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Localization at Semiprime Ideals

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Lambek and Michler [7] and Jategaonkar [4, 5] have studied a notion of localization at a semiprime ideal I of a left Noetherian ring R. The purpose of this paper is to extend certain of their results to rings R satisfying only the condition that R/I is a left Goldie ring, which includes all rings with Krull dimension on the left. Throughout the paper, R will be an associative ring with identity and I will denote a (two-sided) ideal of R. If σ is the largest torsion radical such that R/I is σ-torsionfree, determining the ring of quotients R_σ, then Theorem 3 gives necessary and sufficient conditions under which $R_\sigma \cdot I$ is the Jacobson radical $J(R_\sigma)$ of R_σ and $R_\sigma/J(R_\sigma)$ is semisimple Artinian. In this case, $R_\sigma/J(R_\sigma)$ is isomorphic to the classical left ring of quotients of R/I. Theorem 6 gives conditions under which R_σ is an Artinian classical left ring of quotients of R.

Recall that a torsion radical σ of R-Mod assigns to each module ${}_RM$ a submodule $\sigma(M)$ such that $\sigma(M/\sigma(M)) = 0$ and $\sigma(M') = M' \cap \sigma(M)$ for all submodules $M' \subseteq M$ and moreover, such that $f(\sigma(M)) \subseteq \sigma(N)$ for any $f \in \mathrm{Hom}_R(M, N)$. The module M is called σ-torsion if $\sigma(M) = M$ and σ-torsionfree if $\sigma(M) = 0$ and the left ideal A of R is called σ-dense if R/A is σ-torsion and σ-closed if R/A is σ-torsionfree. The σ-closure $\bar{A}$ of the left ideal $A \subseteq R$ is the intersection of all σ-closed left ideals that contain A; note that $\bar{A}/A = \sigma(R/A)$. The set of σ-dense left ideals of R characterize σ since $m \in \sigma(M)$ if and only if $Dm = 0$ for some σ-dense left ideal D, or equivalently, if and only if $\mathrm{Ann}(m) = \{r \in R \mid rm = 0\}$ is σ-dense.

A torsion radical σ is larger than a torsion radical τ, denoted $\sigma \geqslant \tau$, if $\sigma(M) \supseteq \tau(M)$ for all $M \in R$-Mod. The largest torsion radical σ for which ${}_RX$ is σ-torsionfree is given by $\sigma(M) = \mathrm{rad}_{E(X)}(M) = \{m \in M \mid f(m) = 0$ for all $f \in \mathrm{Hom}_R(M, E(X))\}$, where $E(X)$ is the injective envelope of X. For the torsion radical $\sigma = \mathrm{rad}_{E(R)}$, σ-dense will be abbreviated to dense, σ-torsionfree to torsionfree, etc. If I is an ideal of R and the left ideal $A \supseteq I$ is

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$\mathrm{rad}_{E(R/I)}$-dense in R, then $\mathrm{Hom}_R(R/A, E(R/I)) = 0$ implies $\mathrm{Hom}_{R/I}(R/A, E({}_{R/I}R/I)) = 0$, since the injective envelope $E({}_{R/I}R/I)$ of R/I in R/I-Mod is $\{x \in E(R/I) \mid Ix = 0\}$ and so A/I is dense in R/I.

For an ideal I of R, let $C(I) = \{c \in R \mid cr \in I \text{ or } rc \in I \text{ implies } r \in I\}$. Thus, $C(I)$ denotes the set of elements of R whose images are regular in R/I. The module ${}_RM$ is said to be $C(I)$-torsion if for each $m \in M$ there exists $c \in C(I)$ such that $cm = 0$, and then a torsion radical $\mathrm{rad}_{C(I)}$ can be defined for any module M by letting $\mathrm{rad}_{C(I)}(M)$ be the sum in M of all $C(I)$-torsion submodules. Thus, $\mathrm{rad}_{C(I)}(M) = \{m \in M \mid Rm \text{ is } C(I)\text{-torsion}\}$. To show that $\mathrm{rad}_{C(I)}$ is in fact a torsion radical, we note that if $\mathrm{rad}_{C(I)}(M/\mathrm{rad}_{C(I)}(M)) \neq 0$, then there exists $m \in M$ such that $m \notin \mathrm{rad}_{C(I)}(M)$, but given $r \in R$, there exists $c \in C(I)$ such that $crm \in \mathrm{rad}_{C(I)}(M)$. But then there exists $c' \in C(I)$ such that $c'crm = 0$ and since $c'c \in C(I)$, this implies that $m \in \mathrm{rad}_{C(I)}(M)$, a contradiction. The remaining conditions follow easily from the definition since a submodule of a $C(I)$-torsion module is again $C(I)$-torsion. Observe that ${}_RM$ is $\mathrm{rad}_{C(I)}$-torsionfree (or simply $C(I)$-torsionfree), if for each $0 \neq m \in M$ there exists $r \in R$ such that $\mathrm{Ann}(rm) \cap C(I) = \varnothing$ and so, in particular R/I is $C(I)$-torsionfree and, therefore, $\mathrm{rad}_{E(R/I)} \geqslant \mathrm{rad}_{C(I)}$. If R satisfies the left Ore condition with respect to $C(I)$, that is, if for each $c \in C(I)$ and $r \in R$ there exists $c' \in C(I)$ and $r' \in R$ such that $r'c = c'r$, then for all $M \in R$-Mod, $\mathrm{rad}_{C(I)}(M) = \{m \in M \mid cm = 0 \text{ for some } c \in C(I)\}$. Note that this condition holds if and only if R/Rc is $C(I)$-torsion for all $c \in C(I)$, or equivalently, if and only if the elements of $C(I)$ are not zero-divisors on $E(R/I)$.

Proposition 1. *If every dense left ideal of R/I contains a regular element, then* $\mathrm{rad}_{E(R/I)} = \mathrm{rad}_{C(I)}$.

Proof. Since $\mathrm{rad}_{C(I)} \leqslant \mathrm{rad}_{E(R/I)}$, assume that they are not equal. Then there exists a nonzero module ${}_RM$ that is $C(I)$-torsionfree, but $\mathrm{rad}_{E(R/I)}$-torsion. By definition of $\mathrm{rad}_{C(I)}$, there exisst $m \in M$ such that $A \cap C(I) = \varnothing$ for $A = \mathrm{Ann}(m)$ and then, $A + I/I$ does not contain a regular element of R/I, since $(A + I) \cap C(I) = \varnothing$. Since M is $\mathrm{rad}_{E(R/I)}$-torsion, A and hence $A + I$, are $\mathrm{rad}_{E(R/I)}$-dense in R, and therefore, $A + I/I$ is dense in R/I, contradicting the hypothesis. ▮

The condition of Proposition 1 holds if R/I has zero singular ideal and every essential left ideal contains a regular element. This includes the important case when R/I is a semiprime left Goldie ring.

Recall that the quotient functor Q_σ determined by σ is defined for $M \in R$-Mod as $\{x \in E(M/\sigma(M)) \mid Dx \in M/\sigma(M) \text{ for some } \sigma\text{-dense left ideal } D\}$. $Q_\sigma(M)$ will be denoted by M_σ. The torsion radical σ is called perfect (see Stenström [8]) if one of the following equivalent conditions hold: (i) σ is hereditary (i.e., Q_σ: R-Mod $\to$ R-Mod is exact) and every σ-dense left ideal contains a finitely generated σ-dense left ideal; (ii) Q_σ is naturally isomorphic to $R_\sigma \otimes_R -$; (iii) every R_σ-module is σ-torsionfree; (iv) $R_\sigma \cdot D = R_\sigma$ for all σ-dense left ideals $D \subseteq R$.

If σ is perfect, then for any left ideal $A \subseteq R$, A_σ is naturally isomorphic to $R_\sigma \otimes_R A$, which may be identified with $R_\sigma \cdot A \subseteq R_\sigma$. Furthermore, in this case there is a one-to-one correspondence between σ-closed left ideals of R and left ideals of R_σ.

LEMMA 2. *Assume that R/I satisfies the left Ore condition and that* $\mathrm{rad}_{C(I)} = \mathrm{rad}_{E(R/I)} = \sigma$.

(a) *If σ is hereditary and I_σ is an ideal of R_σ, then $R_\sigma/I_\sigma \simeq Q_{\mathrm{Cl}}(R/I)$, where Q_{Cl} denotes the classical left ring of quotients.*

(b) *$I_\sigma = J(R_\sigma)$ and $R_\sigma/J(R_\sigma)$ is semisimple Artinian if and only if R/I is a semiprime left Goldie ring, σ is perfect, and I_σ is an ideal of R_σ.*

Proof. (a) Applying σ to the exact sequence $0 \to I \to R \to R/I \to 0$ yields the exact sequence $0 \to I_\sigma \to R_\sigma \to (R/I)_\sigma$, so $R_\sigma/I_\sigma \subseteq (R/I)_\sigma$, with equality if Q_σ is exact. The proof of Lemma 2.3 of Lambek and Michler [7] then shows that $R_\sigma/I_\sigma \subseteq Q_{\mathrm{Cl}}(R/I) \subseteq (R/I)_\sigma$ since $\sigma = \mathrm{rad}_{C(I)}$.

(b) As in the proof of part (a), $R_\sigma/I_\sigma \subseteq (R/I)_\sigma \subseteq E(R/I)$ and if $I_\sigma = J(R_\sigma)$ with $R_\sigma/J(R_\sigma)$ semisimple Artinian, then $E(R/I)$ contains an isomorphic copy of each simple R_σ-module, so that $E(R/I)$ is a cogenerator for R_σ-Mod. Hence, every R_σ-module is σ-torsionfree, and σ is perfect. By part (a), $Q_{\mathrm{Cl}}(R/I) \simeq R_\sigma/I_\sigma$ is semisimple Artinian and so R/I must be a semiprime left Goldie ring.

Conversely, by part (a), $R_\sigma/I_\sigma \simeq Q_{\mathrm{Cl}}(R/I)$ is semisimple Artinian since R/I is a semiprime left Goldie ring, and thus $I_\sigma \supseteq J(R_\sigma)$. On the other hand, since σ is perfect, $E(R/I) = E(R_\sigma/I_\sigma)$ must contain an isomorphic copy of each simple R_σ-module since every R_σ-module is σ-torsionfree. Therefore, $I_\sigma \subseteq J(R_\sigma)$, since I_σ annihilates each simple R_σ-module. ▮

THEOREM 3. *Let I be a semiprime ideal such that R/I is a left Goldie ring. Then, the following conditions are equivalent for* $\sigma = \mathrm{rad}_{E(R/I)}$.

(1) *$I_\sigma = J(R_\sigma)$ and $R_\sigma/J(R_\sigma)$ is semisimple Artinian.*

(2) *I_σ is an ideal of R and σ is perfect.*

(3) *R satisfies the left Ore condition with respect to $C(I)$ and for each $c \in C(I)$, there exists $r \in R$ such that $rc \in C(I)$ and such that for each $a \in R$ with $arc = 0$ there exists $c' \in C(I)$ such that $c'ar = 0$.*

Proof. (1) if and only if (2) is Lemma 2(b). (2) implies (3): since R/I is a semiprime left Goldie ring, $\sigma = \mathrm{rad}_{C(I)}$ and $Rc + I/I$ is σ-dense in R/I for all $c \in C(I)$, and so $Rc + I$ is σ-dense for all $c \in C(I)$. Therefore, $R_\sigma \cdot c + I_\sigma = R_\sigma(Rc + I) = R_\sigma$ since σ is perfect, and since $I_\sigma = J(R_\sigma)$ by condition (1), $R_\sigma = R_\sigma \cdot c = R_\sigma \cdot Rc$, which shows that Rc is σ-dense. Thus, R satisfies the left Ore condition with respect to $C(I)$. The second part of condition (3) holds by Proposition 15.3 of Stenström [8] since σ is perfect. (3) implies (2): by [8, Proposition 15.3], σ is perfect. An easy computation using [8, Proposition 15.2] to describe R_σ shows that $I_\sigma = R_\sigma \cdot I$ is an ideal. ▌

If R is left Noetherian, then σ is perfect if and only if it is hereditary and condition (3) holds if and only if R satisfies the Ore condition with respect to $C(I)$. Thus, Theorem 3 reduces to Theorem 2.7 of Lambek and Michler [7]. Lambek and Michler [6] give an example of a left Noetherian ring and a prime ideal I, where condition (3) fails to hold; on the other hand an easy argument, using the fact that a matrix is regular if and only if its determinant is also, shows that condition (3) holds for any ideal in a matrix ring over a commutative ring.

COROLLARY 4. *$Q_{\mathrm{Cl}}(R)$ exists and $Q_{\mathrm{Cl}}(R)/J(Q_{\mathrm{Cl}}(R))$ is semisimple Artinian if and only if there exists a semiprime ideal I of R such that R/I is a left Goldie ring, $C(I) = C(0)$ and R satisfies the left Ore condition with respect to $C(I)$.*

Proof. Since $C(I) = C(0)$, the conditions are sufficient by Theorem (3). Conversely, if $Q = Q_{\mathrm{Cl}}(R)$, $J = J(Q)$ and Q/J is semisimple Artinian, then $R/J \cap R$ is a left order in $Q/J(Q)$ and so $R/J \cap R$ is a semiprime left Goldie ring. The remainder of the proof follows as in [7, Corollary 2.8]. ▌

The following proposition gives conditions under which the rings of Theorem 3 satisfy an Artin–Rees type of property.

PROPOSITION 5. *Let I be a semiprime ideal of R that satisfies the conditions of Theorem 3. Then for each left ideal A of R_σ, there exists a positive integer n such that $A \cap (J(R_\sigma))^n \subseteq J(R_\sigma)A$ if and only if for each left ideal A of R there exists a positive integer n such that $A \cap \overline{I^n} \subseteq \overline{IA}$.*

Proof. Since I_σ is an ideal of R_σ and σ is perfect, $I_\sigma^2 = I_\sigma R_\sigma I = I_\sigma I = R_\sigma II = (I^2)_\sigma$ and this extends to any finite product of I. If A is a left ideal of R, then by assumption, $A_\sigma \cap (I^n)_\sigma = A_\sigma \cap (I_\sigma)^n \subseteq I_\sigma A_\sigma = I_\sigma R_\sigma A = R_\sigma IA = (IA)_\sigma$ for some positive integer n. Using the one-to-one correspondence between closed left ideals of R and ideals of R_σ, which preserves finite intersections and inclusions, $A \cap \overline{I^n} \subseteq \bar{A} \cap \overline{I^n} \subseteq \overline{IA}$.

Conversely, if A is a left ideal of R_σ, then the inverse image $\pi^{-1}(A)$ of A under the induced ring homomorphism $\pi: R \to R_\sigma$ is a σ-closed left ideal of R.

Then there exists a positive integer n such that $\pi^{-1}(A) \cap \overline{I^n} \subseteq \overline{I \cdot \pi^{-1}(A)}$. But since $Q_\sigma(\pi^{-1}(A)) = A$, this implies that $A \cap (I^n)_\sigma = A \cap (I_\sigma)^n \subseteq R_\sigma I \cdot \pi^{-1}(A) = I_\sigma R_\sigma \pi^{-1}(A) = I_\sigma A$. ▮

THEOREM 6. *Let I be a semiprime ideal and let $\sigma = \mathrm{rad}_{E(R/I)}$, with $K = \sigma(R) = \mathrm{Ann}(E(R/I))$. Then R_σ is a left Artinian classical ring of fractions of R with respect to $C(I)$ if and only if*

(i) *$I^k \subseteq K$ for some positive integer k (assume $I^{k-1} \nsubseteq K$),*

(ii) *R/I and R/K are left Goldie rings,*

(iii) *$R/\{r \in R \mid I^m r \subseteq K\}$ has finite uniform dimension for all integers $1 \leqslant m \leqslant k$, and*

(iv) *for each $r \in R$ and $c \in C(I)$, $rc = 0$ implies there exists $c' \in C(I)$ such that $c'r = 0$.*

Proof. Assume that R_σ is a left Artinian classical ring of left fractions with respect to $C(I)$. Since R_σ is a partial classical ring of left quotients of R/K, it follows that $R_\sigma = Q_{\mathrm{Cl}}(R/K)$, since R_σ is left Artinian. Hence, (ii) and (iii) follow from Small's theorem. (See Hajarnavis [2].) By Proposition 15.7 of Stenström [8], condition (iv) holds, and thus the conditions of Theorem 3 are satisfied, since R must satisfy the left Ore condition with respect to $C(I)$. But then $R_\sigma I = I_\sigma = J(R_\sigma)$ is nilpotent and so $I^k \subseteq \sigma(R) = K$ for some positive integer k. This shows that (i) holds and moreover, that I/K is the prime radical of R/K.

Conversely, $\mathrm{rad}_{E(R/I)} = \mathrm{rad}_{C(I)}$ since R/I is a left Goldie ring, and so each element of $C(I)$ is left regular modulo $K = \mathrm{rad}_{C(I)}(R)$. To show this, let $r \in R$ and $c \in C(I)$ and assume that $rc \in K$. Then for each $s \in R$, there exists $c' \in C(I)$ such that $c'src = 0$, so by condition (iv) there exists $c'' \in C(I)$ such that $c''csr = 0$, so that $r \in K$.

To show that R satisfies the left Ore condition with respect to $C(I)$ it suffices to show that R/Rc is $C(I)$-torsion for all $c \in C(I)$. Let $I_0 = K$ and $I_m = \{r \in R \mid I^m r \subseteq K\}$, so that $I_k = R$. If $rc \in I_m$ for $r \in R$ and $c \in C(I)$, then $I^m rc \subseteq K$, and since c is left regular modulo K, $I^m r \subseteq K$ and hence, $r \in I_m$. Thus, each element of $C(I)$ is left regular modulo I_m and so, $I_m \cap I_{m+1}c = I_m c$ for $0 \leqslant m \leqslant k$.

Now for $0 \leqslant m \leqslant k$, $I_{m+1} \supseteq I_m + I_{m+1}c \supseteq I_{m+1}c$ and $I_m + I_{m+1}c/I_{m+1}c \simeq I_m/I_m \cap I_{m+1}c = I_m/I_m c$. Thus, to show that $I_{m+1}/I_{m+1}c$ is σ-torsion, it suffices to show that both $I_{m+1}/I_m + I_{m+1}c$ and $I_m/I_m c$ are σ-torsion, so that R/Rc is σ-torsion if $I_{m+1}/I_m + I_{m+1}c$ is σ-torsion for $0 \leqslant m < k$ ($I_0/I_0 c = K/Kc$ is a factor of $K = \sigma(R)$ and so it must be σ-torsion).

Let $x \in I_{m+1}$, and let $A = \{r \mid rx \in I_m + I_{m+1}c\}$. If A/I is essential in R/I, then since R/I is a semiprime Goldie ring, there exists $c' \in C(I)$ such that

$c' \in A$ and $I_{m+1}/I_m + I_{m+1}c$ will be σ-torsion. Accordingly, to show that A/I is essential in R/I, let $r \in R$ and $r \notin A$. Then $rx \notin I_m$, since $rx \in I_m$ implies that $r \in A$ and so $0 \neq rx + I_m$ and $I_{m+1}c + I_m/I_m$ is essential in I_{m+1}/I_m since c is left regular modulo I_m and R/I_m is, by assumption, finite-dimensional. Thus, there exists $s \in R$ such that $srx \in I_{m+1}c + I_m$, but $srx \notin I_m$. Now $sr \in I$ implies that $I^m srx \subseteq I^{m+1}x \subseteq K$, a contradiction. Thus, $sr \notin I$ and A/I is essential in R/I.

Since it has now been established that R satisfies the left Ore condition with respect to $C(I)$, if $cr \in K$ for $r \in R$, $c \in C(I)$, then there exists $c' \in C(I)$ such that $c'cr = 0$, so $r \in K$. Thus, $C(I) \subseteq C(K)$ and since $I^k \subseteq K$, I/K is the prime radical of R/K, and thus R/K satisfies the hypothesis of Small's theorem as presented in [2]. It follows that $C(I) = C(K)$ and thus, $R_\sigma = Q_{\mathrm{Cl}}(R/K)$ is left Artinian. ∎

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