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# The $\Sigma^2$-conjecture for metabelian groups: the general case

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**Abstract**

The Bieri–Neumann–Strebel invariant $\Sigma^m(G)$ of a group $G$ is a certain subset of a sphere that contains information about finiteness properties of subgroups of $G$. In case of a metabelian group $G$ the set $\Sigma^1(G)$ completely characterizes finite presentability and it is conjectured that it also contains complete information about the higher finiteness properties ($FP_m$-conjecture). The $\Sigma^m$-conjecture states how the higher invariants are obtained from $\Sigma^1(G)$. In this paper we prove the $\Sigma^2$-conjecture.


## Introduction

Let $G$ be a group and $X$ be a $K(G,1)$-complex with finite m-skeleton. A character $\chi : G \to \mathbb{R}$ gives rise to a height function $h : \widetilde{X} \to \mathbb{R}$ on the universal covering of $X$, i.e., $h$ is continuous with $h(gx) = h(x) + \chi(g)$ for all $g \in G$, $x \in \widetilde{X}$. The geometric invariant $\Sigma^m(G)$ consists of the set of equivalence classes of characters for which the positive half $h^{-1}[0,\infty)$ is essentially $(m-1)$-connected, in other words there exists $d < 0$ such that the inclusion $h^{-1}[0,\infty) \to h^{-1}[d,\infty)$ induces the trivial map between the $i$th dimensional homotopy groups for $i \leqslant m-1$. These invariants originated in the work of Bieri and Strebel (1980) on finitely generated metabelian groups $G$ where it was shown that $\Sigma^1(G)$ contains the information as to whether $G$ is finitely presented. In general the $\Sigma$-invariants contain complete information about the finiteness-type of normal subgroups above the commutator subgroup [8]. For the definitions of the homological and homotopical finiteness types $FP_m$ and $F_m$ of groups we refer to [10].

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Although the $\Sigma$-invariants have undergone quite an evolution since 1980 and have been investigated for many different classes of groups, some fundamental open questions remain in the metabelian setting. The $FP_m$-conjecture states that if $G$ is a finitely generated metabelian group then $\Sigma^1(G)$ contains the information as to whether $G$ is of type $FP_m$. The $\Sigma^m$-conjecture says that for such groups $\Sigma^m(G)$ can be obtained from $\Sigma^1(G)$ by a simple process. In this paper we prove the $\Sigma^2$-conjecture. Let us first give precise definitions and statements of the conjectures and our results.

The homological invariants $\Sigma^m(G, M)$ for a finitely generated group $G$ and a $\mathbb{Z}G$-module $M$ were first introduced in [7,8]. By definition

$$\Sigma^m(G, M) = \big\{[\chi] \in S(G) \mid M \text{ is of type } FP_m \text{ over } \mathbb{Z}G_\chi\big\}$$

where $G_\chi = \{g \in G \mid \chi(g) \geqslant 0\}$ and

$$S(G) = \big\{[\chi] = \mathbb{R}_{>0}\chi \mid \chi \in \mathrm{Hom}_{\mathbb{Z}}(G, \mathbb{R}) \setminus \{0\}\big\}$$

is the character sphere of the group $G$. The homotopical version $\Sigma^m(G)$ of $\Sigma^m(G, \mathbb{Z})$ defined for groups $G$ of homotopical type $F_m$ was already given at the beginning of this section. It was first considered by Renz [23] and was later investigated for different classes of groups in [19,21]. In general the homotopical invariant $\Sigma^m(G)$ is a subset of the homological invariant $\Sigma^m(G, \mathbb{Z})$, $\Sigma^1(G) = \Sigma^1(G, \mathbb{Z})$ and there is a Hurewitz type formula

$$\Sigma^m(G) = \Sigma^m(G, \mathbb{Z}) \cap \Sigma^2(G) \quad \text{for } m \geqslant 2.$$

We identify $\mathbb{R}_{>0}\Sigma^1(G)^c$ with $\{\chi \in \mathrm{Hom}(G, \mathbb{R}) \setminus \{0\} \mid [\chi] \in \Sigma^1(G)^c\}$ via the identification of $\Sigma^1(G)^c$ with the unit sphere in $\mathbb{R}^n \simeq \mathrm{Hom}(G, \mathbb{R})$.

**The $\boldsymbol{FP_m}$-conjecture** [3]. *A finitely generated metabelian group $G$ is of type $FP_m$ if and only if*

$$0 \notin \mathrm{conv}_{\leqslant m}\big(\mathbb{R}_{>0}\Sigma^1(G)^c\big)$$

*where the upper index $c$ denotes the complement in $S(G)$ and* $\mathrm{conv}_{\leqslant m} T$ *denotes the convex hull of not more than $m$ elements from $T$.*

By the main result of [9] a metabelian group is finitely presented if and only if it is of type $FP_2$ (this is not so in general [2]). This implies that for metabelian groups the properties $FP_m$ and $F_m$ are the same. In particular the $FP_m$-conjecture suggests a description of the metabelian groups of type $F_m$. For such groups $\Sigma^m(G)$ is conjectured to be determined only by $\Sigma^1(G)$.

**The $\boldsymbol{\Sigma^m}$-conjecture.** *If $G$ is a metabelian group of type $F_m$ then*

$$\mathbb{R}_{>0}\Sigma^m(G)^c = \mathbb{R}_{>0}\Sigma^m(G, \mathbb{Z})^c = \mathrm{conv}_{\leqslant m}\big(\mathbb{R}_{>0}\Sigma^1(G)^c\big).$$

Both conjectures are still open though there is strong evidence that they should hold. The $FP_m$-conjecture is more explored and many cases of it have been proved in the last two decades: $m = 2$ [9], $m = 3$ and $G$ a split extension of abelian groups [5], $G$ of finite Prüfer rank [1], the torsion analogue of Åberg's result [11,16]. In all these cases the proofs have geometric flavour and rely on building $(m-1)$-connected CW-complexes on which $G$ acts cocompactly with polycyclic stabilizers. The 'only if' part of the $FP_m$-conjecture seems to be easier than the 'if' part, it is established in the case when $G$ is an extension of abelian groups $M$ by $Q$ and either the extension is split or $M$ is of finite exponent as abelian group [16,22].

Recently more work was done on the $\Sigma^m$-conjecture. In [19,20] H. Meinert generalises Åberg's approach to show that the $\Sigma^m$-conjecture holds for groups of finite Prüfer rank. An interesting new approach for groups with sufficient commutativity is suggested in [13]. It implies that $\mathrm{conv}_{\leqslant 2}(\mathbb{R}_{>0}\Sigma^1(G)^c) \subseteq \mathbb{R}_{>0}\Sigma^2(G,\mathbb{Z})^c \subseteq \mathbb{R}_{>0}\Sigma^2(G)^c$ for finitely presented abelian-by-nilpotent groups $G$. Recently the inclusion $\mathrm{conv}_{\leqslant 2}(\mathbb{R}_{>0}\Sigma^1(G)^c) \subseteq \mathbb{R}_{>0}\Sigma^2(G)^c$ for finitely presented groups $G$ that do not contain free subgroups of rank two was proved in [18]. In [14] the $\Sigma^m$-conjecture is proved for the class of groups considered in [16]. Until now the case $m = 2$ has been known only for $G$ a split extension of abelian groups [15]. In this paper we establish the $\Sigma^2$-conjecture. It will follow as a corollary from the next result.

**Theorem A.** *Suppose $M \to G \to Q$ is a short exact sequence of groups with $M, Q$ abelian and $G$ finitely presented. If $\chi$ is a real non-trivial character of $G$ such that $\chi \notin \mathrm{conv}_{\leqslant 2}(\mathbb{R}_{>0}\Sigma^1(G)^c)$ then $[\chi] \in \Sigma^2(G)$.*

As already mentioned the inclusion $\mathrm{conv}_{\leqslant 2}(\mathbb{R}_{>0}\Sigma^1(G)^c) \subseteq \mathbb{R}_{>0}\Sigma^2(G)^c$ is proved in [13]. This together with Theorem A implies our main result.

**Corollary B.** *The $\Sigma^2$-conjecture for metabelian groups holds.*

A standard $Q_\chi - K(M,1)$-complex is a $K(M,1)$-complex $X$ with single 0-cell that comes with a $Q_\chi$-action that is free on cells except the 0-cell and makes $\pi_1(X)$ into a $Q_\chi$-module isomorphic to $M$.

**Theorem C.** *Suppose $M \to G \to Q$ is a short exact sequence of groups with $M, Q$ abelian and $G$ finitely presented and $\chi$ is a real character of $G$ with $\chi(M) = 0$. Then the following are equivalent*:

(1) $\chi \notin \mathrm{conv}_{\leqslant 2}(\mathbb{R}_{>0}\Sigma^1(G)^c)$.
(2) *There exists a standard $Q_\chi - K(M,1)$-complex with $Q_\chi$-finite 2-skeleton.*

Theorem C together with the fact that the $\Sigma^2$-conjecture holds implies the following corollary.

**Corollary D.** *Suppose $G$ is a finitely presented group, an extension of $M$ by $Q$ where $M$ and $Q$ are both abelian. Then $[\chi] \in \Sigma^2(G)$ if and only if there exists a standard $Q_\chi - K(M,1)$-complex with $Q_\chi$-finite 2-skeleton.*

## 1. Preliminaries on some geometric properties of $\Sigma$

Throughout this section $Q$ is free abelian group of rank $n$. We view $Q$ as the lattice $\mathbb{Z}^n$ in the euclidean space $\mathbb{R}^n$, $(\ ,\ )$ denotes the scalar product in $\mathbb{R}^n$ and $|\ |$ is the standard norm. $\mathrm{Hom}_{\mathbb{Z}}(Q,\mathbb{R})$ is identified with $\mathbb{R}^n$, where $v \in \mathbb{R}^n$ corresponds to the homomorphism sending $q \in Q = \mathbb{Z}^n$ to $(q,v)$. Under this identification $S(Q)$ corresponds to the unit sphere $S^{n-1}$ in $\mathbb{R}^n$.

In this section we review some geometric properties of $\Sigma$. The first result we quote shows a link between $\Sigma$ for modules and their annihilators. A weaker version of Lemma 1 was one of the core arguments in the proof of the fact that 2-tameness implies finite presentability for metabelian groups [9]. We write $\Sigma_M(Q)$ for $\Sigma^0(Q,M)$ to be consistent with the notations from [9,12] and note that for $G$ a finitely generated group that is an extension of $M$ by $Q$ the projection $G \to Q$ induces a bijection between $\Sigma^1(G)^c$ and $\Sigma_M^c(Q) = S(Q) \setminus \Sigma_M(Q)$, where $M$ is viewed as a left $\mathbb{Z}Q$-module via conjugation.

**Lemma 1** [6]. *Suppose $M$ is a finitely generated $\mathbb{Z}Q$-module. Then there exists a finite subset $\Lambda$ of the centralizer of $M$ in $\mathbb{Z}Q$ and some $\nu > 0$ such that for every $[\mu] \in \Sigma_M(Q)$ there is an element $\lambda$ in $\Lambda$ with*

$$\min\{\mu(q) \mid q \in \operatorname{supp}\lambda\} > \nu.$$

By considering powers of the elements in $\Lambda$ we see that for every $\nu > 0$ there always exists a finite set $\Lambda_\nu$ with the above properties. We continue with a generalization of another geometric lemma from [9].

**Lemma 2.** *For every $\nu > 0$ there is a positive integer $\rho_1(\nu)$ such that for $x \in \mathbb{R}^n$ with $|x| \geqslant \rho_1(\nu)$, $x/|x| \in -\Sigma_M(Q)$ there is $\lambda \in \Lambda_\nu$ such that $x + \operatorname{supp}\lambda$ is a subset of the open ball with centre the origin and radius $|x| - \nu/2$.*

**Proof.** By Lemma 1 there is $\lambda \in \Lambda_\nu$ with the property $\chi(\operatorname{supp}\lambda) > \nu$ for $\chi = -x/|x|$. Let $c$ be the upper bound of the norms of the elements in $\bigcup_{\mu\in\Lambda_\nu}\operatorname{supp}\mu$. Then for $|x| > \max\{\nu/2, (c^2 - \nu^2/4)/\nu\}$ and $q \in \operatorname{supp}\lambda$ we have

$$\begin{aligned}|x+q|^2 &= |x|^2 + |q|^2 - 2|x|(\chi,q) < |x|^2 + c^2 - 2\nu|x| \\ &< |x|^2 - \nu|x| + \frac{\nu^2}{4} = \left(|x| - \frac{\nu}{2}\right)^2.\end{aligned}$$

This completes the proof. □

The next result is a refined version of the obvious observation that if a finite number of points lie in an open half subspace then they can be translated closer to the origin. We restrict to the case when the points are in a cone as we want the translation vector to have integral coordinates.

**Lemma 3** [12, Lemma 3.8]. *Suppose $\varepsilon > 0$ and $\nu \geqslant 0$. There exist positive integers $\rho_2(\varepsilon, \nu) < \rho_3(\varepsilon, \nu)$ with the following property. Suppose $X$ is a finite set in $\mathbb{R}^n$ and $u \in \mathbb{R}^n$ such that for every $x \in X$ of length bigger than $\rho_3(\varepsilon, \nu)$ we have $(u, x) > \varepsilon |u|\,|x|$. Then there exists $v \in \mathbb{Z}^n$ with length smaller than $\rho_2(\varepsilon, \nu)$ and such that for $x \in X$ with $|x| \geqslant \rho_3(\varepsilon, \nu)$*

$$|x| - |v + x| > \nu/2.$$

The geometric structure of $\Sigma^1(G)$ for general finitely generated groups $G$ could be really complicated, for example $\Sigma^1(G)^c$ could have an isolated non-discrete point [7, Section 8]. This cannot happen for metabelian groups where $\Sigma^1(G)^c$ is a rationally defined spherical polyhedron [4] and hence the discrete points in $\Sigma^1(G)^c$ form a dense subset. By definition a rationally defined polyhedron is a finite union $C_1 \cup \cdots \cup C_j$ where every $C_i$ is a finite intersection of affine closed subspaces of $\mathbb{R}^n$, where $n$ is the torsion free rank of the abelianization $G/[G, G]$ and all closed subspaces are given by equations with rational coefficients. The projection of a rationally defined polyhedron to the unit sphere $S^{n-1}$ is a rationally defined spherical polyhedron, i.e., finite union of finite intersections of closed half subspheres, where every subsphere is defined by a rational point in $S^{n-1}$.

**Theorem 4** [4]. *If $G$ is a finitely generated metabelian group $\mathbb{R}_{\geqslant 0}\Sigma^1(G)^c$ is a rationally defined polyhedron, in particular $\Sigma^1(G)^c$ is a rationally defined spherical polyhedron.*

We finish this section with Corollary 5 that is an immediate consequence of the polyhedral structure of $\Sigma^1(G)^c$ and the Bieri–Strebel criterion that a finitely generated metabelian group $G$ is finitely presented if and only if $\Sigma^1(G)^c$ does not contain antipodal points [9].

**Corollary 5.** *Suppose $M \to G \to Q$ is a short exact sequence of groups with $M$, $Q$ abelian and $G$ finitely presented and $\chi$ is a real character of $G$ such that $\chi \notin \operatorname{conv}_{\leqslant 2}(\mathbb{R}_{>0}\Sigma^1(G)^c)$ and $\chi(M) = 0$. Then there exists a positive real number $\varepsilon$ depending on $\chi$ such that for every two elements $x_1, x_2 \in \Sigma^1(G)^c$ there is $u \in \mathbb{R}^n$ (depending on $x_1, x_2, \chi$) with*

$$\frac{(u, v)}{|u|\,|v|} > \varepsilon \quad \textit{for all } v \in \{x_1, x_2, -\chi\}.$$

**Proof.** Since $\operatorname{conv}_{\leqslant 2}(\mathbb{R}_{>0}\Sigma^1(G)^c) \cap \{0, \chi\} = \emptyset$ we have that every triple $x_1, x_2, -\chi$ lies in an open hemisphere of $\operatorname{Hom}(G, \mathbb{R})$. Then there exists $u$ (depending on $x_1, x_2, \chi$) such that $(u, v)/(|u|\,|v|) > 0$ for every $v \in \{x_1, x_2, -\chi\}$. Furthermore we can choose $u$ in such a way that $\min\{(u, v)/(|u|\,|v|) \mid v \in x_1, x_2, -\chi\}$ is as big as possible. Finally the existence of $\varepsilon$ comes from the polyhedral structure of $\Sigma^1(G)^c$ and the fact that $\chi$ is fixed. $\square$

## 2. A special subset $B(m, z_0)$ of $\mathbb{R}^n$

### 2.1. A generating set for $G$

From now on to the end of Section 5 we work on the proof of Theorem A. Assume $\chi$ is a real non-trivial character of $G$ satisfying the assumptions of Theorem A, i.e., $\chi \notin \mathrm{conv}_{\leqslant 2}(\mathbb{R}_{>0}\Sigma^1(G)^c)$. Furthermore without loss of generality we can assume that $\chi$ has length 1. By the main result of [17] whenever $\chi(M) \neq 0$ we have $[\chi] \in \Sigma^\infty(G) = \bigcap_{m\in\mathbb{N}} \Sigma^m(G)$. As our aim is to show that $[\chi] \in \Sigma^2(G)$ we can assume $\chi(M) = 0$.

By [19, Introduction] for a general finitely presented group $G$ the invariant $\Sigma^2(G)$ is invariant under taking subgroups of finite index, i.e., if $H$ is a subgroup of $G$ of finite index then $[\chi] \in \Sigma^2(G)$ if and only if $[\chi|_H] \in \Sigma^2(H)$. Thus we can assume that $Q = G/M$ is free abelian and so $G$ has a generating set

$$X = \{a_1, \dots, a_s, g_1, \dots, g_n\}$$

where $a_1, \dots, a_s$ generate $M$ as a normal subgroup of $G$, $g_1, \dots, g_n$ modulo $M$ is a basis of the free abelian group $Q = G/M$. Furthermore we assume that

$$g g_i^{\varepsilon_i} g_j^{\varepsilon_j} g_i^{-\varepsilon_i} g_j^{-\varepsilon_j} g^{-1} = a_{\alpha(i,j,\varepsilon_i,\varepsilon_j,g)} \in \{a_1, \dots, a_s\}$$

for $1 \leqslant j \neq i \leqslant n$, $\varepsilon_i, \varepsilon_j \in \{\pm 1\}$ and $g$ a word on $g_1^{\pm 1}, \dots, g_n^{\pm 1}$ of length at most $d$, where $d$ is a natural number to be defined in Section 3, just before Proposition 6.

### 2.2. The construction of $B(m, z_0)$

As before we identify $Q$ with the integral lattice $\mathbb{Z}^n$ in $\mathbb{R}^n$ and assume $e_1 = g_1 M, \dots, e_n = g_n M$ is an orthonormal basis of $\mathbb{R}^n$. Define $I(m)$ to be the halfball in $\mathbb{R}^n$ that is the intersection with the closed ball with centre the origin and radius $m$ and the half subspace $\mathbb{R}^n_{\chi \geqslant 0} = \{r \in \mathbb{R}^n \mid \chi(r) \geqslant 0\}$. By definition $B(m, z_0)$ is the union of all closed balls with radius $z_0$ and centre in $I(m)$. For every two points $v, w$ in $\bigcup_{m\geqslant 0} B(m, z_0)$ we define the "distance" $d(v, w)$ to be the smallest non-negative real number $m$ such that for some $q \in Q_\chi = \{q \in Q \mid \chi(q) \geqslant 0\}$ both $v$ and $w$ are in $q + B(m, z_0)$. Loosely speaking the function $d$ will be used as inductive parameter in the main part of the proof of Theorem A, though we will not strictly refer to it.

From now on we fix $\varepsilon$ to be the positive real number given by Corollary 5 and set

$$\nu = \max\bigl\{2\sqrt{n} + 4, 2\rho_2(\varepsilon, 0) + 1\bigr\}, \qquad z_0 = \max\left\{\rho_1(\nu), \rho_3(\varepsilon, \nu), \frac{\nu}{2} + 2\sqrt{n}, \rho_3(\varepsilon, 0)\right\}$$

where $\rho_1$, $\rho_2$ and $\rho_3$ are defined in Lemmas 2 and 3 and write $B(m)$ for $B(m, z_0)$. Furthermore we fix a finite set $\Lambda_\nu$ of the centralizer of $M$ in $\mathbb{Z}Q$ given by Lemma 1 for the already fixed value of $\nu$. It will become clear from the proof of Theorem A why we define $z_0$ and $\nu$ in this way.

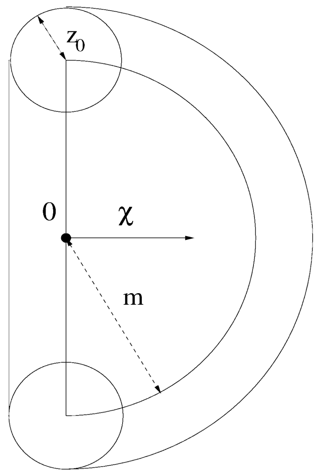


Fig. 1. $B(m, z_0)$.

Though the choice of the set $B(m)$ might look strange it is motivated by the following two properties. The sets $\{B(m)\}_{m\geqslant 0}$ exhaust the affine space $\mathbb{R}^n_{\chi\geqslant -z_0} = \{r \in \mathbb{R}^n \mid \chi(r) \geqslant -z_0\}$, i.e., $\bigcup_{m\geqslant 0} B(m) = \mathbb{R}^n_{\chi\geqslant -z_0}$. And locally $B(m)$ looks like a ball, so locally it is possible to do contractions in all possible directions not only in directions in $\mathbb{R}^n_{\chi\geqslant 0}$.

We say that a non-negative real number $m$ is "special"if there is an integral point on the boundary of $B(m)$. It is easy to see that the set $A_0$ of all special numbers is discrete in $\mathbb{R}$ because every set $B(m)$ contains only finitely many integral points. Thus for every $m \geqslant \alpha_0 = \min\{\alpha \mid \alpha \in A_0\}$ there exist elements $\alpha(m), \beta(m) \in A_0$ such that $\alpha(m) \leqslant m < \beta(m)$ and there is no other element of $A_0$ between $\alpha(m)$ and $\beta(m)$. Then

$$B(m) \cap Q = B\big(\alpha(m)\big) \cap Q.$$

## 3. Free groups and some commutator calculations

Let $F$ be the free group on $\{b_1, \dots, b_s, h_1, \dots, h_n\}$ and

$$\mu : F \to G$$

be the surjective homomorphism sending $b_i$ to $a_i$ and $h_i$ to $g_i$. Let $H$ be the subgroup of $F$ generated by $h_1, \dots, h_n$ and

$$\theta : H \to Q = G/M, \qquad \widetilde{\theta} : H \to G$$

be the homomorphisms sending $h_i$ to $g_i M$ and $h_i$ to $g_i$. By definition

$$\overline{H} = \big\{h_1^{z_1} \dots h_n^{z_n} \mid z_i \in \mathbb{Z}\big\}$$

is the set of ordered words and

$$\left\{h_{\pi(1)}^{z_1} \dots h_{\pi(n)}^{z_n} \mid \pi \in S_n, z_j \in \mathbb{Z}\right\}$$

the set of semiordered words.

Every element $h$ of $H$ can be written in a unique way as an irreducible word $h_{i_1}^{\varepsilon_1} \dots h_{i_j}^{\varepsilon_j}$ where $1 \leqslant i_1, \dots, i_j \leqslant n$ and $\varepsilon_i \in \{-1, 1\}$, i.e., if $i_k = i_{k+1}$ then $\varepsilon_k \neq -\varepsilon_{k+1}$. Every such word corresponds to a path $\gamma(h)$ in the 1-skeleton $\bigcup_{0 \leqslant k \leqslant n-1} \mathbb{Z}^k \times \mathbb{R} \times \mathbb{Z}^{n-k-1}$ of $\mathbb{R}^n$ that starts from the origin and finishes at $\theta(h_{i_1}^{\varepsilon_1} \dots h_{i_j}^{\varepsilon_j})$. More precisely $\gamma(h)$ is the path $\gamma(h_{i_1}^{\varepsilon_1} \dots h_{i_{j-1}}^{\varepsilon_{j-1}})$ followed by the edge with ends $\theta(h_{i_1}^{\varepsilon_1} \dots h_{i_{j-1}}^{\varepsilon_{j-1}})$ and $\theta(h)$.

Now for every choice of a positive real number $m$ and every point $q \in Q \cap B(m)$ we fix an element $w(m, q) \in H$ such that $\theta(w(m, q)) = q$ and $\gamma(w(m, q))$ is a simple path in $B(m)$, i.e., a path that does not intersect itself. Note that when $m_1 < m_2$ the element $w(m_1, q)$ is not necessary the same as $w(m_2, q)$.

Now we impose some strong restrictions on the elements $w(m + \sqrt{n}, q)$ of $H$ that are not necessarily in the proof of Proposition 6 but are really important for the proof of Theorem A. We want to construct $w(m + \sqrt{n}, q)$ so that there exists a natural number $d$ such that for every $m \geqslant 0$ and $q \in B(m + \sqrt{n}) \cap Q$ the beginning of $\gamma(w(m + \sqrt{n}, q))$ that excludes the last $d$ vertices of $\gamma(w(m + \sqrt{n}, q))$ is a simple path inside the union $U$ of all closed balls with centre in $I(m)$ and radius $z_0 - \nu/2$. In particular as $\nu \geqslant 2$ all these beginnings are inside $B(m - 1)$. This is easy to arrange as for every $q \in B(m + \sqrt{n})$ there exists an element $y \in I(m + \sqrt{n})$ such that the closed ball $B$ with centre $y$ and radius $z_0$ is inside $B(m + \sqrt{n})$ and this ball contains $q$. Then we can find an element $q_1$ from $Q$ that is as close as possible to $y$, so $q_1$ is in the union $U$ as $2\sqrt{n} \leqslant z_0 - \nu/2$ implies $q_1 \in U$. Now we can link $q_1$ with $q$ by a simple path $\gamma$ inside the ball $B$ and link the origin with $q_1$ by a simple path inside $U \setminus \gamma$. Now $d$ is the upper bound of the length of a simple path (i.e., without intersections) inside any ball $B$ in $\mathbb{R}^n$ with radius $z_0$. Such an upper bound exists because for any such $B$ there exists an element $\tilde{q}$ from $Q$ with the property that $\tilde{q} + B$ is inside the closed ball $B_1$ in $\mathbb{R}^n$ with centre the origin and radius $z_0 + \sqrt{n}$ and $d$ is not bigger than the upper bound of the lengths of simple paths inside $B_1 \cap$ the 1-skeleton of $\mathbb{R}^n$.

From now on we fix the number $d$ used in Section 2.1 as the number $d$ constructed in the above paragraph.

**Proposition 6.** *Let $h$ be an element of the derived subgroup of $H$ such that $\gamma(h)$ is inside $B(m)$. Then*

$$h = \left({}^{g_1}c_1\right) \dots \left({}^{g_m}c_m\right)$$

*where $\gamma(({}^{g_1}c_1) \dots ({}^{g_m}c_m))$ is inside $\mathbb{R}^n_{\lambda \geqslant -z_0}$, $c_k \in \{h_i^{\varepsilon_i} h_j^{\varepsilon_j} h_i^{-\varepsilon_i} h_j^{-\varepsilon_j} \mid 1 \leqslant i \neq j \leqslant n$, $\varepsilon_i, \varepsilon_j \in \{\pm 1\}\}$ and $g_i \in \{w(m + \sqrt{n}, q) \mid q \in B(m + \sqrt{n}) \cap Q\}$.*

**Proof.** Instead of working with elements of $H$ we will work with the corresponding paths in $\mathbb{R}^n$ and will prove the assertion of the proposition in terms of closed paths attached at $1_G$.

Let $V$ be the union of all standard cubes in $\mathbb{R}^n$ (i.e., $Q$-translates of $[0,1]^n$) that intersect $B(m)$. Then $V$ is simply connected and $\gamma(h)$ is contractible in $V$.

Note that $V$ is obtained from its 1-skeleton by gluing standard cubes $J$ with boundaries some $Q$-translates of $\gamma(c_k)$ for some $k$. These cubes give a contraction of their boundaries to a point. The same could be achieved if we glue to the 1-skeleton of $V$ 2-cells attached at the origin with boundary $\gamma(w(m+\sqrt{n},q))\partial(J)\gamma(w(m+\sqrt{n},q))^{-1}$, where $q$ is a point of the boundary of $J$ that is inside $B(m+\sqrt{n})$ and $\partial(J)$ is the boundary of a 2-cell of $J$ considered as a path attached at the vertex $q$. Note this is exactly the assertion of the proposition stated in terms of paths and so the proof is completed. $\square$

## 4. Building spaces equipped with free and cocompact $G_\chi$-action and with "small" fundamental groups

### 4.1. *The definition of the spaces* $\{W_m\}_{m\geqslant 0}$

In this section we define the spaces $W_m$ and $V_m$ and formulate Theorem 7 that will be the main block in the proof of Theorem A. The spaces $\{W_m\}_{m\geqslant 0}$ could be viewed as approximations to the space we want to build: a 1-connected CW-complex acted on cocompactly and freely by $G_\chi$.

By definition $V_m$ is a 2-dimensional combinatorial complex with vertices $G$, edges $G\times\{b_1^{\pm 1},\ldots,b_s^{\pm 1},h_1^{\pm 1},\ldots,h_{n-1}^{\pm 1},h_n^{\pm 1}\}$ and 2-cells $G\times(R_1\cup R_2\cup R_{3,m})$ for a finite subset $R_1\cup R_2\cup R_{3,m}$ of the free group $F$ defined in Section 3. The group $G$ acts on $V_m$ via left multiplication.

The description of the boundary maps in $V_m$ is as in the Cayley complex associated to a presentation of $G$. The edge $(g,f)$ has label $f$ and vertices $g$ and $g\mu(f)$ and the label of a path is the product in $F$ of the consecutive labels of the edges in the path. The edges $(g,f)$ and $(g\mu(f),f^{-1})$ are identified. If $(g,r)$ is a 2-cell its boundary is the path at $g$ with label $r$. By definition

$$R_1=\bigl\{g\bigl[h_i^{\varepsilon_i},h_j^{\varepsilon_j}\bigr]g^{-1}b^{-1}_{\alpha(i,j,\varepsilon_i,\varepsilon_j,\widetilde{\theta}(g))}\mid 1\leqslant j\neq i\leqslant n,\ \varepsilon_i,\varepsilon_j\in\{\pm 1\}\text{ and}$$
$$g\text{ is an element of }H\text{ such that the length of }\gamma(g)\text{ is at most }d\bigr\}$$

where $\alpha(i,j,\varepsilon_i,\varepsilon_j,\widetilde{\theta}(g))$ was defined in Section 2.1 and by definition $[x,y]=xyx^{-1}y^{-1}$.

The description of the 2-cells $G\times R_2$ is a bit more complicated. Suppose $\lambda\in\Lambda_\nu$ and $g_{1,\lambda},\ldots,g_{m,\lambda}$ are semiordered words in $H$ such that

$$\bigl\{\theta(g_{1,\lambda}),\ldots,\theta(g_{m,\lambda})\bigr\}=\operatorname{supp}\lambda,$$
$$\sum_i z_{i,\lambda}\theta(g_{i,\lambda})=\lambda.$$

By definition $R_2$ is the set of all expressions

$$\bigl({}^{g_{1,\lambda}}b_i\bigr)^{z_{1,\lambda}}\ldots\bigl({}^{g_{m,\lambda}}b_i\bigr)^{z_{m,\lambda}}(b_i)^{-1}$$

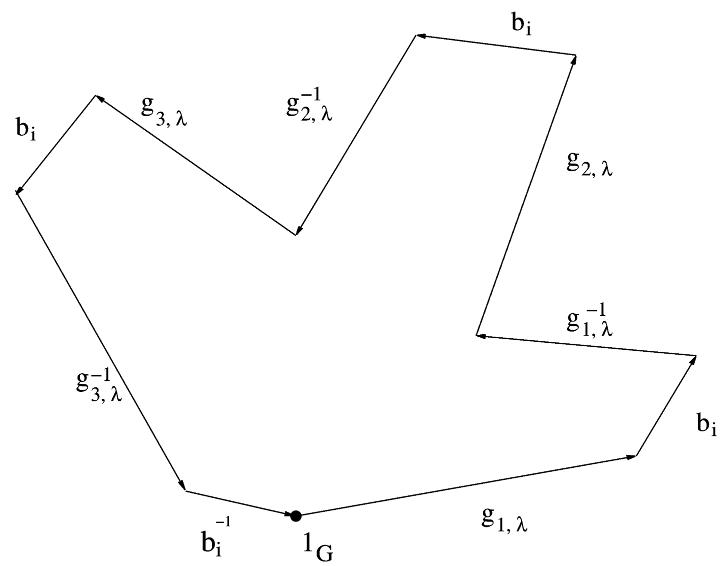


Fig. 2. Relations of the second type.

for all possible $\lambda \in \Lambda_\nu$ and all possible semiordered words $g_{i,\lambda}$ in $H$ with the properties described above and all $i \leqslant s$. The set $R_2$ is finite as $\Lambda_\nu$ is finite and for a fixed $q \in Q$ there are only finitely many semiordered elements $h$ in $H$ with $\theta(h) = q$.

Finally we define cells that are responsible for some of the commutator relations in $M$.

$$R_{3,m} = \bigl\{\bigl[{}^{h'}b_i, {}^{h''}b_j\bigr] \mid h', h'' \in H,\ \gamma(h'), \gamma(h'') \text{ are simple paths in } B(m),\ 1 \leqslant i, j \leqslant s\bigr\}.$$

Now we define $h_m : V_m \to \mathbb{R}$ to be a regular height function associated to the character $\chi$ of $G$, i.e., $h_m$ is a continuous function such that $h_m(gv) = \chi(g) + h_m(v)$ and the restriction of $h_m$ to every cell attains its extremes on the boundary of the cell. In addition we assume that the restriction of $h_m$ on the vertex set $G$ of $V_m$ is the character $\chi$. The subcomplex $W_m$ is defined as the maximal subcomplex in $h_m^{-1}[-z_0, \infty)$. Remember that for $m \geqslant \alpha_0$ we have $B(m) \cap Q = B(\alpha(m)) \cap Q$ and hence $W_m$ is the same as $W_{\alpha(m)}$.

Note that for $m_1 < m_2$ the complex $W_{m_2}$ is obtained from $W_{m_1}$ by gluing on additional 2-dimensional cells and there is a natural map $W_{m_1} \to W_{m_2}$.

The following theorem is the core of the proof of Theorem A and will be discussed in details in Section 4.3.

**Theorem 7.** *For $m \geqslant \nu$ there exists a real positive number $\delta(m)$ such that the homomorphism*

$$\pi_1(W_{m-\delta(m)}) \to \pi_1(W_m)$$

*induced by the natural map $W_{m-\delta(m)} \to W_m$ is an isomorphism.*

Note Theorem 7 is equivalent to the following result: for $h', h'' \in H, \gamma(h'), \gamma(h'')$ simple paths in $B(m)$ and $1 \leqslant i, j \leqslant s$ every path attached at the vertex $1_G$ with label $[{}^{h'}b_i, {}^{h''}b_j]$ is contractible in $W_{m-\delta(m)}$. Since $A_0$ (the set of special numbers defined at the end of Section 2) is a discrete set Theorem 7 implies that for every $\nu \leqslant m_1 < m_2$ the map $\pi_1(W_{m_1}) \to \pi_1(W_{m_2})$ is an isomorphism.

*4.2. Some contractible paths in $W_m$*

**Lemma 8.** (a) *Let $h_1, h_2 \in H, \theta(h_1) = \theta(h_2)$ and $\gamma(h_1)$ and $\gamma(h_2)$ be paths in $B(m)$. Then the path in $W_m$ attached at $1_G$ and with label $({}^{h_1}b_i)({}^{h_2}b_i)^{-1}$ is contractible in $W_m$.*

(b) *Suppose $\gamma$ is a closed path in $W_m$ at $1_G$ with label $f \in F$, $\gamma$ is contractible in $W_m$ and $g$ is an element of $F$ with $\mu(g) \in G_\chi$ and $\gamma(\alpha(g)) \subset \mathbb{R}^n_{\chi \geqslant -z_0}$ where $\alpha : F \to H$ is the homomorphism that is identity map on all $h_i$'s and sends all $b_i$'s to $1_H$. Then the path at $1_G$ with label $gfg^{-1}$ is contractible in $W_m$.*

**Proof.** (a) Note it is sufficient to prove the lemma when $\gamma(h_2)$ is a simple path. Indeed suppose we have proved the case when $\gamma(h_2)$ is a simple path. If in general $\gamma(h_2)$ is not simple consider some element $h$ in $H$ such that $\theta(h) = \theta(h_1)$ and $\gamma(h)$ is a simple path in $B(m)$. Then the paths at $1_G$ with labels $({}^{h_1}b_i)({}^{h}b_i)^{-1}$ and $({}^{h_2}b_i)({}^{h}b_i)^{-1}$ are contractible in $B(m)$ and hence the path at $1_G$ with label $({}^{h_1}b_i)({}^{h_2}b_i)^{-1} = ({}^{h_1}b_i)({}^{h}b_i)^{-1}({}^{h}b_i)({}^{h_2}b_i)^{-1}$ is contractible in $B(m)$ as required. Thus without loss of generality we can restrict to the case when $\gamma(h_2)$ is a simple path in $B(m)$. By Proposition 6 in the free group $F$

$$h_1 = \left({}^{g_1}c_1\right) \dots \left({}^{g_k}c_k\right)h_2$$

and hence

$$\left({}^{h_1}b_i\right)^{-1}\left({}^{g_1}c_1\right) \dots \left({}^{g_k}c_k\right)\left({}^{h_2}b_i\right)\left({}^{g_k}c_k\right)^{-1} \dots \left({}^{g_1}c_1\right)^{-1} = 1$$

where $\gamma(({}^{g_1}c_1) \dots ({}^{g_k}c_k))$ is inside $\mathbb{R}^n_{\lambda \geqslant -z_0}$, $g_i = g_i' g_i''$, $\gamma(g_i'')$ has length at most $d$, $\gamma(g_i')$ is a simple path in $B(m-1)$ and $c_1, \dots, c_k \in \{h_i^{\varepsilon_i} h_j^{\varepsilon_j} h_i^{-\varepsilon_i} h_j^{-\varepsilon_j} \mid 1 \leqslant i \neq j \leqslant n, \varepsilon_i, \varepsilon_j \in \{\pm 1\}\}$. Then the path $\widetilde{\gamma}$ at $1_G$ with label

$$\left({}^{h_1}b_i\right)^{-1}\left({}^{g_1}c_1\right) \dots \left({}^{g_k}c_k\right)\left({}^{h_2}b_i\right)\left({}^{g_k}c_k\right)^{-1} \dots \left({}^{g_1}c_1\right)^{-1}$$

is contractible in every $W_s$. Using the 2-cells of first type $G \times R_1$ we can substitute the ${}^{g_j''}c_j$'s with some $b_k$'s. Using the 2-cells of third type $G \times R_{3,m}$ we see that the paths at $1_G$ and hence at every vertex that is an element in $G_\chi$ with labels $[{}^{g_j}c_j, {}^{h_2}b_i]$ are homotopic in $W_m$. This implies that the path $\widetilde{\gamma}$ is contractible in $W_m$ to the path at $1_G$ with label $({}^{h_1}b_i)^{-1}({}^{h_2}b_i)$.

(b) The path at $1_G$ with label $gfg^{-1}$ is the concatenation of the paths $\gamma_1, \mu(g).\gamma$ and $\gamma_1^{-1}$, where $\mu(g) \in G_\chi$, $\mu(g).\gamma$ is the image of $\gamma$ under the action of $\mu(g)$ and $\gamma_1$ is the path in $W_m$ with label $g$ that starts at $1_G$. Finally, as $\gamma$ is contractible in $W_m$ and $W_m$ is $G_\chi$-invariant, the path $\mu(g).\gamma$ is contractible in $W_m$ too. □

**Proposition 9.** *Suppose $m \geqslant \nu$ and $h', h''$ are elements of $H$, such that $\gamma(h')$ and $\gamma(h'')$ are simple paths in $B(m)$ and $B(m_1)$, respectively, for some $m_1 < m$. Furthermore $\theta(h'')$ is in a closed ball $B$ with centre $y''$ in $I(m)$ and radius $z_0 - \nu/2$. Then there exists $m_2 \in [m-1, m)$ such that the path at $1_G$ with label $[{}^{h'}b_i, {}^{h''}b_j]$ is contractible in $W_{m_2}$.*

**Proof.** I. First we consider the case when there exists an element $q \in Q_\chi$ such that $|q| \leqslant z_0$ and $\theta(h'), \theta(h'') \in q + B(m_0)$ for some $m_0 < m$. As every element in a closed ball could be linked with the centre via a semiordered path there is a semiordered element $v \in H$ with $\theta(v) = q$ and $\gamma(v^{-1}) \subset B(m_0)$. As $\theta(v) = q$ we have that both $\theta(v^{-1}h'), \theta(v^{-1}h'')$ are in $B(m_0)$.

Let $\widetilde{h}', \widetilde{h}''$ be elements in $H$ such that $\theta(\widetilde{h}') = \theta(h')$, $\theta(\widetilde{h}'') = \theta(h'')$ and both paths $\gamma(v^{-1}\widetilde{h}')$ and $\gamma(v^{-1}\widetilde{h}'')$ are in $B(m_0)$. We consider elements $\overline{v^{-1}\widetilde{h}'}$ and $\overline{v^{-1}\widetilde{h}''}$ in $H$ (not necessarily ordered) such that for $h \in \{\widetilde{h}', \widetilde{h}''\}$ the path $\gamma(\overline{v^{-1}h})$ is a simple path in $B(m_0)$ with end $\theta(v^{-1}h)$. Then by Lemma 8(a) the paths ${}^{v^{-1}\widetilde{h}'}b_i({}^{\overline{v^{-1}\widetilde{h}'}}b_i)^{-1}$ and ${}^{v^{-1}\widetilde{h}''}b_j({}^{\overline{v^{-1}\widetilde{h}''}}b_j)^{-1}$ are contractible in $W_{m_0}$ and by the definition of the cells of third type the path at $1_G$ with label $[{}^{\overline{v^{-1}\widetilde{h}'}}b_i, {}^{\overline{v^{-1}\widetilde{h}''}}b_j]$ is contractible in $W_{m_0}$. In particular the path at $1_G$ with label $[{}^{v^{-1}\widetilde{h}'}b_i, {}^{v^{-1}\widetilde{h}''}b_j]$ is contractible in $W_{m_0}$ and since $\theta(v) \in Q_\chi$ by Lemma 8(b) the path at $1_G$ with label

$$\left[{}^{\widetilde{h}'}b_i, {}^{\widetilde{h}''}b_j\right]$$

is contractible in $W_{m_0}$.

Note that by assumption $\theta(h'') \in B(m_1)$ and $\gamma(h'')$ is a simple path in $B(m_1)$. By Lemma 8(a), ${}^{\widetilde{h}''}b_j({}^{h''}b_j)^{-1}$ is contractible in $W_{m_2}$ and so the path at $1_G$ with label

$$\left[{}^{\widetilde{h}'}b_i, {}^{h''}b_j\right] \tag{1}$$

is contractible in $W_{m_2}$, where $m_2 = \max\{m_0, m_1, m-1\}$. Finally by Proposition 6 we have in the free group $F$

$$\widetilde{h}' = \left({}^{g_1}c_1\right) \ldots \left({}^{g_k}c_k\right)h'$$

where $\gamma(({}^{g_1}c_1) \ldots ({}^{g_k}c_k))$ is inside $\mathbb{R}^n_{\lambda \geqslant -z_0}$, $g_i = g_i' g_i''$, $\gamma(g_i'')$ has length at most $d$, $\gamma(g_i')$ is a simple path in $B(m-1)$ and $c_1, \ldots, c_k \in \{h_i^{\varepsilon_i} h_j^{\varepsilon_j} h_i^{-\varepsilon_i} h_j^{-\varepsilon_j} \mid 1 \leqslant i \neq j \leqslant n, \varepsilon_i, \varepsilon_j \in \{\pm 1\}\}$. Thus the path at $1_G$ with label

$${}^{h'}b_i\left({}^{g_k}c_k\right)^{-1} \ldots \left({}^{g_1}c_1\right)^{-1}\left({}^{\widetilde{h}'}b_i\right)^{-1}\left({}^{g_1}c_1\right) \ldots \left({}^{g_k}c_k\right) \tag{2}$$

is contractible in every $W_s$. Using the cells of type 1 we can substitute in the above path the labels ${}^{g_t}c_t$ with ${}^{g_t'}b_{k(t)}$ for some $k(t)$. Using the 2-cells of third type we see that every path at a vertex in $G_\chi$ with label $[{}^{g_t'}b_{k(t)}, {}^{h''}b_j]$ is contractible in $W_{m_2}$ and so every path at

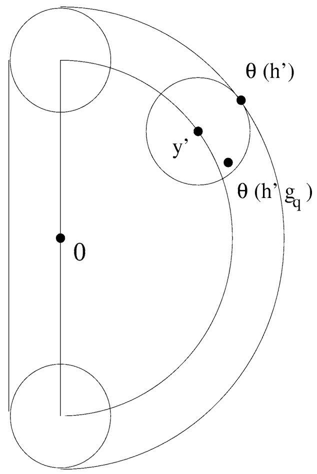


Fig. 3.

$1_G$ and hence at an element in $G_\chi$ with label $[{}^{g_k}c_k, {}^{h''}b_j]$ is contractible in $W_{m_2}$. Then (1) and (2) imply that the path at $1_G$ with label

$$\left[{}^{h'}b_i, {}^{h''}b_j\right]$$

is contractible in $W_{m_2}$, as required.

II. Now we assume that an element $q \in Q$ with the properties stated in the first case does not exist. We choose $y'$ in $I(m)$ so that $\theta(h')$ lies on the boundary of the closed ball of radius $z_0$ and center $y'$. We remind the reader that $z_0 \geqslant \rho_3(\varepsilon, \nu)$ (the latter number is defined in Lemma 3). We will first show that in this case $(y' - \theta(h'))/|y' - \theta(h')|$ is in $\Sigma_M(Q)$. Suppose that this is not so. Then $\chi_1 = (y' - \theta(h'))/|y' - \theta(h')| \in \Sigma_M^c(Q)$ and by Corollary 5 there exists $u$ such that $(u, v)/(|u|\,|v|) > \varepsilon$ for $v \in \{\chi_1, -\chi\}$. Now we apply Lemma 3 for the set $X = \{y' - \theta(h') = z_0\chi_1, -\rho_3(\varepsilon, 0)\chi\}$ to obtain the existence of an element $q \in Q$ such that $|q| \leqslant \rho_2(\varepsilon, 0) \leqslant z_0$, $|q + y' - \theta(h')| < |y' - \theta(h')| \leqslant z_0$ and $|q - \rho_3(\varepsilon, 0)\chi| < |\rho_3(\varepsilon, 0)\chi|$. The latter implies that $q \in Q_\chi$. Furthermore $|-q + \theta(h'') - y''| \leqslant |q| + |\theta(h'') - y''| \leqslant \rho_2(\varepsilon, 0) + z_0 - \nu/2 < z_0$ and so $-q + \theta(h')$ and $-q + \theta(h'')$ are elements of $B(m')$ for some $m' < m$. Then $q \in Q_\chi$ has all the properties required in case I, a contradiction.

Since $(y' - \theta(h'))/|y' - \theta(h')|$ is in $\Sigma_M(Q)$ there is (by Lemma 2) a centralizer $\lambda$ in $\Lambda_\nu$ such that for every $q$ in the support of $\lambda$ we have $\theta(h') + q$ is in the open ball with centre $y'$ and radius $z_0 - \nu/2$. For every $q$ in the support of $\lambda$ choose a semiordered word $g_q \in H$ such that $\theta(g_q) = q$ and $\gamma(h'g_q)$ is the concatenation of $\gamma(h')$ with a path inside the closed ball with centre $y'$ and radius $z_0$. This is possible because every two integral points in a closed ball could be linked with a path inside the ball whose label is a semiordered word.

Using the 2-cells of second type corresponding to $\lambda$ and the semiordered words $g_q$ we see that for $m_2 < m$ the path at $1_G$ with label $[{}^{h'}b_i, {}^{h''}b_j]$ is contractible in $W_{m_2}$ if the paths at $1_G$ with labels $[{}^{h'g_q}b_i, {}^{h''}b_j]$ are contractible in $W_{m_2}$.

We fix $q \in \operatorname{supp} \lambda$ and write $g$ for $g_q$. Let $\overline{h'g}$ be an element from $H$ (not necessarily ordered) such that $\gamma(\overline{h'g})$ is a simple path in $B(m-1)$. By Proposition 6

$$h'g = \left({}^{g_1}c_1\right)\ldots\left({}^{g_k}c_k\right)\overline{h'g} \tag{3}$$

where $\gamma(({}^{g_1}c_1)\ldots({}^{g_k}c_k))$ is inside $\mathbb{R}^n_{\lambda \geqslant -z_0}$, $g_i = g_i' g_i''$ are elements of $H$, $\gamma(g_i'')$ has length at most $d$, $\gamma(g_i')$ is a simple path in $B(m-1)$ and $c_1, \ldots, c_k \in \{h_i^{\varepsilon_i} h_j^{\varepsilon_j} h_i^{-\varepsilon_i} h_j^{-\varepsilon_j} \mid 1 \leqslant i \neq j \leqslant n,\ \varepsilon_i, \varepsilon_j \in \{\pm 1\}\}$. Using the relations of first type every path at an element of $G_\chi$ with label

$$\left[{}^{g_k}c_j, {}^{\overline{h'g}}b_i\right]$$

is homotopic in $W_{m-1}$ to a path with label $[{}^{g_k'}b_{t(j,k)}, {}^{\overline{h'g}}b_i]$ which is contractible in $W_{m-1}$ via the 2-cells of third type. This combined with (3) shows that the path at $1_G$ with label

$${}^{h'g}b_i\left({}^{\overline{h'g}}b_i\right)^{-1}$$

is contractible in $W_{m-1}$. Finally we note that for $m_2 = \max\{m_1, m-1\}$ the path at $1_G$ with label

$$\left[{}^{\overline{h'g}}b_i, {}^{h''}b_j\right]$$

is contractible in $W_{m_2}$ and hence the path at $1_G$ with label $[{}^{h'g}b_i, {}^{h''}b_j]$ is contractible in $W_{m_2}$. $\square$

### 4.3. *Proof of Theorem 7*

We fix elements $h', h''$ in $H$ such that $\gamma(h'), \gamma(h'')$ are simple paths in $B(m)$ and elements $y', y'' \in I(m) \subset \mathbb{R}^n$ such that $|y' - \theta(h')| = z_0$, $|y'' - \theta(h'')| = z_0$. Our aim is to show that there exists $m_2 < m$ such that the path at $1_G$ with label $[{}^{h'}b_i, {}^{h''}b_j]$ is contractible in $W_{m_2}$.

There are three cases to consider:

1. $\dfrac{y' - \theta(h')}{|y' - \theta(h')|} \in \Sigma_M(Q)$.
2. $\dfrac{y'' - \theta(h'')}{|y'' - \theta(h'')|} \in \Sigma_M(Q)$.
3. $\dfrac{y' - \theta(h')}{|y' - \theta(h')|}, \dfrac{y'' - \theta(h'')}{|y'' - \theta(h'')|} \notin \Sigma_M(Q)$.

**Case 1.** The proof of this case is the same as the proof of case II of Proposition 9. Let us sketch it again. By Lemma 2 there is $\lambda$ in $\Lambda_\nu$ such that for every $q$ in the support of $\lambda$ we have $\theta(h') + q$ is in the open ball with centre $y'$ and radius $z_0 - \nu/2$. For every $q$ in the support of $\lambda$ choose a semiordered word $g_q \in H$ such that $\theta(g_q) = q$ and $\gamma(h'g_q)$ is the concatenation of $\gamma(h')$ with a path in the closed ball with centre $y'$ and radius $z_0$. It is sufficient to show that the paths at $1_G$ with labels $[{}^{h'g_q}b_i, {}^{h''}b_j]$ are contractible in $W_{m_2}$ for some $m_2 < m$.

We fix $q \in \operatorname{supp}\lambda$ and write $g$ for $g_q$. Let $\overline{h'g}$ be an element from $H$ such that $\gamma(\overline{h'g})$ is a simple path in $B(m-1)$. By Proposition 6

$$h'g = \left({}^{g_1}c_1\right)\dots\left({}^{g_k}c_k\right)\overline{h'g}$$

where $\gamma(({}^{g_1}c_1)\dots({}^{g_k}c_k))$ is inside $\mathbb{R}^n_{\lambda\geqslant -z_0}$, $g_i = g_i'g_i''$ are elements of $H$, $\gamma(g_i'')$ has length at most $d$, $\gamma(g_i')$ is a simple path in the union $U$ of all closed balls with centres in $I(m)$ and radius $z_0 - \nu/2$ and $c_1, \dots, c_k \in \{h_i^{\varepsilon_i}h_j^{\varepsilon_j}h_i^{-\varepsilon_i}h_j^{-\varepsilon_j} \mid 1 \leqslant i \neq j \leqslant n, \varepsilon_i, \varepsilon_j \in \{\pm 1\}\}$. Note $U \subseteq B(m-1)$. As in the proof of Proposition 9 this shows that the path at $1_G$ with label

$${}^{h'g}b_i\left({}^{\overline{h'g}}b_i\right)^{-1}$$

is contractible in $W_{m-1}$. Using Proposition 9 for $m_1 = m-1$ we see that the path at $1_G$ with label

$$\left[{}^{\overline{h'g}}b_i, {}^{h''}b_j\right]$$

is contractible in $W_{m_2}$ for some $m-1 \leqslant m_2 < m$ and hence the path at $1_G$ with label $[{}^{h'g}b_i, {}^{h''}b_j]$ is contractible in $W_{m_2}$, as required.

**Case 2.** This is the same as Case 1.

**Case 3.** By Corollary 5 for $x_1 = y' - \theta(h')$, $x_2 = y'' - \theta(h'')$ and Lemma 3 for $X = \{-z_0\chi, x_1, x_2\}$ there exists an ordered element $v \in \overline{H}$ such that

$$\left|\theta(v)\right| < \rho_2(\varepsilon, \nu) \leqslant z_0, \qquad \left|\theta(v) + y' - \theta(h')\right| < z_0 - \frac{\nu}{2},$$
$$\left|\theta(v) + y'' - \theta(h'')\right| < z_0 - \frac{\nu}{2}, \qquad \left|-z_0\chi + \theta(v)\right| < |-z_0\chi| \quad \text{and so} \quad \chi\left(\theta(v)\right) > 0$$

Now we repeat a trick used in case I of the proof of Proposition 9. As

$$\theta\left(v^{-1}h'\right) \in B(m-1) \quad \text{and} \quad \gamma\left(v^{-1}\right) \subset B(m-1)$$

there is an element $\widetilde{h}' \in H$ such that $\theta(h') = \theta(\widetilde{h}')$ and $\gamma(v^{-1}\widetilde{h}')$ is a simple path in $B(m-1)$. Similarly there is an element $\widetilde{h}''$ in $H$ with the property that $\theta(h'') = \theta(\widetilde{h}'')$ and $\gamma(v^{-1}\widetilde{h}'')$ is a simple path in $B(m-1)$.

By the definition of $W_{m-1}$ the path at $1_G$ with label $[{}^{v^{-1}\widetilde{h}'}b_i, {}^{v^{-1}\widetilde{h}''}b_j]$ is contractible in $W_{m-1}$. This together with Lemma 8(b) implies that the path at $1_G$ with label

$$\left[{}^{\widetilde{h}'}b_i, {}^{\widetilde{h}''}b_j\right]$$

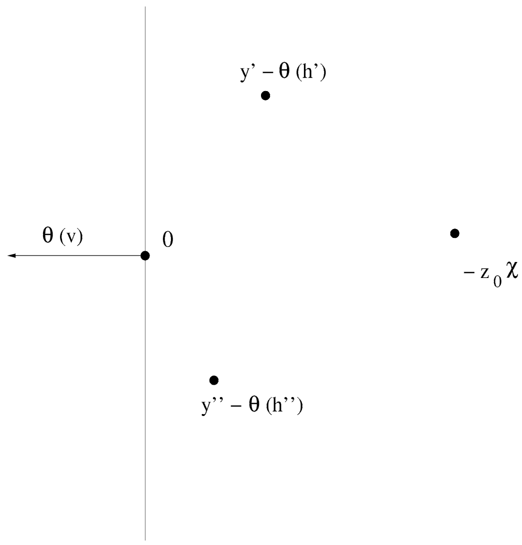


Fig. 4.

is contractible in $W_{m-1}$. Again by Proposition 6

$$\begin{aligned}({}^{g'_1}c'_1)\dots({}^{g'_k}c'_k)h' &= \widetilde{h}',\\ ({}^{g''_1}c''_1)\dots({}^{g''_r}c''_r)h'' &= \widetilde{h}''\end{aligned}$$

where $g'_j, g''_j$ are elements of $H$ such that $\gamma(({}^{g'_1}c'_1)\dots({}^{g'_k}c'_k))$ is inside $\mathbb{R}^n_{\lambda \geqslant -z_0}$, $\gamma(({}^{g''_1}c''_1)\dots({}^{g''_r}c''_r))$ is inside $\mathbb{R}^n_{\lambda \geqslant -z_0}$, $g'_j = \alpha_j\beta_j$, $g''_j = \mu_j\nu_j$, $\gamma(\alpha_j)$ and $\gamma(\mu_j)$ are simple paths in the union $U$ of all closed balls with centre in $I(m)$ and radius $z_0 - \nu/2$, the simple paths $\gamma(\beta_j)$ and $\gamma(\nu_j)$ have length at most $d$ and $c'_i, c''_j \in \{[h_p^{\varepsilon_p}, h_q^{\varepsilon_q}] \mid 1 \leqslant p \neq q \leqslant n;\ \varepsilon_p, \varepsilon_q = \pm 1\}$. Then

$$\begin{aligned}{}^{h'}b_i &= \big(({}^{g'_1}c'_1)\dots({}^{g'_k}c'_k)\big)^{-1}\big({}^{\widetilde{h}'}b_i\big)\big({}^{g'_1}c'_1\big)\dots\big({}^{g'_k}c'_k\big),\\ {}^{h''}b_j &= \big(({}^{g''_1}c''_1)\dots({}^{g''_r}c''_r)\big)^{-1}\big({}^{\widetilde{h}''}b_j\big)\big({}^{g''_1}c''_1\big)\dots\big({}^{g''_r}c''_r\big).\end{aligned}$$

Using the relations of first type we can substitute in the above expressions ${}^{g'_j}c'_j$ with ${}^{\alpha_j}b_t$ for some $t$ and ${}^{g''_j}c''_j$ with ${}^{\mu_j}b_s$ for some $s$. By Proposition 9 there exists $m - 1 \leqslant m_2 < m$ such that every path in $W_{m_2}$ at some element of $G_\chi$ with label $[{}^{\alpha_j}b_t, {}^{\widetilde{h}'}b_i]$ or $[{}^{\mu_j}b_s, {}^{\widetilde{h}''}b_j]$ is contractible in $W_{m_2}$. Thus the paths at $1_G$ with labels ${}^{h'}b_i({}^{\widetilde{h}'}b_i)^{-1}$ and ${}^{h''}b_i({}^{\widetilde{h}''}b_i)^{-1}$ are contractible in $W_{m_2}$. This completes the proof.

## 5. Proof of Theorem A

We first show that we can attach $G_\chi$-finitely many 2-cells $(G_\chi, R_4)$ to $W_\nu$ to obtain a simply connected complex $\widetilde{W}$. For this consider the covering $W_\nu \to M\backslash W_\nu$. Associated with it comes a short exact sequence

$$\pi_1(W_\nu, 1_G) \to \pi_1(M\backslash W_\nu, 1_Q) \to M.$$

Because of the relations $R_1$ the group $\pi_1(M\backslash W_\nu, 1_Q)$ is generated by closed paths with label of the form ${}^h b_i$, where $h \in H$ such that $\gamma(h)$ is a path in $\mathbb{R}^n_{\chi \geqslant -z_0}$. Since closed paths with labels $[{}^{h'} b_i, {}^{h''} b_j]$, where $\gamma(h'), \gamma(h'')$ are paths (not necessarily simple) in $\mathbb{R}^n_{\chi \geqslant -z_0}$ are contractible in $W_\nu$ (by Theorem 7 and Lemma 8(a)) we see that $\pi_1(M\backslash W_\nu, 1_Q)$ is a finitely generated $Q_\chi$-module and the above sequence is an exact sequence of $Q_\chi$-modules. Furthermore as $\pi_1(M\backslash W_\nu, 1_Q)$ is abelian it is isomorphic to the homology group $H_1(M\backslash W_\nu)$. Using the description of $M\backslash W_\nu$ given by vertices $V = M\backslash V(W_\nu)$, edges $E = M\backslash E(W_\nu)$ and 2-cells $C = M\backslash C(W_\nu)$ together with the sequence $0 \to \mathbb{Z}[R] \to \mathbb{Z}[E] \to \mathbb{Z}[V] \to \mathbb{Z} \to 0$ we deduce that $H_1(M\backslash W_\nu)$ is finitely presented over $\mathbb{Z}Q_\chi$. Since $[\chi] \in \Sigma_M(Q)$, the module $M$ is a finitely generated and hence finitely presented $Q_\chi$-module (in fact of type $FP_\infty$ by [15, Lemma 5.1]). Then by dimension shifting argument $\pi_1(W_\nu, 1_G)$ is a finitely generated $Q_\chi$-module, i.e., $G_\chi$-module (with trivial $M$-action) and the result follows. So we can indeed attach $G_\chi$-finitely many 2-cells $(G_\chi, R_4)$ to obtain a simply connected complex $\widetilde{W}$.

We can now quickly finish the proof of Theorem A. Attach to $V_\nu$ $G$ -finitely many 2-cells $(G, R_4)$ to obtain a simply connected complex $\widetilde{V}$. The height function $h_\nu : V_\nu \to \mathbb{R}$ extends in a unique way to a regular $\chi$-equivariant height function $h$ of $\widetilde{V}$ and by construction the maximal subcomplex $\widetilde{V}^{[-z_0,\infty)}$ in $h_\nu^{-1}[-z_0,\infty)$ is $\widetilde{W}$. Note that $\widetilde{V}$ is the Cayley complex of $G$ with respect to the finite presentation $\langle X \mid R_1 \cup R_2 \cup R_{3,\nu} \cup R_4\rangle$ of $G$ and the half subspace $\widetilde{V}^{[-z_0,\infty)}$ is 1-connected. So $[\chi] \in \Sigma^2(G)$ and the proof of Theorem A is completed.

## 6. Proof of Theorem C

### 6.1. The construction of W

The rest of the paper is devoted to the proof of Theorem C. The proof we present follows the main ideas of the proof of Theorem A, in fact it is much simpler as there will be no need to work with simple and non-simple paths in free groups. In this section we assume that the first condition of Theorem C holds and aim to build a space $W$ that will be the 2-skeleton of a standard $Q_\chi - K(M,1)$ complex and $Q_\chi / W$ will be compact.

Before constructing the complex $W$ we construct some approximations $W_m$ of $W$. By definition $W_m$ is a 2-dimensional CW-complex acted on by $Q_\chi$ with 1-vertex and edges the elements of the disjoint union of free $Q_{\chi \geqslant -z_0}$-orbits $\bigcup_{1 \leqslant i \leqslant s} {}^{Q_{\chi \geqslant -z_0}} b_i$. The 2-cells of $W_m$ are the disjoint union of free $Q_\chi$-orbits ${}^{Q_\chi} C_1 \cup {}^{Q_\chi} C_{2,m}$, where $C_1$ and $C_{2,m}$ are finite sets of 2-cells. The definitions of $C_1$ and $C_{2,m}$ resemble the definition of $R_2$ and $R_{3,m}$ in Section 4.1. The boundaries of the cells in $C_1$ are the paths

$$\left({}^{q_0 q_{1,\lambda}}b_i\right)^{z_{1,\lambda}}\dots\left({}^{q_0 q_{m,\lambda}}b_i\right)^{z_{m,\lambda}}\left({}^{q_0}b_i\right)^{-1}$$

where $\lambda \in \Lambda_\nu$, $\lambda = \sum_i z_{i,\lambda} q_{i,\lambda}$ and $q_0$ is an element of $Q$ depending on $\lambda$ with the property that $\beta_{0,\lambda} = \min\{\chi(q_0), \chi(q_0 g_{i,\lambda}) \mid 1 \leqslant i \leqslant m\} \geqslant -z_0$ and $\beta_{0,\lambda}$ is as close to $-z_0$ as possible. Then for every $\lambda \in \Lambda_\nu$, $\lambda = \sum_i z_{i,\lambda} q_{i,\lambda}$ and $\{q\} \cup q(\operatorname{supp}\lambda) \subset Q_{\chi \geqslant -z_0}$ the path with label

$$\left({}^{q q_{1,\lambda}}b_i\right)^{z_{1,\lambda}}\dots\left({}^{q q_{m,\lambda}}b_i\right)^{z_{m,\lambda}}\left({}^{q}b_i\right)^{-1}$$

is contractible via the 2-cells ${}^{Q_\chi}C_1$. The boundaries of the cells in $C_{2,m}$ are paths with labels

$$\left[{}^{q'}b_i, {}^{q''}b_j\right], \quad \text{for } q', q'' \in B(m),\ 1 \leqslant i, j \leqslant s.$$

Similarly to Theorem 7 we have the following result.

**Theorem 12.** *For $m \geqslant \nu$ there exists $\delta(m) > 0$ such that the map*

$$\pi_1(W_{m-\delta(m)}) \to \pi_1(W_m)$$

*is an isomorphism.*

Once Theorem 12 is proved we construct $W$ from $W_\nu$ by gluing finitely many free $Q_\chi$-orbits of 2-cells. Note Theorem 12 follows from the following lemma in the same way as Theorem 7 follows from Proposition 9. In fact in this case the proof is slightly easier as our exponents are in $Q$, not in $H$ and we do not worry about commutators of exponents.

**Lemma 13.** *Suppose $m \geqslant \nu$ and $q', q''$ are elements of $Q$ such that $q' \in B(m)$ and $q''$ is in a closed ball $B$ with centre in $I(m)$ and radius $z_0 - \nu/2$. Then there exists $m - 1 \leqslant m_2 < m$ such that the path at $1_G$ with label $[{}^{q'}b_i, {}^{q''}b_j]$ is contractible in $W_{m_2}$.*

*6.2. A corollary of the existence of W*

Suppose $W$ is the 2-skeleton of a standard $Q_\chi - K(M,1)$-complex such that $Q_\chi$ acts cocompactly on $W$, i.e., the edges and 2-cells of $W$ form disjoint unions of finitely many free $Q_\chi$-orbits, in particular the set of edges is $\bigcup_{1 \leqslant i \leqslant s} {}^{Q_\chi}b_i$.

Assume Theorem C does not hold and $\chi = \chi_1 + \chi_2$ for some $[\chi_i] \in \Sigma^1(G)^c$, in particular $\chi_i(M) = 0$. We split $W$ as a union $W_{\chi_1} \cup W_{\chi_2}$ where $W_{\chi_i}$ is the subcomplex of $W$ containing all cells with edge support in $\bigcup_{1 \leqslant t \leqslant s} {}^{Q_\chi \cap Q_{\chi_i \geqslant c_i}} b_t$. In addition we choose $c_1, c_2$ in such a way that the intersection $W_{\chi_1} \cap W_{\chi_2}$ is sufficiently big so that every cell of $W$ is either in $W_{\chi_1}$ or in $W_{\chi_2}$ (note this is possible because $Q_\chi$ acts cocompactly on $W$).

By van Kampen's theorem $M \simeq \pi_1(W)$ is the push-out of the maps $i_1 : \pi_1(W_{\chi_1} \cap W_{\chi_2}) \to \pi_1(W_{\chi_1})$ and $i_2 : \pi_1(W_{\chi_1} \cap W_{\chi_2}) \to \pi_1(W_{\chi_2})$ induced by the inclusions of the relevant spaces. As $M$ does not contain free subgroups of rank two either the image of $i_j$ has index two in $\pi_1(W_{\chi_j})$ for both $j = 1$ and $j = 2$ or one of the maps $i_1$ and $i_2$ is an epimorphism. The first case can be avoided by changing $c_i$'s. In the second case we can assume that $i_1$ is epimorphism. Then as $\pi_1(W)$ is the push-out of $i_1$ and $i_2$ we get that the inclusion of spaces $W_{\chi_2} \to W$ induces epimorphisms $\pi_1(W_{\chi_2}) \to \pi_1(W) \simeq M$ and $H_1(W_{\chi_2}) \to H_1(W) \simeq M$. As in [9, Lemma 4.7] we see that $H_1(W_{\chi_2})$ is finitely generated over $\mathbb{Z}(Q_\chi \cap Q_{\chi_2})$ and hence $M$ is finitely generated over $\mathbb{Z}Q_{\chi_2}$, so $[\chi_2]$ is in $\Sigma^1(G)$, a contradiction.

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