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A Class of Monotone Decreasing Rearrangements

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We consider monotone decreasing rearrangement with respect to the finite measure $d\mu(x) = \varphi(x)\,dx$ on $\mathbf{R}$, where φ is a strictly positive, symmetric decreasing, log-concave function.

1. INTRODUCTION

The study of monotone equimeasurable rearrangements was initiated by Hardy and Littlewood [9] in the course of their work on fractional integrals. For a good introduction to the general subject of rearrangements, see the monograph of B. Kawohl [10].

Monotone decreasing rearrangements with respect to a Gaussian measure were studied by A. Ehrhard [5, 6]. In reference [11] M. Ledoux found an interesting application to the logarithmic Sobolev inequality of L. Gross. The purpose of this paper is to extend known rearrangement inequalities to a broad class of finite measures containing the Gaussian measure.

2. PRELIMINARIES

Let μ be a finite Borel measure on $\mathbf{R}$. The *distribution function* μ_f of a real-valued Borel measurable function f is defined for $x \in \mathbf{R}$ by

$$\mu_f(x) = \mu\{y \in \mathbf{R} : f(y) > x\}. \tag{1}$$

It is immediate that μ_f is a decreasing right-continuous function with left-hand limits everywhere. We assume throughout that functions are finite μ-a.e. Hence $\mu_f(x) \nearrow \mu(\mathbf{R})$ as $x \searrow -\infty$, and $\mu_f(x) \searrow 0$, as $x \nearrow \infty$.

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PROPOSITION 1. *Let f, g, f_n $(n=1, 2, ...)$ be a real-valued Borel measurable functions. The distribution function has the properties*:

1. $f \leqslant g$ *μ-a.e. implies* $\mu_f \leqslant \mu_g$;
2. $f_n \nearrow f$ *μ-a.e. implies* $\mu_{f_n} \nearrow \mu_f$.

Proof. Property 1 follows directly from the definition (1). Property 2 is an easy consequence of the monotone convergence theorem.

Now suppose that μ has no pure point support. The *monotone decreasing rearrangement* f^* of a Borel measurable function f with respect to the measure μ is defined for $x \in \mathbf{R}$ by

$$f^*(x) = \inf\{y \in \mathbf{R} : \mu_f(y) \leqslant \mu(-\infty, x)\}. \tag{2}$$

We summarize some of the properties of f^* now.

PROPOSITION 2. *Let f, g, f_n $(n=1, 2, ...)$ be real-valued Borel measurable functions. The monotone decreasing rearrangement f^* is a decreasing, right-continuous function on* $\mathbf{R}$. *Furthermore,*

1. $f \leqslant g$ *μ-a.e. implies* $f^* \leqslant g^*$;
2. $f_n \nearrow f$ *μ-a.e. implies* $f_n^* \nearrow f^*$;
3. *if f_n $(n=1, 2, ...)$ denotes the sequence of lower-cutoff functions* $f \vee (-n)$, *then* $f_n^* \searrow f^*$;
4. *f and f^* are μ-equimeasurable.*

Proof. Clearly f^* is decreasing. Since $x \mapsto \mu(-\infty, x)$ is a continuous increasing function, $f^*(x)$ is right-continuous in case the decreasing function

$$F^*(x) = \inf\{y \in \mathbf{R} : \mu_f(y) \leqslant x\}$$

is right-continuous on the interval $[0, \mu(\mathbf{R}))$. Note that $\mu_f(F^*(x)) \leqslant x$, owing to the right-continuity of μ_f. Now let $z = \lim_{x \to x_0^+} F^*(x)$, and suppose that $z < F^*(x_0)$; then $\mu_f(z) > x_0$, by the definition of F^*. On the other hand, using the fact that μ_f is decreasing,

$$\mu_f(z) \leqslant \lim_{x \to x_0^+} \mu_f(F^*(x)) \leqslant \lim_{x \to x_0^+} x = x_0,$$

which is a contradiction. Therefore $\lim_{x \to x_0^+} F^*(x) = F^*(x_0)$, establishing the right-continuity of f^*.

If $f \leqslant g$ μ-a.e., then $\mu_f \leqslant \mu_g$. It follows from definition (2) that $\mu_f \leqslant \mu_g$ implies $f^* \leqslant g^*$. This establishes statement 1. Similarly, if $f_n \nearrow f$ μ-a.e., then $\mu_{f_n} \nearrow \mu_f$, which implies that $f_n^* \nearrow f^*$. This establishes statement 2.

Next we show that f and f^* have the same distribution function. Since f^* is a decreasing right-continuous function, we know that for each $y \in \mathbf{R}$, there exists some $z_0 \in \mathbf{R}$ such that $\{z \in \mathbf{R} : f^*(z) > y\} = (-\infty, z_0)$. We have $f^*(z_0) \leqslant y$ automatically. Hence

$$\mu_f(y) \leqslant \mu_f(f^*(z_0)) \leqslant \mu(-\infty, z_0),$$

in which the second inequality follows from the definition of f^* and the right-continuity of μ_f. Suppose now that $\mu_f(y) < \mu(-\infty, z_0)$. Then there exists some $w_0 < z_0$ such that $\mu_f(y) = \mu(-\infty, w_0)$. Consequently,

$$\begin{aligned} f^*(w_0) &= \inf\{z \in \mathbf{R} : \mu_f(z) \leqslant \mu(-\infty, w_0)\} \\ &= \inf\{z \in \mathbf{R} : \mu_f(z) \leqslant \mu_f(y)\} \\ &\leqslant y, \end{aligned}$$

which is a contradiction, since $f^*(z) > y$ for all $z \in (-\infty, z_0)$. Therefore $\mu_f(y) = \mu(-\infty, z_0) = \mu_{f^*}(y)$.

Finally, statement 3 is clear in case f is lower-bounded. Otherwise, we note that $f_n^* = f^*$ on the interval $(-\infty, x_n)$, where x_n is defined by $\mu(-\infty, x_n) = \mu_f(-n)$. Since f is finite μ-a.e., $x_n \to \infty$ as $n \to \infty$.

3. Some Properties of the Rearrangement

Next we follow G. Chiti [2] in showing that the *-operation is non-expansive in certain Orlicz spaces. For a good reference on Orlicz spaces, rearrangements, and related topics, see [1]. Let F be an increasing convex function on $[0, \infty)$, with $F(0) = 0$. The *Orlicz space* $L_F(\mathbf{R}, d\mu)$ consists of the real-valued functions on $\mathbf{R}$ such that

$$\exists r, r > 0 : \int F\left(\frac{|f(x)|}{r}\right) d\mu(x) < \infty,$$

with the norm

$$\|f\|_F = \inf\left\{r > 0 : \int F\left(\frac{|f(x)|}{r}\right) d\mu(x) < 1\right\}.$$

Since f and f^* share the same distribution function, $\|f\|_F = \|f^*\|_F$. Of particular concern is the case of $L^p(\mathbf{R}, d\mu)$ spaces, corresponding to $F(x) = x^p$.

First we recall a lemma used by Chiti.

Lemma 1. *Let $F(x)$, $x \geqslant 0$, be convex, increasing. If $x_1 \geqslant x_2$, $y_1 \geqslant y_2$, then*

$$F(|x_1 - y_1|) + F(|x_2 - y_2|) \leqslant F(|x_1 - y_2|) + F(|x_2 - y_1|).$$

Proof. The function $c(x)=F(|x|)$ is convex on $\mathbf{R}$. Consequently, for $a, b \geqslant 0$, $x \in \mathbf{R}$,

$$c(x)-c(x-b) \leqslant c(x+a)-c(x+a-b).$$

The lemma follows by choosing $x=x_2-y_2$, $a=x_1-x_2$, $b=y_1-y_2$.

THEOREM 1. *Let $F(x)$, $x>0$, be convex, increasing, with $F(0)=0$, and let f, g be real-valued Borel measurable functions. Then*

$$\int F(|f^*(x)-g^*(x)|)\, d\mu(x) \leqslant \int F(|f(x)-g(x)|)\, d\mu(x). \tag{3}$$

Proof. Without loss of generality we may assume that $\mu(\mathbf{R})=1$. If f and g are simple functions of the form

$$f=\sum_{k=1}^{p} a_k \chi_{E_k}, \qquad g=\sum_{k=1}^{p} b_k \chi_{E_k},$$

in which $\mu(E_k)$ is rational for each k, then (3) follows immediately from the lemma. For f and g bounded Borel functions, there exist sequences $f_n \nearrow f$, $g_n \nearrow g$ $(n=1, 2, ...)$ of the form

$$f_n=\sum_{k=1}^{p_n} a_{n,k} \chi_{E_{n,k}}, \qquad g_n=\sum_{k=1}^{p_n} b_{n,k} \chi_{E_{n,k}},$$

such that $\mu(E_{n,k})$ is rational for each n, k, and such that $f_n \geqslant \inf f$, $g_n \geqslant \inf g$ for all n. Note that the uniform lower bounds on f_n, g_n imply the uniform bound

$$|f_n(x)-g_n(x)| \leqslant \max\{|\sup f-\inf g|, |\sup g-\inf f|\}.$$

Now using Proposition 2, Fatou's lemma, and Lebesgue's dominated convergence theorem, we have

$$\begin{aligned} \int F(|f^*-g^*|)\, d\mu &\leqslant \liminf_{n\to\infty} \int F(|f_n^*-g_n^*|)\, d\mu \\ &\leqslant \liminf_{n\to\infty} \int F(|f_n-g_n|)\, d\mu \\ &= \int F(|f-g|)\, d\mu. \end{aligned} \tag{4}$$

For f and g lower-bounded functions we use upper-cutoff approximations $f_n=f \wedge n$, $g_n=g \wedge n$ in argument (4). Finally, for f and g arbitrary Borel measurable functions, we use lower-cutoff approximations $f_n = f \vee (-n)$, $g_n = g \vee (-n)$ and Proposition 2.3 in argument (4).

Now let $d\mu(x) = \varphi(x)\,dx$ be a finite measure on $\mathbf{R}$ given by a continuous, strictly positive density φ. Then all previous results extend to Lebesgue measurable functions. Since the function $E(x) = \mu(-\infty, x)$ is strictly increasing, its inverse E^{-1} is well-defined on the interval $(0, \mu(\mathbf{R}))$. If $A \subset \mathbf{R}$, $r > 0$, then we define $A_r \subset \mathbf{R}$ to be the set

$$A_r = \{x + y : x \in A, |y| \leqslant r\}.$$

We say that monotone decreasing rearrangement (with respect to μ) is *regularizing* in case

$$E^{-1} \circ \mu(A_r) \geqslant E^{-1} \circ \mu(A) + r \tag{5}$$

for all closed Borel sets $A \subset \mathbf{R}$ and all $r \geqslant 0$.

THEOREM 2. *Let $d\mu(x) = \varphi(x)\,dx$ be a finite Borel measure on $\mathbf{R}$, where φ is a strictly positive, symmetric decreasing, log-concave function. Then monotone decreasing rearrangement with respect to μ is regularizing.*

A. Ehrhard [5] first proved this assertion in the important special case where μ is a Gaussian measure.

Proof. We follow Ehrhard's method of proof here; see Ref. [5, Prop. 1.3] for full detail. First, μ inherits from Lebesgue measure the property of being regular, so it suffices to prove (5) for A a finite disjoint union of closed intervals. Our task is to prove (5) for A a single closed interval $[a, b]$; we omit Ehrhard's inductive proof on the number of closed intervals.

Equality holds in (5) in the cases $a = -\infty$, $b = \infty$, and $r = 0$. The inequality is also trivial when $a = b$, for then the right side is identically $-\infty$. Therefore we assume that $-\infty < a < b < \infty$ and $r > 0$.

Since φ is positive, log-concave, it is continuous; it follows that E and $h(r) = \mu(a - r, b + r)$ are continuously differentiable functions. We must show that

$$\frac{d}{dr} E^{-1} \circ \mu(a - r, b + r) \geqslant 1$$

for all $r > 0$. The derivative is easily calculated:

$$\frac{d}{dr} E^{-1} \circ \mu(a - r, b + r)$$
$$= [\varphi(E^{-1} \circ \mu(a - r, b + r))]^{-1} [\varphi(a - r) + \varphi(b + r)],$$

so we must show, equivalently, that for all $-\infty < a < b < \infty$,

$$\varphi(t) \leqslant \varphi(a) + \varphi(b), \qquad \text{where} \quad t := E^{-1} \circ \mu(a, b). \tag{6}$$

Suppose that $0 \leqslant a < b < \infty$. In this case $\mu(a, b) < \mu(a, \infty)$, so that by symmetry $t < -a$, and thus $\varphi(t) < \varphi(a)$. The case $-\infty < a < b \leqslant 0$ is similar. Now suppose that $a < 0 < b$. We define the function

$$\Phi(a) = \varphi(t_b(a)) - \varphi(a), \qquad a \in (-\infty, 0).$$

Here $t_b(a) := E^{-1} \circ \mu(a, b)$. As $a \searrow -\infty$, $\Phi(a) \to \varphi(b)$; we want to show that $\Phi(a) \leqslant \varphi(b)$ for all $a < 0$. Since φ is strictly positive, log-concave, it is differentiable almost everywhere, with left and right derivatives uniformly bounded on compact sets. It follows that Φ is absolutely continuous on finite intervals, so we can write for $c < a < 0$

$$\Phi(a) = \Phi(c) + \int_c^a \Phi'(x)\, dx. \tag{7}$$

Now we have μ-a.e.

$$\begin{aligned} \Phi'(x) &= \varphi'(t_b(x)) \frac{dt_b}{dx} - \varphi'(x) \\ &= -\varphi'(t_b(x)) \frac{\varphi(x)}{\varphi(t_b(x))} - \varphi'(x). \end{aligned}$$

Since φ is symmetric, log-concave, the condition

$$\frac{\varphi'(t_b(x))}{\varphi(t_b(x))} + \frac{\varphi'(x)}{\varphi(x)} \geqslant 0 \ \mu\text{-a.e.}$$

is equivalent to the condition $t_b(x) + x \leqslant 0$. Suppose first that $\mu(x, b) \leqslant \frac{1}{2}\mu(\mathbf{R})$. Then $t_b(x) \leqslant 0$, so that $t_b(x) + x < 0$. Suppose next that it is possible to decrease x until $t_b(x) = -x$. Then

$$\begin{aligned} & \mu(-\infty, t_b(x)) = \mu(x, b), \qquad \text{or} \\ & \mu(-\infty, x) + \mu(x, t_b(x)) = \mu(x, t_b(x)) + \mu(t_b(x), b) \\ \Rightarrow \quad & \mu(-\infty, x) = \mu(t_b(x), b) \\ \Rightarrow \quad & \mu(-\infty, x) + \mu(x, 0) = \mu(t_b(x), b) + \mu(x, 0) \\ \Rightarrow \quad & \mu(-\infty, x) + \mu(x, 0) = \mu(t_b(x), b) + \mu(0, t_b(x)) \qquad \text{(by symmetry of } \varphi) \\ \Rightarrow \quad & \mu(-\infty, 0) = \mu(0, b) \end{aligned}$$

which is impossible since φ is strictly positive.

We have shown that $\Phi'(x) \leqslant 0$ for all $x < 0$. From (7) we conclude that $\Phi(a) \leqslant \Phi(c)$ for all $c < a$. Letting $c \to -\infty$ completes the proof.

In Ref. [6] Ehrhard deduced several consequences of the regularizing property of a rearrangement, one of which we state below as a corollary. First we need some notation. Let $A \subset \mathbf{R}$ be a Borel set, and let χ_A be the characteristic function of A. We define $\mathscr{S}(A)$ to be the open interval $\{\chi_A^* > 0\} = (-\infty, E^{-1} \circ \mu(A))$. Clearly, if $A \subset B$ are Borel sets, then $\mathscr{S}(A) \subset \mathscr{S}(B)$. Now let f be a measurable function. Since f and f^* are μ-equimeasurable, $\mathscr{S}\{f > t\} = \mathscr{S}\{f^* > t\} = \{f^* > t\}$. In terms of the $\mathscr{S}$-operation, inequality (5) is equivalent to the inequality $\mu(A_r) \geqslant \mu((\mathscr{S}A)_r)$.

COROLLARY 1. *Let μ be as in Theorem 2. Then monotone decreasing rearrangement with respect to μ reduces the Lipschitz constant of a Lipschitz function.*

Proof. Suppose the corollary is false for some Lipschitz continuous function f having Lipschitz constant L; that is

$$L = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|} < \infty,$$

while there exist $x_1, x_2 \in \mathbf{R}$ such that

$$f^*(x_2) + L\,|x_1 - x_2| < f^*(x_1).$$

Then there exist $t_1, t_2 \in \mathbf{R}$ such that

$$f^*(x_2) + L\,|x_1 - x_2| < t_2 + L\,|x_1 - x_2| < t_1 < f^*(x_1). \tag{8}$$

Let $r = |x_1 - x_2|$. We note that

$$\{f > t_2\} \supset \{f > t_1\}_r. \tag{9}$$

(Otherwise, there exist x, z, $|z| \leqslant r$, such that $f(x) > t_1$, while $f(x+z) \leqslant t_2$. Then $|f(x) - f(x+z)| > Lr$, which contradicts the hypothesis on f.) Applying $\mathscr{S}$ to both sides of inclusion (9),

$$\mathscr{S}\{f > t_2\} \supset \mathscr{S}(\{f > t_1\}_r) \Rightarrow \{f^* > t_2\} \supset \{f^* > t_1\}_r,$$

since $\mathscr{S}(\{f > t_1\}_r) \supset (\mathscr{S}\{f > t_1\})_r$ according to inequality (5). This inclusion says that if $f^*(x) > t_1$ and $|z| \leqslant r$, then $f^*(x+z) > t_2$. In particular, $f^*(x_1) > t_1$ implies $f^*(x_2) > t_2$, which contradicts (8).

COROLLARY 2. *Let A be a closed Borel set such that $\mu(\partial A) = 0$. Then we have for all $r \geqslant 0$ the boundary inequality*

$$\mu((\partial A)_r) \geqslant \mu((\partial \mathscr{S} A)_r). \tag{10}$$

Proof. We express $(\partial A)_r$ as the disjoint union

$$(\partial A)_r = (A_r \setminus A) \cup ((\overline{\sim A})_r \setminus \overline{\sim A}) \cup \partial A.$$

Here the sign $\sim$ means complement. Taking measures,

$$\mu((\partial A)_r) = \mu(A_r) - \mu(A) + \mu((\overline{\sim A})_r) - \mu(\overline{\sim A}).$$

According to the remark preceding Corollary 1, $\mu(A_r) \geqslant \mu((\mathscr{S}A)_r)$ and $\mu((\overline{\sim A})_r) \geqslant \mu((\mathscr{S}(\overline{\sim A}))_r)$, so we are finished if we show that

$$\mu((\mathscr{S}(\overline{\sim A}))_r) = \mu((\overline{\sim \mathscr{S}A})_r) \tag{11}$$

$$\mu(\overline{\sim A}) = \mu(\overline{\sim \mathscr{S}A}). \tag{12}$$

By symmetry of φ, (11) is true if and only if $\mu(\mathscr{S}(\overline{\sim A})) = \mu(\overline{\sim \mathscr{S}A})$. Note that $\overline{\sim \mathscr{S}A} = \sim \mathscr{S}A$, since $\mathscr{S}A$ is open. Now

$$\mathscr{S}(\overline{\sim A}) = \mathscr{S}(\sim A \cup \partial(\sim A)) = \mathscr{S}(\sim A \cup \partial A) = \mathscr{S}(\sim A),$$

so that

$$\mu(\mathscr{S}(\overline{\sim A})) = \mu(\mathscr{S}(\sim A)) = \mu(\sim A),$$

while

$$\mu(\overline{\sim \mathscr{S}A}) = \mu(\sim \mathscr{S}A) = \mu(\mathbf{R}) - \mu(\mathscr{S}A) = \mu(\mathbf{R}) - \mu(A) = \mu(\sim A).$$

This establishes (11). In the course of the proof we saw that

$$\mu(\overline{\sim \mathscr{S}A}) = \mu(\sim \mathscr{S}A) = \mu(\sim A) = \mu(\sim A \cup \partial A) = \mu(\overline{\sim A}),$$

establishing (12).

4. Rearrangement Inequalities for the Derivative

The inequalities established in this section are similar to those found in [3, 4] concerning monotone rearrangement with respect to Lebesgue measure on an interval. The method of proof of the following theorem is implicit in a paper by M. Ledoux [11]. Using a version of Theorem 3 in the case where μ is a Gaussian measure, Ledoux derives the logarithmic Sobolev inequality of L. Gross [8].

Theorem 3. *Let f be a Lipschitz continuous function, and let μ be as in Theorem 2. Then for every Borel set $A \subseteq \mathbf{R}$,*

$$\int_{f^{-1}(A)} |Df| \, d\mu \geqslant \int_{f^{*-1}(A)} |Df^*| \, d\mu. \tag{13}$$

Note that according to Rademacher's theorem (see [7, Sect. 3.1]), the derivative of a Lipschitz function is an essentially bounded Borel measurable function. Hence, by Corollary 1 both sides of (13) are finite.

Proof. It suffices to prove the theorem for the case in which A is an interval $[a, b]$. Since f is Lipschitz, so is the function $f_{ab} := (f \vee a) \wedge b$. Clearly

$$\int_{f^{-1}[a,b]} |Df| \, d\mu = \int_{\mathbf{R}} |Df_{ab}| \, d\mu,$$

and

$$\int_{f^{*-1}[a,b]} |Df^*| \, d\mu = \int_{\mathbf{R}} |D(f^*)_{ab}| \, d\mu = \int_{\mathbf{R}} |Df^*_{ab}| \, du.$$

Hence, without loss of generality we may take the domain of integration in (13) to be $\mathbf{R}$.

According to the simple one-dimensional version of the Hausdorff area formula (see [7, Theorem 3.2.3]

$$\int |Df| \, d\mu = \int \Bigg(\sum_{x \in f^{-1}(y)} \varphi(x) \Bigg) \, dy. \tag{14}$$

Let f_n denote the approximation

$$f_n(x) = \begin{cases} f(n), & \text{if } \; x \geqslant n, \\ f(x), & \text{if } \; -n \leqslant x \leqslant n, \\ f(-n), & \text{if } \; x \leqslant -n. \end{cases}$$

This approximation scheme has the property that if $y \neq f(\pm n)$, then $f_n^{-1}(y) \subset (-n, n)$; furthermore, if

$$\sum_{x \in f_n^{-1}(y)} \varphi(x) < \infty,$$

then the sum is over a finite set $\{x_i\}_{i=1}^m \subset (-n, n)$. We have in this case

$$\begin{aligned} \sum_{i=1}^{m} \varphi(x_i) &= \sum_{i=1}^{m} \lim_{r \to 0} \frac{1}{2r} \mu(\{x_i\}_r) \\ &= \lim_{r \to 0} \frac{1}{2r} \mu(\{f_n^{-1}(y)\}_r) \\ &= \lim_{r \to 0} \frac{1}{2r} \mu((\partial\{f_n \geqslant y\})_r). \end{aligned}$$

Since f is continuous, the set $\{f_n \geqslant y\}$ is closed, so that according to Corollary 2

$$\lim_{r \to 0} \frac{1}{2r} \mu((\partial\{f_n \geqslant y\})_r) \geqslant \lim_{r \to 0} \frac{1}{2r} \mu((\partial\mathscr{S}\{f_n \geqslant y\})_r) = \lim_{r \to 0} \frac{1}{2r} \mu((\partial\{f_n^* \geqslant y\})_r).$$

Now $\partial\{f_n^* \geqslant y\} = E^{-1} \circ \mu\{f_n \geqslant y\}$, so that

$$\lim_{r \to 0} \frac{1}{2r} \mu((\partial\{f_n^* \geqslant y\})_r) = \varphi \circ E^{-1} \circ \mu\{f_n \geqslant y\} = \varphi \circ E^{-1} \circ \mu\{f_n > y\},$$

in which the last step uses the fact that $\{f_n = y\}$ is a μ-null set. Inserting this result into the Hausdorff area formula, we obtain

$$\int |Df_n| \, d\mu \geqslant \int \varphi \circ E^{-1} \circ \mu\{f_n > y\} \, dy.$$

By the monotone convergence theorem,

$$\lim_{n \to \infty} \int |Df_n| \, d\mu = \int |Df| \, d\mu,$$

so that

$$\begin{aligned} \int |Df| \, d\mu &\geqslant \liminf_{n \to \infty} \int \varphi \circ E^{-1} \circ \mu\{f_n > y\} \, dy \\ &\geqslant \int \liminf_{n \to \infty} \varphi \circ E^{-1} \circ \mu\{f_n > y\} \, dy \\ &= \int \varphi \circ E^{-1} \circ \mu\{f > y\} \, dy, \end{aligned}$$

in which we have applied Fatou's lemma in the second inequality. Now we apply the Hausdorff area formula (14) to the Lipschitz continuous function f^*. If

$$\sum_{x \in f^{*-1}(y)} \varphi(x) < \infty,$$

then the sum is over a single term

$$x = E^{-1} \circ \mu\{f^* > y\} = E^{-1} \circ \mu\{f > y\}.$$

Thus

$$\int |Df^*| \, d\mu = \int \varphi \circ E^{-1} \circ \mu\{f > y\} \, dy,$$

which completes the proof.

The following theorem is stated in [6] for the special case of μ being a Gaussian measure.

THEOREM 4. *Let F be an increasing convex function on $[0, \infty)$, with $F(0)=0$, and let μ be as in Theorem 2. Then for every Lipschitz continuous function f and every Borel set $A \subseteq \mathbf{R}$,*

$$\int_{f^{-1}(A)} F(|Df|)\, d\mu \geqslant \int_{f^{*-1}(A)} F(|Df^*|\, d\mu. \tag{15}$$

Proof. It suffices to prove the theorem for A an interval $[a, b]$. Let

$$t_{n,i} = a + \frac{i}{n}(b-a), \qquad i = 0, ..., n-1$$

denote the points partitioning $[a, b]$ into n intervals of length $(b-a)/n$, and let $I_{n,i}$ denote the associated intervals

$$I_{n,i} = \begin{cases} [t_{n,i}, t_{n,i+1}] & \text{if } i \neq n-1, \\ [t_{n,n-1}, b] & \text{if } i = n-1. \end{cases}$$

By Jensen's inequality,

$$\begin{aligned} &\frac{1}{\mu(f^{-1}(I_{n,i}))} \int_{f^{-1}(I_{n,i})} F(|Df|)\, d\mu \\ &\qquad \geqslant F\left(\frac{1}{\mu(f^{-1}(I_{n,i}))} \int_{f^{-1}(I_{n,i})} |Df|\, d\mu\right). \end{aligned} \tag{16}$$

Since $\mu(f^{-1}(I_{n,i})) = \mu(f^{*-1}(I_{n,i}))$ and F is increasing we conclude from Theorem 3 that

$$\begin{aligned} &F\left(\frac{1}{\mu(f^{-1}(I_{n,i}))} \int_{f^{-1}(I_{n,i})} |Df|\, d\mu\right) \\ &\qquad \geqslant F\left(\frac{1}{\mu(f^{*-1}(I_{n,i}))} \int_{f^{*-1}(I_{n,i})} |Df^*|\, d\mu\right). \end{aligned} \tag{17}$$

Let $F_n(y)$ denote the simple function on $f^{*-1}[a, b]$ whose value is equal to that of the right side of inequality (17) when $y \in f^{*-1}(I_{n,i})$. Combining (16) and (17) we see that

$$\begin{aligned} \int_{f^{-1}[a,b]} F(|Df|)\, d\mu &\geqslant \liminf_{n \to \infty} \int_{f^{*-1}[a,b]} F_n(y)\, d\mu(y) \\ &\geqslant \int_{f^{*-1}[a,b]} \liminf_{n \to \infty} F_n(y)\, d\mu(y) \end{aligned} \tag{18}$$

by an application of Fatou's lemma.

Now suppose that $y \in J = f^{*-1}(x)$, where J is an interval. Let $K_{n,x}$, $n = 1, 2, \ldots$ denote the sequence of intervals $I_{n,i}$ containing x. Then

$$\liminf_{n\to\infty} \frac{1}{\mu(f^{*-1}(K_{n,x}))} \int_{f^{*-1}(K_{n,x})} |Df^*|\, d\mu = \frac{1}{\mu(J)} \int_J |Df^*|\, d\mu = 0,$$

since f^* is flat on J. Therefore, since F is continuous and $F(0) = 0$, $\liminf_{n\to\infty} F_n(y) = 0$ in this case.

Suppose on the other hand that $y = f^{*-1}(x)$ identically; that is, that $f^{*-1}(x)$ consists of the single point y. Again, let $K_{n,x}$ denote the sequence of intervals $I_{n,i}$ containing x. Since φ is uniformly continuous on compact sets and $|Df^*|$ is essentially bounded,

$$\begin{aligned}\liminf_{n\to\infty} F_n(y) &= F\left(\liminf_{n\to\infty} \frac{1}{\mu(f^{*-1}(K_{n,x}))} \int_{f^{*-1}(K_{n,x})} |Df^*|\, d\mu\right)\\ &= F(|Df^*(y)|)\end{aligned}$$

for μ-almost all such y, according to Lebesgue's differentiation theorem. This concludes the proof of the theorem.

COROLLARY 3. *Let μ and F be as in Theorem 4. Then for every nonnegative Borel measurable function G and every Lipschitz continuous function f,*

$$\int (G \circ f) F(|Df|)\, d\mu \geq \int (G \circ f^*)\, F(|Df^*)\, d\mu. \tag{19}$$

Note that because G is Borel measurable, $G \circ f$ and $G \circ f^*$ are both measurable functions.

Proof. If G is a simple function, then the corollary follows immediately from Theorem 4. In the general case we approximate G by a sequence of nonnegative Borel functions $G_n \nearrow G$, and apply the monotone convergence theorem.

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