma)\langle F_\sigma f, F_{-\bar\sigma-1}h\rangle_S|_{\sigma=-(1/2)+i\rho}\,d\rho.$$

Therefore, the correspondence above can be extended to $L^2(D,d\mu)$.

Proof. Let us assign to any function $f\in\mathcal{D}(D)$ the function $(Mf)(c)$ on $[1,\infty)$:

$$(Mf)(c)=\int_S f(rs)\,ds,\qquad c=x_1=\frac{1+r^2}{1-r^2}.$$

This averaging map carries $\mathcal{D}(D)$ to $\mathcal{D}([1,\infty))$. In particular, we have $(Mf)(1)=2\pi f(0)$. If a function Φ on D depends only on c: $\Phi(z)=F(c)$, then $(\Phi,f)=(1/4)\langle\!\langle F, Mf\rangle\!\rangle$ where

$$\langle\!\langle F,H\rangle\!\rangle=\int_1^{\infty}F(c)\overline{H(c)}\,dc.$$

Mehler–Fock formulae [2, 3.14(8), (9)] give:

$$(3.5)\qquad \langle\!\langle F,H\rangle\!\rangle=\int_{-\infty}^{\infty}\pi^2\omega(\sigma)\langle\!\langle F,P_\sigma\rangle\!\rangle\langle\!\langle P_\sigma,H\rangle\!\rangle|_{\sigma=-(1/2)+i\rho}\,d\rho,$$

where P_σ is the Legendre function. Formula (3.5) implies

$$\delta=\int_{-\infty}^{\infty}\omega(\sigma)\Psi_\sigma|_{\sigma=-(1/2)+i\rho}\,d\rho,$$

where $(\delta,f)=\overline{f(0)}$. It gives (3.3), (3.4). □

4. SPHERICAL REPRESENTATIONS OF THE GROUP SL(2, ℂ)

In this section we give some material about spherical representations of the group $\widetilde{G}=\mathrm{SL}(2,\mathbb{C})$, see, for example, [8]—in a form adapted for us here.

For $\lambda\in\mathbb{C}$, let us denote by $\mathcal{D}_\lambda(\mathbb{C})$ the space of functions f in $C^\infty(\mathbb{C})$ such that the function $|z|^{2\lambda}f(-1/z)$ belongs to $C^\infty(\mathbb{C})$ too. The representation $\widetilde{R}_\lambda$ of $\widetilde{G}$ acts on the space $\mathcal{D}_{-\lambda-2}(\mathbb{C})$ by

$$(4.1)\qquad \big(\widetilde{R}_\lambda(g)f\big)(z)=f(z\cdot g)|\beta z+\delta|^{-2\lambda-4},$$

where

$$z\cdot g=\frac{\alpha z+\gamma}{\beta z+\delta},\qquad g=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in\widetilde{G}.$$

It is useful to realize the space $\mathcal{D}_\lambda(\mathbb{C})$ in homogeneous functions on the cone $\mathcal{C}$, see Section 1, and in functions on its sections. Let $\mathcal{D}_\lambda(\mathcal{C})$ be the space of functions f in $C^\infty(\mathcal{C})$ homogeneous of order λ:

$$f(tx) = |t|^\lambda f(x), \quad x \in \mathcal{C},\ t \neq 0.$$

This space can be realized as a space of functions on each of sections $\mathbb{C}$, Ω, $\mathcal{L}$ of the cone $\mathcal{C}$. For $\mathbb{C}$ we obtain just $\mathcal{D}_\lambda(\mathbb{C})$, for Ω we obtain the whole space $\mathcal{D}(\Omega)$, for $\mathcal{L}$ we obtain some space $\mathcal{D}_\lambda(\mathcal{L})$. It allows to insert the topology from $\mathcal{D}(\Omega)$ into $\mathcal{D}_\lambda(\mathbb{C})$ and $\mathcal{D}_\lambda(\mathcal{L})$. The space $\mathcal{D}_\lambda(\mathcal{L})$ contains $\mathcal{D}(\mathcal{L})$ and is contained in $C^\infty(\mathcal{L})$. For $f \in \mathcal{D}_\lambda(\mathcal{C})$, its restrictions $f(z)$ and $f(x)$ to sections $\mathbb{C}$ and $\mathcal{L}$ are connected as follows:

$$f(x) = |p|^{-\lambda} f(z). \tag{4.2}$$

The representation $\widetilde{R}_\lambda$ of the group $\widetilde{G}$ acts on the space $\mathcal{D}_{-\lambda-2}(\mathcal{C})$ by translations: $(\widetilde{R}_\lambda(g)f)(x) = f(\bar{g}'xg)$, we use the matrix realization of the space $\mathbb{R}^4$, see Section 1. Restricting functions in $\mathcal{D}_{-\lambda-2}(\mathcal{C})$ to different sections, we obtain different realizations of the representation $\widetilde{R}_\lambda$. For $\mathbb{C}$ it is exactly (4.1).

Let us return to the $\mathbb{C}$-realization. The Hermitian form

$$\langle f, h\rangle_{\mathbb{C}} = \int_{\mathbb{C}} f(z)\overline{h(z)}\,dx\,dy, \quad z = x + iy, \tag{4.3}$$

is invariant with respect to the pair $(\widetilde{R}_\lambda, \widetilde{R}_{-\bar{\lambda}-2})$, i.e.,

$$\langle \widetilde{R}_\lambda(g) f, h\rangle_{\mathbb{C}} = \langle f, \widetilde{R}_{-\bar{\lambda}-2}(g^{-1})h\rangle_{\mathbb{C}}.$$

It allows to extend $\widetilde{R}_\lambda$ to the space $\mathcal{D}'_{-\lambda-2}(\mathbb{C})$ of distributions—antilinear continuous functionals on $\mathcal{D}_{\bar{\lambda}}(\mathbb{C})$.

In particular, two functions $\theta_{\lambda,\nu}(z) = p^{\lambda,\nu}$, where $\nu = 0, 1$, belong to $\mathcal{D}'_\lambda(\mathbb{C})$, they are invariant with respect to $G = \mathrm{SU}(1,1)$ in the representation $\widetilde{R}_{-\lambda-2}$ (and any G-invariant function is a linear combination of these functions).

The operator B_λ defined by

$$(B_\lambda f)(z) = \int_{\mathbb{C}} |z - w|^{2\lambda} f(w)\,du\,dv, \quad w = u + iv,$$

maps $\mathcal{D}_{-\lambda-2}(\mathbb{C})$ in $\mathcal{D}_\lambda(\mathbb{C})$ and intertwines $\widetilde{R}_\lambda$ with $\widetilde{R}_{-\lambda-2}$. With the form (4.3) it interacts as follows: $\langle B_\lambda f, h\rangle_{\mathbb{C}} = \langle f, B_{\bar{\lambda}} h\rangle_{\mathbb{C}}$. It allows to extend B_λ to the space $\mathcal{D}'_{-\lambda-2}(\mathbb{C})$. In particular, it carries the function $\theta_{-\lambda-2,\nu}$ to the function $\theta_{\lambda,\nu}$ with a factor which we denote by $c(\lambda)^{-1}$:

$$B_\lambda \theta_{-\lambda-2,\nu} = c(\lambda)^{-1}\theta_{\lambda,\nu}, \quad c(\lambda) = -(\lambda+1)/\pi.$$

The composition $B_{-\lambda-2}B_\lambda$ is a scalar operator:

$$B_{-\lambda-2}B_\lambda = \left[c(\lambda)c(-\lambda-2)\right]^{-1} E. \tag{4.4}$$

Let R_λ, $\lambda \in \mathbb{C}$, denote the restriction of $\widetilde{R}_\lambda$ to $G = \mathrm{SU}(1,1)$. Let I_λ be the following operator on $\mathcal{D}_\lambda(\mathbb{C})$:

$$(I_\lambda f)(z) = |z|^{2\lambda} f(1/\bar{z}). \tag{5.1}$$

It is an involution. Let us denote by $\mathcal{D}_{\lambda,\nu}(\mathbb{C})$, $\nu = 0, 1$, the subspace of functions $f \in \mathcal{D}_\lambda(\mathbb{C})$ of parity ν with respect to I_λ, i.e., $I_\lambda f = (-1)^\nu f$, or:

$$f(z) = (-1)^\nu |z|^{2\lambda} f(1/\bar{z}). \tag{5.2}$$

Operators $R_\lambda(g)$, $g \in G$, commute with $I_{-\lambda-2}$, so that they preserve both subspaces $\mathcal{D}_{-\lambda-2,\nu}(\mathbb{C})$, $\nu = 0, 1$. Therefore, R_λ splits into two representations, we denote them by $R_{\lambda,\nu}$ and call the *canonical representations on the two-sheeted hyperboloid.*

Thus, the canonical representation $R_{\lambda,\nu}$, $\lambda \in \mathbb{C}$, $\nu = 0, 1$, of the group G acts on the space $\mathcal{D}_{-\lambda-2,\nu}(\mathbb{C})$ by

$$\big(R_{\lambda,\nu}(g) f\big)(z) = f(z \cdot g) |bz + \bar{a}|^{-2\lambda-4},$$

where $g \in G$ is the matrix (1.3).

Let us define an operator $Q_{\lambda,\nu} : \mathcal{D}_{-\lambda-2,\nu}(\mathbb{C}) \to \mathcal{D}_{\lambda,\nu}(\mathbb{C})$ by

$$Q_{\lambda,\nu} = c(\lambda) B_\lambda \circ I_{-\lambda-2},$$

so that

$$(Q_{\lambda,\nu} f)(z) = c(\lambda) \int_{\mathbb{C}} |1 - z\overline{w}|^{2\lambda} f(w)\, du\, dv, \qquad w = u + iv. \tag{5.3}$$

It intertwines $R_{\lambda,\nu}$ with $R_{-\lambda-2,\nu}$. In virtue of (4.4) we have

$$Q_{-\lambda-2,\nu} Q_{\lambda,\nu} = E. \tag{5.4}$$

As in Section 4, we can extend $R_{\lambda,\nu}$ and $Q_{\lambda,\nu}$ to $\mathcal{D}'_{-\lambda-2,\nu}(\mathbb{C})$.

Let $(f, h)_{\lambda,\nu}$ be a sesqui-linear form:

$$(f, h)_{\lambda,\nu} = \langle Q_{\lambda,\nu} f, h \rangle_{\mathbb{C}}. \tag{5.5}$$

It is invariant with respect to the pair $(R_{\lambda,\nu}, R_{\bar{\lambda},\nu})$. So for λ real, it is a Hermitian form invariant with respect to $R_{\lambda,\nu}$. Using the property of parity (5.2), we can rewrite (5.3) and (5.5) as integrals over D and $D \times D$:

$$(Q_{\lambda,\nu} f)(z) = \int_D K_{\lambda,\nu}(z, w) f(w)\, du\, dv, \tag{5.6}$$

$$(f, h)_{\lambda,\nu} = \int_{D\times D} K_{\lambda,\nu}(z, w) f(z) \overline{h(w)}\, dx\, dy\, du\, dv,$$

where

$$K_{\lambda,\nu}(z,w)=c(\lambda)\big\{|1-z\overline{w}|^{2\lambda}+(-1)^{\nu}|z-w|^{2\lambda}\big\}.$$

We shall call the operator $Q_{\lambda,\nu}$ and the form $(f,h)_{\lambda,\nu}$ as well as the form $\mathcal{B}_{\lambda,\nu}$ below (Section 6) the *Berezin transform* and the *Berezin form* respectively.

For a function f in $\mathcal{D}_{\lambda,\nu}(\mathbb{C})$, consider the following function on the disk D:

$$f_0(z)=p^{-\lambda}f(z). \tag{5.7}$$

By (1.4) we have

Lemma 5.1. *Let* $f\in\mathcal{D}_{-\lambda-2,\nu}(\mathbb{C})$. *Then if* $\operatorname{Re}\lambda>-3/2$, *then* f_0 *belongs to* $L^2(D,d\mu)$.

Let us pass in (5.6) to functions (cf. (5.7))

$$f_0(z)=p^{\lambda+2}f(z),\qquad h_0(z)=p^{\bar{\lambda}+2}h(z).$$

Then the form $(1/2)(f,h)_{\lambda,\nu}$ becomes the form $\mathcal{B}_{\lambda,\nu}(f_0,h_0)$, see (6.1) and (6.2) below.

In the $\mathcal{L}$-realization the operator I_λ corresponds to the symmetry $x\mapsto -x$, $x\in\mathbb{R}^3$. Therefore, the subspaces $\mathcal{D}_{\lambda,\nu}(\mathcal{L})$, $\nu=0,1$, consist of functions f in $\mathcal{D}_\lambda(\mathcal{L})$ such that $f(-x)=(-1)^{\nu}f(x)$, $x\in\mathcal{L}\subset\mathbb{R}^3$. By (1.6), (4.2) the form $(f,h)_{\lambda,\nu}$ becomes:

$$(f,h)_{\lambda,\nu}=2^{-\lambda-4}c(\lambda)\int_{\mathcal{L}\times\mathcal{L}}\big|-[x,y]+1\big|^{\lambda}f(x)\overline{h(y)}\,dx\,dy,$$

where $f\in\mathcal{D}_{-\lambda-2,\nu}(\mathcal{L})$, $h\in\mathcal{D}_{-\bar{\lambda}-2,\nu}(\mathcal{L})$.

Let us denote by L_λ the restriction of R_λ to a space Σ of distributions on $\mathbb{C}$ concentrated at the unit circle S, this space is the union of the spaces Σ_k, $k\in\mathbb{N}$, consisting of distributions

$$\varphi_0(s)\delta(p)+\varphi_1(s)\delta'(p)+\cdots+\varphi_k(s)\delta^{(k)}(p),$$

where $\delta(p)$ is the Dirac delta function on the real line, $\varphi_m\in\mathcal{D}(S)$.

Denote by $\Sigma_{\lambda,\nu}$ (resp. $\Sigma_{\lambda,\nu,k}$) the subspace of distributions $\zeta\in\Sigma$ (resp. $\zeta\in\Sigma_k$) of parity ν with respect to the involution I_λ and by $L_{\lambda,\nu}$ the restriction of L_λ (or R_λ) to $\Sigma_{-\lambda-2,\nu}$.

For a function f of class C^∞ in a neighbourhood of S, consider its Taylor series $a_0+a_1p+a_2p^2+\cdots$. Here $a_k=a_k(f)$ are functions in $\mathcal{D}(S)$. Let $a(f)$ be the column $(a_0(f),a_1(f),\ldots)$. A representation M_λ of G acts on the space of these columns: $M_\lambda(g)a(f)=a(R_\lambda(g)f)$.

Let $M_{\lambda,\nu}$ be the subrepresentation of M_λ acting on the subspace of sequences $a(f)$ with $f\in\mathcal{D}_{-\lambda-2,\nu}(\mathbb{C})$.

Decompositions of L_λ and M_λ were given in [6]. We do not turn here our attention to decompositions of $L_{\lambda,\nu}$ and $M_{\lambda,\nu}$.

In this section we consider the Berezin form

$$\mathcal{B}_{\lambda,\nu}(f,h)=\int\limits_{D\times D} E_{\lambda,\nu}(z,w)f(z)\overline{h(w)}\,d\mu(z)\,d\mu(w), \tag{6.1}$$

defined on the space $\mathcal{D}(D)$ by the *Berezin kernel*

$$\begin{aligned} E_{\lambda,\nu}(z,w)=c(\lambda)\Bigg\{&\left[\frac{(1-z\overline{w})(1-w\overline{z})}{(1-z\overline{z})(1-\overline{w}w)}\right]^{\lambda} \\ &+(-1)^{\nu}\left[\frac{(z-w)(\overline{z}-\overline{w})}{(1-z\overline{z})(1-w\overline{w})}\right]^{\lambda}\Bigg\}. \end{aligned} \tag{6.2}$$

Notice that the first summand in (6.2) with the factor $c(\lambda)$ is just the Berezin kernel from [6]. Let us decompose this form into invariant sesqui-linear forms corresponding to the representations T_σ.

Theorem 6.1. *Let $f,h\in\mathcal{D}(D)$. Then for* $\operatorname{Re}\lambda<-1/2+\nu$ *and for* $\lambda=-1/2+\nu$ *we have*

$$\mathcal{B}_{\lambda,\nu}(f,h)=\int\limits_{-\infty}^{\infty}\omega(\sigma)\Lambda(\lambda,\nu,\sigma)\langle F_\sigma f,F_{-\overline{\sigma}-1}h\rangle_S|_{\sigma=-1/2+i\rho}\,d\rho; \tag{6.3}$$

for $-1/2+\nu+2r<\operatorname{Re}\lambda<3/2+\nu+2r$ *and for* $\lambda=3/2+\nu+2r$, $r\in\mathbb{N}$, *we have*

$$\mathcal{B}_{\lambda,\nu}(f,h)=\int\limits_{-\infty}^{\infty}+\sum\frac{M(\lambda,m)}{j(\lambda-m)}\langle A_{\lambda-m)}F_{\lambda-m}f,F_{\overline{\lambda}-m}h\rangle_S, \tag{6.4}$$

where $m=\nu+2s$, *the integral denotes the same integral as in* (6.3) *and the sum is taken over* $s=0,1,\dots,r$; *finally, for* $\operatorname{Re}\lambda=-1/2+\nu+2r$, $\lambda\neq-1/2+\nu+2r$ *we have*

$$\mathcal{B}_{\lambda,\nu}(f,h)=\int\limits_{-\infty}^{\infty}+\sum+\frac{1}{2}\cdot\frac{M(\lambda,k)}{j(\lambda-k)}\langle A_{\lambda-k)}F_{\lambda-k}f,F_{\overline{\lambda}-k}h\rangle_S, \tag{6.5}$$

where $k=\nu+2r$, *the sum in* (6.5) *contains the same summands as in* (6.4), *but the sum is taken over* $s=0,\dots,r-1$ (*for* $r=0$ *the sum is absent*), *the factor* $\omega(\sigma)$ *is given by* (2.2),

$$\Lambda(\lambda,\nu,\sigma)=\frac{\Gamma(-\lambda-\sigma-1)\Gamma(-\lambda+\sigma)}{\Gamma(-\lambda-1)\Gamma(-\lambda)}\cdot\frac{\sin\lambda\pi+(-1)^{\nu}\sin\sigma\pi}{\sin\lambda\pi}, \tag{6.6}$$

$$M(\lambda,m)=-\pi^{-2}(2\lambda-2m+1)\cdot\frac{\Gamma(\lambda+2)\Gamma(\lambda+1)}{m!\Gamma(2\lambda+2-m)}. \tag{6.7}$$

Proof. The kernel $E_{\lambda,\nu}(z,w)$ is invariant with respect to the diagonal action of the group G. Therefore, it can be obtained by translations $z \mapsto z \cdot g$ (for $z \cdot g$, see (1.3)) from a function of one variable:

$$E_{\lambda,\nu}(z) = E_{\lambda,\nu}(z,0) = c(\lambda)\left\{(1 - z\bar{z})^{-\lambda} + (-1)^{\nu}\left(\frac{z\bar{z}}{1 - z\bar{z}}\right)^{\lambda}\right\}.$$

This function depends only on $c = (1 + z\bar{z})/(1 - z\bar{z}) = x_1$, see (1.2), c runs through the interval $[1, \infty)$. We have:

$$E_{\lambda,\nu}(z) = 2^{-\lambda}c(\lambda)\left\{(c+1)^{\lambda} + (-1)^{\nu}(c-1)^{\lambda}\right\}.$$

This distribution is meromorphic in λ with simple poles at $\lambda \in -2 - \mathbb{N}$. Let us expand it into spherical functions Ψ_{λ}, see formula (6.9) below.

Let us set in (3.5) $F = E_{\lambda,\nu}$ and $H = Mf$, $f \in \mathcal{D}(D)$. In order to find $\langle\langle E_{\lambda,\nu}, P_{\sigma}\rangle\rangle$ with $\sigma = -1/2 + i\rho$, we compute the integral

$$\int_{-\infty}^{\infty} E_{\lambda,\nu}(c) P_{\sigma}(c)\, dc. \tag{6.8}$$

It converges absolutely for $\operatorname{Re}\lambda > -1$, $\operatorname{Re}(\lambda + \sigma + 1) < 0$, $\operatorname{Re}(\lambda - \sigma) < 0$, so that if $\operatorname{Re}\sigma = -1/2$, then $-1 < \operatorname{Re}\lambda < -1/2$. The integral (6.8) can be found by [4, 7.135(2), 7.127], it is equal to $(2/\pi)\Lambda$, where Λ is given by (6.6). For $\sigma = -1/2 + i\rho$, the integral $\langle\langle E_{\lambda,\nu}, P_{\sigma}\rangle\rangle$ is just the integral (6.8) since for such σ we have $\overline{P_{\sigma}} = P_{\bar{\sigma}} = P_{-\sigma-1} = P_{\sigma}$. Further, by Section 3 we have $\langle\langle P_{\sigma}, Mf\rangle\rangle = (2/\pi)(\Psi_{\sigma}, f)$. Substituting all this in (3.5), we obtain the desired expansion

$$E_{\lambda,\nu} = \int_{-\infty}^{\infty} \omega(\sigma)\Lambda(\lambda, \nu, \sigma)\Psi_{\sigma}|_{\sigma=-1/2+i\rho}\, d\rho \tag{6.9}$$

for $-1 < \operatorname{Re}\lambda < -1/2$. By analyticity we extend this expansion to $\operatorname{Re}\lambda < -1/2$.

As a function of σ, the function Λ has poles at points

$$\sigma = \lambda - k, \qquad \sigma = -\lambda - 1 + l, \tag{6.10}$$

and zeros at points

$$\sigma = -\lambda - k - 2, \qquad \sigma = \lambda + 1 + l, \tag{6.11}$$

where $k, l \in \mathbb{N}$ and $k \equiv \nu$, $l \equiv \nu$.

Now we make the analytic continuation of (6.9) from the half plane $\operatorname{Re}\lambda < -1/2$. When poles (6.10) intersect (or meet) the integration line—the line $\operatorname{Re}\sigma = -1/2$, they give additional terms.

The obtained expansions of $E_{\lambda,\nu}$ give formulae (6.3)–(6.5). □

The function Λ has the following properties:

$$(6.12)\qquad \Lambda(\lambda,\nu,-\sigma-1)=\Lambda(\lambda,\nu,\sigma),$$

$$(6.13)\qquad \Lambda(\lambda,\nu,\sigma)\Lambda(-\lambda-2,\nu,\sigma)=1.$$

They reflect the equivalence of T_σ and $T_{-\sigma-1}$ and property (5.4).

Theorem 6.2. *For $-1\leqslant \operatorname{Re}\lambda<-1/2$, the Berezin form $\mathcal{B}_{\lambda,\nu}$ and the operator $Q_{\lambda,\nu}$ are bounded in $L^2(D,d\mu)$.*

Proof. It is sufficient to show that Λ with $\sigma=-1/2+i\rho$ as a function of $\rho\in\mathbb{R}$ is bounded on $\mathbb{R}$. But it follows from its behaviour at infinity:

$$\Lambda(\lambda,\nu,-1/2+i\rho)\sim \mathrm{const}\cdot|\rho|^{-2\lambda-2},$$

see [2, 1.18(6)]. □

Let us consider the form $c(\lambda)^{-1}\mathcal{B}_{\lambda,\nu}$ with real λ. Formulae of Theorem 6.1 show that the form $c(\lambda)^{-1}\mathcal{B}_{\lambda,\nu}(f,f)$ has the sign $(-1)^{m+\nu}$ for $-m-1<\lambda<-m$, $m\in\mathbb{N}$. Therefore, the representation $R_{\lambda,\nu}$ considered on $\mathcal{D}(D)$ is unitarizable with respect to this form with a suitable sign (unitary canonical representations).

7. POISSON AND FOURIER TRANSFORMS ASSOCIATED WITH CANONICAL REPRESENTATIONS

Let $\lambda,\sigma\in\mathbb{C}$, $\nu=0,1$. We define Poisson transforms $P_{\lambda,\nu,\sigma}$ and Fourier transforms $F_{\lambda,\nu,\sigma}$ associated with the canonical representation $R_{\lambda,\nu}$ as compositions of Poisson and Fourier transforms from Section 3 with the multiplication by a power of p:

$$(P_{\lambda,\nu,\sigma}\varphi)(z)=p^{-\lambda-2-\sigma,\nu}\int_S|1-z\bar{s}|^{2\sigma}\varphi(s)\,ds,$$

$$(F_{\lambda,\nu,\sigma}f)(s)=\int_{\mathbb{C}}|1-z\bar{s}|^{2\sigma}p^{\lambda-\sigma,\nu}f(z)\,dx\,dy,$$

where $z=x+iy$, $p=1-z\bar{z}$, $\varphi\in\mathcal{D}(S)$, $f\in\mathcal{D}_{-\lambda-2,\nu}(\mathbb{C})$. They are conjugate to each other:

$$(7.1)\qquad \langle F_{\lambda,\nu,\sigma}f,\varphi\rangle_S=\langle f,P_{-\bar{\lambda}-2,\nu,\bar{\sigma}}\varphi\rangle_{\mathbb{C}}.$$

The transforms $P_{\lambda,\nu,\sigma}$ and $F_{\lambda,\nu,\sigma}$ intertwine $T_{-\sigma-1}$ with $R_{\lambda,\nu}$ and $R_{\lambda,\nu}$ with T_σ respectively. With the intertwining operators A_σ and $Q_{\lambda,\nu}$ (see Sections 2 and 5) they interact as follows:

$$(7.2)\qquad P_{\lambda,\nu,\sigma}A_\sigma=j(\sigma)P_{\lambda,\nu,-\sigma-1},$$

$$(7.3)\qquad Q_{\lambda,\nu}P_{\lambda,\nu,\sigma}=\Lambda(\lambda,\nu,\sigma)P_{-\lambda-2,\nu,\sigma},$$

$$(7.4)\qquad A_\sigma F_{\lambda,\nu,\sigma}=j(\sigma)F_{\lambda,\nu,-\sigma-1},$$

$$(7.5)\qquad F_{-\lambda-2,\nu,\sigma}Q_{\lambda,\nu}=\Lambda(\lambda,\nu,\sigma)F_{\lambda,\nu,\sigma}.$$

Formulae (7.2) and (7.4) follow immediately from (3.1) and (3.2), formula (7.5) will be proved in Section 8, formula (7.3) follows from (7.5) by (7.1).

Theorem 7.1. *For a K-finite function $\varphi \in \mathcal{D}(S)$ and $\sigma \notin (1/2)+\mathbb{Z}$ the Poisson transform has the following decomposition in powers of p:*

$$(7.6)\qquad (P_{\lambda,\nu,\sigma}\varphi)(z) = p^{-\lambda-\sigma-2,\nu}\sum_{k=0}^{\infty}(C_{\sigma,k}\varphi)(s)\cdot p^k + p^{-\lambda+\sigma-1,\nu}\sum_{k=0}^{\infty}(D_{\sigma,k}\varphi)(s)\cdot p^k.$$

The theorem follows from [6, Theorem 3.1]. Operators $C_{\sigma,k}$ and $D_{\sigma,k}$ on $\mathcal{D}(S)$ are expressed in terms of a differential operator $W_{\sigma,k}$ on $\mathcal{D}(S)$:

$$(7.7)\qquad C_{\sigma,k} = A_{-\sigma-1}W_{-\sigma-1,k}, \qquad D_{\sigma,k} = j(\sigma)W_{\sigma,k},$$

which is defined by a generating function—as follows. Let us consider the following power series in powers of p:

$$(7.8)\qquad (1-p)^{m/2}F(\sigma+1,\sigma+1+m;2\sigma+2;p) = \sum_{k=0}^{\infty} w_k(\sigma,-m^2)p^k,$$

where F is the Gauss hypergeometric function. The coefficients w_k of this series are polynomials in $-m^2$ of degree $[k/2]$ with coefficients rational in σ. Functions $\exp(im\alpha)$ are eigenfunctions of the operator $d^2/d\alpha^2$ with eigenvalues $-m^2$. We set

$$W_{\sigma,k} = w_k\left(\sigma,\frac{d^2}{d\alpha^2}\right).$$

Notice that $W_{\sigma,0}=1$.

Here is an explicit expression of $W_{\sigma,k}$ (it was absent in [6]). Let us denote

$$L_{\sigma,s} = \prod_{r=0}^{s-1}\left[(\sigma+1+2r)^2 + \frac{d^2}{d\alpha^2}\right],$$

then

$$W_{\sigma,k} = \sum_{0\leqslant s\leqslant k/2}(-1)^s 2^{-4s}\frac{(s+(\sigma+1)/2)^{[k-2s]}}{s!(k-2s)!(\sigma+3/2)^{[s]}}L_{\sigma,s}.$$

This formula is proved as follows. By [2, 2.11(26)] we transform the left-hand side of (7.8) to

$$(1-p)^{-(\sigma+1)/2}F\left(\frac{\sigma+1+m}{2},\frac{\sigma+1-m}{2};\sigma+\frac{3}{2};-\frac{p^2}{4(1-p)}\right),$$

then in each term of this series we expand $(1-p)^{-(\sigma+1)/2-s}$ into the binomial series and finally collect similar terms.

Recall [3] that the distribution $x^{\alpha,\nu}$ ($\alpha \in \mathbb{C}$, $\nu = 0, 1$) on the real line has poles of the first order at $\alpha = -n-1$, $n \in \mathbb{N}$, $n \equiv \nu$, with residues

$$2(-1)^n \frac{1}{n!} \delta^{(n)}(x).$$

Therefore, because of the leading factors $p^{-\lambda-2-\sigma,\nu}$ and $p^{-\lambda+\sigma-1,\nu}$ in (7.6) the Poisson transform $P_{\lambda,\nu,\sigma}$ has poles just at poles (6.10) of the function $\Lambda(\lambda, \nu, \sigma)$ and Laurent coefficients for $P_{\lambda,\nu,\sigma}$ are the same as for $P_{\lambda,\sigma}$ in [6], multiplied by 2. For example, if a pole belongs only to one of series (6.10), then it is simple and the residue is as follows:

$$\widehat{P}_{\lambda,\nu,\lambda-k} = 2\frac{(-1)^k}{k!} j(\lambda-k) \cdot \xi_{\lambda,k}, \quad k \equiv \nu,$$
$$\widehat{P}_{\lambda,\nu,-\lambda-1+l} = -2\frac{(-1)^l}{l!} \xi_{\lambda,l} \circ A_{\lambda-l}, \quad l \equiv \nu,$$

where $\xi_{\lambda,k}$ is an operator $\mathcal{D}(S) \to \Sigma_k$ defined in [6]:

$$\xi_{\lambda,k}(\varphi) = \sum_{m=0}^{k} (-1)^m \frac{k!}{(k-m)!} W_{\lambda-k,m}(\varphi) \cdot \delta^{(k-m)}(p).$$

Let $V_{\lambda,k}$ be the image of $\xi_{\lambda,k}$. It is contained in $\Sigma_{-\lambda-2,\nu,k}$, $\nu \equiv k$.

Taking the residue of (7.3) at $\sigma = \lambda - k$, $k \equiv \nu$, we obtain

$$Q_{\lambda,\nu}\xi_{\lambda,k} = \frac{1}{4}(-1)^\nu k! j(-\lambda-1+k) \cdot M(\lambda,k) \cdot P_{-\lambda-2,\nu,\lambda-k}, \tag{7.9}$$

where $M(\lambda, k)$ is given by (6.7).

The Fourier transform $F_{\lambda,\nu,\sigma}$ has poles in σ at zeros (6.11) of the function Λ. The Laurent coefficients are the same as in [6], multiplied by 2. For example, if a pole belongs only to one of the series (6.11), then it is simple and

$$\widehat{F}_{\lambda,\nu,-\lambda-2-k} = j(-\lambda-2-k) \cdot b_{\lambda,k}, \quad k \equiv \nu,$$
$$\widehat{F}_{\lambda,\nu,\lambda+1+l} = -A_{-\lambda-2-l} \circ b_{\lambda,l}, \quad l \equiv \nu,$$

where $b_{\lambda,k}$ is the boundary operator $\mathcal{D}_{-\lambda-2,\nu}(\mathbb{C}) \to \mathcal{D}(S)$ defined in terms of Taylor coefficients (see [6]):

$$b_{\lambda,k}(f) = \sum_{m=0}^{k} W_{-\lambda-2-k,k-m}\big(a_m(f)\big).$$

The operator $b_{\lambda,k}$ intertwines $R_{\lambda,\nu}$ with $T_{-\lambda-2-k}$. The conjugacy (7.1) gives the conjugacy for residues:

$$\big\langle b_{\lambda,k}(f), \varphi\big\rangle_S = 2\frac{(-1)^k}{k!}\big\langle f, \xi_{-\bar{\lambda}-2,k}(\varphi)\big\rangle_{\mathbb{C}}. \tag{7.10}$$

We extend the space $\mathcal{D}_{\lambda,\nu}(\mathbb{C})$ as follows. Let $\mathcal{T}_{\lambda,\nu,k}$, $k \in \mathbb{N}$, be the space of functions f of class C^∞ outside of S, having parity ν with respect to I_λ: $I_\lambda f = (-1)^\nu f$, and having the Taylor decomposition of order k:

$$f(z) = a_0 + a_1 p + \cdots + a_k p^k + \mathrm{o}(p^k),$$

where $a_m \in \mathcal{D}(S)$. The operators $b_{\lambda,m}$ with $m \leqslant k$ can be extended in the natural way to the space $\mathcal{T}_{-\lambda-2,\nu,k}$.

Let us consider the Poisson and Fourier transforms at poles of each other: $P_{\lambda,\nu,\lambda+1+m}$, $P_{\lambda,\nu,-\lambda-2-m}$, $F_{\lambda,\nu,-\lambda-1+m}$, $F_{\lambda,\nu,\lambda-m}$, where $m \equiv \nu$. Then these transforms have some special properties.

For the Poisson transforms $P_{\lambda,\nu,\lambda+1+m}$ and $P_{\lambda,\nu,-\lambda-2-m}$ the leading factors are p^m and $p^{-2\lambda-3-m,\nu}$. The first of them is a polynomial in p. Let us fix $k \in \mathbb{N}$ and let $\operatorname{Re}\lambda < -k - 3/2$. Then for any $m = 0, 1, \ldots, k$ the second leading factor is $o(p^k)$, so that, in particular, the image of the transforms $P_{\lambda,\nu,\lambda+1+m}$ and $P_{\lambda,\nu,-\lambda-2-m}$ lie in $\mathcal{T}_{-\lambda-2,\nu,k}$ and we can apply to them the boundary operators $b_{\lambda,r}$, $r \leqslant k$. These boundary operators turn out to be the inverse operators for the Poisson transforms—up to a factor or the operator A_σ. Namely, we have

Theorem 7.2. *Let* $k \in \mathbb{N}$. *Let* $\operatorname{Re}\lambda < -k - 3/2$. *Then for* $m \leqslant k$ *and* $m \equiv \nu$ *we have*:

$$b_{\lambda,m} \circ P_{\lambda,\nu,-\lambda-2-m} = A_{\lambda+1+m}, \tag{7.11}$$

$$b_{\lambda,m} \circ P_{\lambda,\nu,\lambda+1+m} = j(\lambda + 1 + m) \cdot E, \tag{7.12}$$

and for $r, m \leqslant k$, $r \neq m$, we have

$$b_{\lambda,r} \circ P_{\lambda,\nu,-\lambda-2-m} = 0, \tag{7.13}$$

$$b_{\lambda,r} \circ P_{\lambda,\nu,\lambda+1+m} = 0. \tag{7.14}$$

Proof. Consider (7.11) and (7.13). The composition $b_{\lambda,r} \circ P_{\lambda,\nu,-\lambda-2-m}$ is an operator $\mathcal{D}(S) \to \mathcal{D}(S)$ intertwining $T_{\lambda+1+m}$ with $T_{-\lambda-2-r}$. Therefore, if $r \neq m$, then this operator is equal to zero. If $r = m$, then it is the operator $A_{\lambda+1+m}$ up to a factor. By (7.6) the coefficient for $P_{\lambda,\nu,-\lambda-2-m}\varphi$ in front of p^m is $C_{-\lambda-2-m,0}\varphi$ which is equal to $A_{\lambda+1+m}\varphi$, see (7.7). On the other hand, the dominant term in $b_{\lambda,m}$ is the Taylor coefficient a_m. Therefore, the composition $b_{\lambda,m} \circ P_{\lambda,\nu,-\lambda-2-m}$ is exactly $A_{\lambda+1+m}$. Similarly we prove (7.12) and (7.14). □

Under the same conditions as in Theorem 7.2 we can apply distributions from $\Sigma_{\lambda,\nu,k}$ to the images of the Poisson transforms occurring in Theorem 7.2. Therefore, we can take as f in (7.10) a function from these images. Then, using Theorem 7.2, we obtain the following action of basic distributions from $\Sigma_{\lambda,\nu,k}$.

Theorem 7.3. *Let* $k \in \mathbb{N}$. *Let* $\operatorname{Re}\lambda < -k - 3/2$. *Then for* $m \leqslant k$ *and* $m \equiv \nu$ *we have*:

$$\bigl\langle \xi_{-\lambda-2,m}(\psi), P_{\bar{\lambda},\nu,-\bar{\lambda}-2-m}\varphi \bigr\rangle_{\mathbb{C}} = \frac{1}{2}(-1)^m m! \langle A_{\lambda+1+m}\psi, \varphi \rangle_S,$$

$$\langle \xi_{-\lambda-2,m}(\psi), P_{\bar\lambda,\nu,\bar\lambda+1+m}\varphi\rangle_{\mathbb{C}} = \frac{1}{2}(-1)^m m!\, j(\lambda+1+m)\langle\psi,\varphi\rangle_S,$$

and for $r, m \leqslant k$, $r \neq m$, we have

$$\langle \xi_{-\lambda-2,r}(\psi), P_{\bar\lambda,\nu,-\bar\lambda-2-m}\varphi\rangle_{\mathbb{C}} = 0,$$
$$\langle \xi_{-\lambda-2,r}(\psi), P_{\bar\lambda,\nu,\bar\lambda+1+m}\varphi\rangle_{\mathbb{C}} = 0.$$

Now, using conjugacy (7.1), we can extend the Fourier transforms $F_{\lambda,\nu,-\lambda-1+m}$ and $F_{\lambda,\nu,\lambda-m}$ to distributions ζ in $\Sigma_{\lambda,\nu,k}$, $m \leqslant k$. Namely, for $\operatorname{Re}\lambda > k - 1/2$ (we replaced λ by $-\lambda-2$) and $m \leqslant k$, $m \equiv \nu$, we set:

$$\langle F_{\lambda,\nu,\lambda-m}\zeta, \varphi\rangle_S = \langle \zeta, P_{-\bar\lambda-2,\nu,\bar\lambda-m}\varphi\rangle_{\mathbb{C}},$$
$$\langle F_{\lambda,\nu,-\lambda-1+m}\zeta, \varphi\rangle_S = \langle \zeta, P_{-\bar\lambda-2,\nu,-\bar\lambda-1+m}\varphi\rangle_{\mathbb{C}}.$$

Then Theorem 7.3 implies:

Theorem 7.4. *The Fourier transforms $F_{\lambda,\nu,-\lambda-1+m}$ and $F_{\lambda,\nu,\lambda-m}$ are inverse maps to $\xi_{\lambda,m}$ up to a factor or the operator A_σ, more exactly, there are relations—for $m, r \leqslant k$ and $m \equiv r \equiv \nu$:*

$$F_{\lambda,\nu,\lambda-m} \circ \xi_{\lambda,m} = \frac{1}{2}(-1)^m m! A_{-\lambda-1+m},$$
$$F_{\lambda,\nu,-\lambda-1+m} \circ \xi_{\lambda,m} = \frac{1}{2}(-1)^m m!\, j(-\lambda-1+m)\cdot E,$$
$$F_{\lambda,\nu,\lambda-m} \circ \xi_{\lambda,r} = 0, \quad r \neq m,$$
$$F_{\lambda,\nu,-\lambda-1+m} \circ \xi_{\lambda,r} = 0, \quad r \neq m.$$

These formulae show that the maps $F_{\lambda,\nu,-\lambda-1+m}$ and $F_{\lambda,\nu,\lambda-m}$ being defined originally as maps to the space $\mathcal{D}'(S)$ of distributions on S are actually maps to the space $\mathcal{D}(S)$.

Finally we have the following property (a similar property was absent for the Lobachevsky plane [6]). We may put $\sigma = \lambda+1+m$ in (7.3): at this point the Poisson transform $P_{-\lambda-2,\nu,\sigma}$ has a pole, but the function $\Lambda(\lambda,\nu,\sigma)$ has a zero, so that the left-hand side has a limit when σ tends to $\lambda+1+m$. Evaluating this indeterminacy, we obtain—for $m \equiv \nu$:

$$Q_{\lambda,\nu}P_{\lambda,\nu,\lambda+1+m} = T(\lambda,m)\cdot \xi_{-\lambda-2,m} \circ A_{-\lambda-2-m}, \tag{7.15}$$

where

$$T(\lambda,m) = \frac{\Gamma(\lambda+2)\Gamma(\lambda+1)}{\Gamma(2\lambda+3+m)}. \tag{7.16}$$

As in [6], we restrict ourselves to a generic case: we consider λ lying in strips $I_k = \{-3/2 + k < \operatorname{Re}\lambda < -1/2 + k\}$, $k \in \mathbb{Z}$.

Case (A): $\lambda \in I_0$. Let us take functions $f \in \mathcal{D}_{-\lambda-2,\nu}(\mathbb{C})$ and $h \in \mathcal{D}_{\bar{\lambda},\nu}(\mathbb{C})$ and consider functions $f_0(z) = p^{\lambda+2} f(z)$ and $h_0(z) = p^{-\bar{\lambda}} h(z)$ on D, cf. (5.7). Since $\lambda \in I_0$, by Lemma 5.1 both functions f_0 and h_0 belong to $L^2(D, d\mu)$. Applying the Plancherel formula (3.4) to the pair of functions f_0 and h_0, we obtain:

$$(f_0, h_0) = \int_{-\infty}^{\infty} \omega(\sigma) \langle F_\sigma f_0, F_{-\bar{\sigma}-1} h_0 \rangle_S |_{\sigma=-1/2+i\rho} \, d\rho, \tag{8.1}$$

on the left-hand side the inner product (1.5) in $L^2(D, d\mu)$ stands. Now we pass in (8.1) to functions f and h. The left-hand side becomes $\langle f, h \rangle_D$ which is $(1/2)\langle f, h \rangle_{\mathbb{C}}$. The Fourier components $F_\sigma f_0$ and $F_{-\bar{\sigma}-1} h_0$ become $(1/2) F_{\lambda,\nu,\sigma} f$ and $(1/2) F_{-\bar{\lambda}-2,\nu,-\bar{\sigma}-1} h$. So, we obtain

$$\langle f, h \rangle_{\mathbb{C}} = \int_{-\infty}^{\infty} \frac{1}{2} \omega(\sigma) \langle F_{\lambda,\nu,\sigma} f, F_{-\bar{\lambda}-2,\nu,-\bar{\sigma}-1} h \rangle_S |_{\sigma=-1/2+i\rho} \, d\rho. \tag{8.2}$$

Here, using conjugacy (7.1), we transfer the Fourier transform from h to f—as the Poisson transform. We obtain a formula which gives an expansion of f considered as a distribution (in $\mathcal{D}'_{-\lambda-2,\nu}(\mathbb{C})$):

$$f = \int_{-\infty}^{\infty} \frac{1}{2} \omega(\sigma) P_{\lambda,\nu,-\sigma-1} F_{\lambda,\nu,\sigma} f |_{\sigma=-1/2+i\rho} \, d\rho. \tag{8.3}$$

Now let us take functions $f \in \mathcal{D}_{-\lambda-2,\nu}(\mathbb{C})$, $h \in \mathcal{D}_{-\bar{\lambda}-2,\nu}(\mathbb{C})$. Then for $\lambda \in I_0$ both functions $f_0(z) = p^{\lambda+2} f(z)$ and $h_0(z) = p^{\bar{\lambda}+2} h(z)$ belong to $L^2(D, d\mu)$ again. Let us take the Berezin form $\mathcal{B}_{\lambda,\nu}(f_0, h_0)$. In virtue of Theorem 6.2 we can write the decomposition (6.3) of this form for λ in the strip $I_0^+ : -1 \leqslant \operatorname{Re}\lambda < -1/2$. Coming back to f and h as above, we obtain the decomposition of the form $(f, h)_{\lambda,\nu}$—still for I_0^+:

$$(f, h)_{\lambda,\nu} = \int_{-\infty}^{\infty} \frac{1}{2} \omega(\sigma) \Lambda(\lambda, \nu, \sigma) \langle F_{\lambda,\nu,\sigma} f, F_{\bar{\lambda},\nu,-\bar{\sigma}-1} h \rangle_S |_{\sigma=-1/2+i\rho} \, d\rho. \tag{8.4}$$

Now we are able to prove (7.5). Let us take (8.2) with $-\lambda - 2$ instead of λ for the functions $Q_{\lambda,\nu} f$ and h. We obtain

$$\langle Q_{\lambda,\nu} f, h \rangle_{\mathbb{C}} = \int_{-\infty}^{\infty} \frac{1}{2} \omega(\sigma) \langle F_{-\lambda-2,\nu,\sigma} Q_{\lambda,\nu} f, F_{\bar{\lambda},\nu,-\bar{\sigma}-1} h \rangle_S |_{\sigma=-1/2+i\rho} \, d\rho, \tag{8.5}$$

where $f \in \mathcal{D}_{-\lambda-2,\nu}(\mathbb{C})$, $h \in \mathcal{D}_{-\bar{\lambda}-2,\nu}(\mathbb{C})$ and $\lambda \in I_0$. Comparing (8.4) with (8.5) and taking into account (5.5) we obtain (7.5) first for I_0^+. Then by analyticity we extend it for all possible λ.

Now extend (8.4) to $-3/2 < \operatorname{Re}\lambda \leqslant -1$. Let λ belong to this strip. Then $-\lambda - 2$ belongs to I_0^+. Let us write (8.4) with λ replaced by $-\lambda - 2$ for the pair $Q_{\lambda,\nu} f$ and $Q_{\bar{\lambda},\nu} h$. Using (5.5), (5.4), (7.5) and (6.12), (6.13) we obtain (8.4) for $-3/2 < \operatorname{Re}\lambda \leqslant 1$. Thus, we have:

Theorem 8.1. *Let $\lambda \in I_0$. Then the canonical representation $R_{\lambda,\nu}$ decomposes into the integral of representations of the continuous series with multiplicity one. Namely, let us assign to any $f \in \mathcal{D}_{\lambda,\nu}(\mathbb{C})$ the family $\{F_{\lambda,\nu,\sigma} f\}$, $\sigma = -1/2 + i\rho$, of its Fourier components. This correspondence is G-equivariant. There is an inversion formula, see* (8.3), *and a decomposition of the Berezin form $(f,h)_{\lambda,\nu}$, see* (8.4).

Case (B): $\lambda \in I_{k+1}$, $k \in \mathbb{N}$. We make the analytic continuation of (8.3) in λ from the strip I_0 to the right—to the strip I_{k+1}. Here poles of the Poisson transform give additional terms. As in [6], we obtain

$$(8.6) \qquad f = \int_{-\infty}^{\infty} + \sum_{\substack{m=0,\\ m\equiv\nu}}^{k} \pi_{\lambda,m}(f)$$

where the integral means the right-hand side of (8.3) and

$$(8.7) \qquad \pi_{\lambda,m} = 2\frac{(-1)^m}{m!} j(-\lambda - 1 + m)^{-1} \cdot \xi_{\lambda,m} \circ F_{\lambda,\nu,-\lambda-1+m}.$$

Similarly, the analytic continuation of (8.4) gives (here additional terms are given by poles of Λ which are the same as poles of $P_{\lambda,\nu,\sigma}$):

$$(8.8) \qquad (f,h)_{\lambda,\nu} = \int_{-\infty}^{\infty} + \sum_{\substack{m=0,\\ m\equiv\nu}}^{k} \frac{1}{2} M(\lambda,m) \langle F_{\lambda,\nu,\lambda-m} f, F_{\bar{\lambda},\nu,-\bar{\lambda}-1+m} h\rangle_S,$$

where the integral means the right-hand side of (8.4).

The operators (8.7) with $m \leqslant k$ can be extended to $\Sigma_{-\lambda-2,\nu,k}$ since the Fourier transform $F_{\lambda,\nu,-\lambda-1+m}$ occurring in $\pi_{\lambda,m}$ is already extended, see Section 7. Thus, the operators $\pi_{\lambda,m}$ with $m \leqslant k$ are defined on the space

$$\mathcal{D}_{-\lambda-2,\nu,k}(\mathbb{C}) = \mathcal{D}_{-\lambda-2,\nu}(\mathbb{C}) + \Sigma_{-\lambda-2,\nu,k}.$$

Decomposition (8.8) can be also extended to the space $\mathcal{D}_{-\lambda-2,\nu,k}(\mathbb{C})$. In particular, for distributions in $\Sigma_{-\lambda-2,\nu,k}$ integrals in (8.6) and (8.8) disappear.

Theorem 8.2. *The operators $\pi_{\lambda,m}$, $m \leqslant k$, acting on the space $\mathcal{D}_{-\lambda-2,\nu,k}(\mathbb{C})$, are projection operators onto the spaces $V_{\lambda,m}$, i.e., there are relations*

$$(8.9) \qquad \pi_{\lambda,m}\pi_{\lambda,m} = \pi_{\lambda,m},$$

$$(8.10) \qquad \pi_{\lambda,m}\pi_{\lambda,r} = 0, \quad r \neq m.$$

Moreover, there are "orthogonality relations":

$$(8.11) \qquad \left(\pi_{\lambda,m}(f), \pi_{\bar{\lambda},m}(h)\right)_{\lambda,\nu} = \frac{1}{2}M(\lambda,m)\langle F_{\lambda,\nu,\lambda-m}f, F_{\bar{\lambda},\nu,-\bar{\lambda}-1+m}h\rangle_S,$$

$$(8.12) \qquad \left(\pi_{\lambda,m}(f), \pi_{\bar{\lambda},r}(h)\right)_{\lambda,\nu} = 0, \quad r \neq m.$$

Proof. By definition we write

$$\pi_{\lambda,m}\pi_{\lambda,r} = 4\frac{(-1)^{m+r}}{m!r!}j(-\lambda-1+m)^{-1}j(-\lambda-1+r)^{-1} \cdot \xi_{\lambda,m} \circ F_{\lambda,\nu,-\lambda-1+m} \circ \xi_{\lambda,r} \circ F_{\lambda,\nu,-\lambda-1+r}.$$

Now we apply Theorem 7.4 to $F_{\lambda,\nu,-\lambda-1+m} \circ \xi_{\lambda,r}$ and obtain (8.9) and (8.10).

Let us prove (8.11) and (8.12). In virtue of (5.5) consider $Q_{\lambda,\nu}\pi_{\lambda,m}f$. By (8.7) and (7.9) it is equal to $(1/2)M(\lambda,m)P_{-\lambda-2,\nu,\lambda-m}F_{\lambda,\nu,-\lambda-1+m}f$. So we have

$$\left(\pi_{\lambda,m}(f), \pi_{\bar{\lambda},r}(h)\right)_{\lambda,\nu} = M(\lambda,m)\frac{1}{r!}(-1)^r j(-\lambda-1+r)^{-1} \cdot \langle P_{-\lambda-2,\nu,\lambda-m}F_{\lambda,\nu,-\lambda-1+m}f, \xi_{\bar{\lambda},r}F_{\bar{\lambda},\nu,-\bar{\lambda}-1+r}h\rangle_{\mathbb{C}}.$$

By conjugacy (7.10) it is equal to

$$\frac{1}{2}M(\lambda,m)j(-\lambda-1+r)^{-1} \cdot \langle b_{-\lambda-2,r}P_{-\lambda-2,\nu,\lambda-m}F_{\lambda,\nu,-\lambda-1+m}f, F_{\bar{\lambda},\nu,-\bar{\lambda}-1+r}h\rangle_S.$$

By Theorem 7.2 it is equal to zero for $r \neq m$, and for $r = m$ it is equal to

$$\frac{1}{2}M(\lambda,m)j(-\lambda-1+m)^{-1}\langle A_{-\lambda-1+m}F_{\lambda,\nu,-\lambda-1+m}f, F_{\bar{\lambda},\nu,-\bar{\lambda}-1+m}h\rangle_S.$$

In virtue of (7.4) it is just equal to the right-hand side of (8.11). □

We see that the decomposition (8.8) is a "Pythagorean theorem" for decomposition (8.6). Thus, in Case (B) we have:

Theorem 8.3. *Let $\lambda \in I_{k+1}$, $k \in \mathbb{N}$. Then the space $\mathcal{D}_{-\lambda-2,\nu}(\mathbb{C})$ has to be completed to $\mathcal{D}_{-\lambda-2,\nu,k}(\mathbb{C})$. On this space the representation $R_{\lambda,\nu}$ splits into the sum of two terms: the first one decomposes as $R_{\lambda,\nu}$ does in Case (A), the second one decomposes into the sum of irreducible representations $T_{-\lambda-1+m}$ with $m = 0, 1, \ldots, k$, $m \equiv \nu$ (there are no Jordan blocks). Namely, let us assign to any f in $\mathcal{D}_{-\lambda-2,\nu,k}(\mathbb{C})$ the family $\{F_{\lambda,\nu,\sigma}f, \pi_{\lambda,m}(f)\}$, where $\sigma = -1/2 + i\rho$ and $m = 0, 1, \ldots, k$, $m \equiv \nu$. This correspondence is G-equivariant. The function f is recovered by the inversion formula* (8.6)*. Moreover, there is a "Plancherel formula"* (8.8) *for the form $(.,.)_{\lambda,\nu}$.*

For $-1/2 < \lambda < 0$, this theorem gives the decomposition of the *unitary* canonical representations, see Section 6.

Case (C): $\lambda \in I_{-k-1}$, $k \in \mathbb{N}$. We make the analytic continuation of (8.3) from I_0 to the left—to I_{-k-1}. Here the poles

$$\sigma = -\lambda - 2 - m, \quad \sigma = \lambda + 1 + m, \quad m \leqslant k, \quad m \equiv \nu, \tag{8.13}$$

of the integrand (they are poles of $F_{\lambda,\nu,\sigma}$) intersect the line $\mathrm{Re}\,\sigma = -1/2$ and give additional terms. As in [6] we obtain

$$f = \int_{-\infty}^{\infty} + \sum_{\substack{m=0,\\ m\equiv\nu}}^{k} \Pi_{\lambda,m}(f), \tag{8.14}$$

where the integral means the right-hand side of (8.3) and

$$\Pi_{\lambda,m} = j(\lambda + 1 + m)^{-1} P_{\lambda,\nu,\lambda+1+m} \circ b_{\lambda,m}. \tag{8.15}$$

Now let us continue (8.4). Here the poles (8.13) are poles of both Fourier transforms. Fortunately, the function Λ as a function of σ has zeros at poles of the Fourier transform. Therefore, the poles (8.13) of the integrand are simple. The pair of poles (8.13) with the same m gives the addition

$$\lim_{\sigma\to-\lambda-2-m} 2\pi\left\{\omega(\sigma)\cdot(\sigma+\lambda+2+m)^{-1}\Lambda(\lambda,\nu,\sigma)\right\}$$
$$\cdot\,\langle \widehat{F}_{\lambda,\nu,-\lambda-2-m} f, -\widehat{F}_{\bar{\lambda},\nu,\bar{\lambda}+1+m} h\rangle_S,$$

which is equal to

$$N(\lambda, m)\langle A_{-\lambda-2-m} b_{\lambda,m} f, b_{\bar{\lambda},m} h\rangle_S,$$

where

$$N(\lambda, m) = \frac{1}{2}(-1)^m j(\lambda + 1 + m)^{-1} m!\, T(\lambda, m), \tag{8.16}$$

for $T(\lambda, m)$, see (7.16). Thus, after continuation we obtain

$$(f, h)_{\lambda,\nu} = \int_{-\infty}^{\infty} + \sum_{\substack{m=0,\\ m\equiv\nu}}^{k} N(\lambda, m)\langle A_{-\lambda-2-m} b_{\lambda,m} f, b_{\bar{\lambda},m} h\rangle_S, \tag{8.17}$$

where the integral means the right-hand side of (8.4).

Denote by $\mathcal{P}_{\lambda,m}$ the image of $\mathcal{D}_{-\lambda-2,\nu}(\mathbb{C})$, $m \equiv \nu$, under $\Pi_{\lambda,m}$. The operators $\Pi_{\lambda,m}$ with $m \leqslant k$ can be extended to the space $\mathcal{T}_{-\lambda-2,\nu,k}$ since the operators $b_{\lambda,m}$ with $m \leqslant k$ are defined on this space. In particular, $\Pi_{\lambda,m}$ can be applied to $\mathcal{P}_{\lambda,r}$, $r \leqslant k$, and we may consider products $\Pi_{\lambda,m}\Pi_{\lambda,r}$ where $m, r \leqslant k$.

Theorem 8.4. *The operators* $\Pi_{\lambda,m}$, $m \leqslant k$, *are projection operators on* $\mathcal{P}_{\lambda,m}$, *namely, there are relations*:

$$\Pi_{\lambda,m}\Pi_{\lambda,m} = \Pi_{\lambda,m}, \tag{8.18}$$

$$\Pi_{\lambda,m}\Pi_{\lambda,r} = 0, \quad r \neq m. \tag{8.19}$$

They are "orthogonal" with respect to the form $(f,h)_{\lambda,\nu}$, *namely,*

$$\begin{aligned}(8.20)\qquad &\bigl(\Pi_{\lambda,m}(f),\Pi_{\bar{\lambda},m}(h)\bigr)_{\lambda,\nu}\\ &\quad = N(\lambda,m)\bigl\langle A_{-\lambda-2-m}b_{\lambda,m}(f), b_{\bar{\lambda},m}(h)\bigr\rangle_S,\\ (8.21)\qquad &\bigl(\Pi_{\lambda,m}(f),\Pi_{\bar{\lambda},r}\bigr)_{\lambda,\nu} = 0, \quad r \neq m,\end{aligned}$$

where $m \equiv \nu$ *and* $N(\lambda,m)$ *is given by* (8.16).

Proof. By definition we write

$$\begin{aligned}\Pi_{\lambda,m}\Pi_{\lambda,r} &= j(\lambda+1+m)^{-1}j(\lambda+1+r)^{-1}\\ &\quad\cdot P_{\lambda,\nu,\lambda+1+m}\circ b_{\lambda,m}\circ P_{\lambda,\nu,\lambda+1+r}\circ b_{\lambda,r}.\end{aligned}$$

Applying formulae (7.12) and (7.14) to $b_{\lambda,m}\circ P_{\lambda,\lambda+1+r}$ we obtain (8.18) and (8.19).

Now consider the left-hand side of (8.21). By (5.5), (8.15), (7.15) it is equal to

$$\begin{aligned}\bigl\langle Q_{\lambda,\nu}\Pi_{\lambda,m}(f),\Pi_{\bar{\lambda},r}(h)\bigr\rangle_{\mathbb{C}} &= j(\lambda+1+m)^{-1}j(\lambda+1+r)^{-1}\\ &\quad\cdot\bigl\langle Q_{\lambda,\nu}P_{\lambda,\nu,\lambda+1+m}b_{\lambda,m}(f), P_{\bar{\lambda},\nu,\bar{\lambda}+1+r}b_{\bar{\lambda},r}(h)\bigr\rangle_{\mathbb{C}}\\ &= j(\lambda+1+m)^{-1}j(\lambda+1+r)^{-1}T(\lambda,m)\\ &\quad\cdot\bigl\langle \xi_{-\lambda-2,m}A_{-\lambda-2-m}b_{\lambda,m}(f), P_{\bar{\lambda},\nu,\bar{\lambda}+1+r}b_{\bar{\lambda},r}(h)\bigr\rangle_{\mathbb{C}}.\end{aligned}$$

Transfer $\xi_{-\lambda-2,m}$ to the second factor in the form $\langle\cdot,\cdot\rangle_{\mathbb{C}}$ by (7.10) and use (7.12), (7.14) and (8.16). We obtain 0 for $r \neq m$ and the right-hand side of (8.20) for $r = m$. $\square$

Formulae (8.20), (8.21) show that (8.17) is a "Pythagorean theorem" for the decomposition (8.14). Thus, in Case (C) we have

Theorem 8.5. *Let* $\lambda \in I_{-k-1}$, $k \in \mathbb{N}$. *Then the representation* $R_{\lambda,\nu}$ *considered on the space* $\mathcal{T}_{-\lambda-2,\nu,k}$ *splits into the sum of two terms. The first one acts on the subspace of functions* f *such that their Taylor coefficients* $a_m(f)$ *are equal to* 0 *for* $m \leqslant k$ *and decomposes as* $R_{\lambda,\nu}$ *does in Case* (A), *the second one decomposes into the direct sum of irreducible* $T_{-\lambda-2-m}$, $m \leqslant k$, $m \equiv \nu$, *acting on the sum of the spaces* $\mathcal{P}_{\lambda,m}$ *(there is no Jordan blocks). There is an inversion formula, see* (8.14), *and a "Plancherel formula" for the form* $(f,h)_{\lambda,\nu}$, *see* (8.17).

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