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Fractional Metric Dimension of Tree and Unicyclic Graph

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Abstract

A vertex v in a simple connected graph G *resolves* two vertices x and y in G if the distance from x to v is not equal to distance from y to v. The vertex set $R\{x,y\}$ is defined as the set of vertices in G which resolve x and y. A function $f : V(G) \to [0,1]$ is called a *resolving function* of G if $f(R\{x,y\}) \geq 1$ for any two distinct vertices x and y in G. The minimal value of $f(V(G))$ for all resolving functions f of G is called the *fractional metric dimension* of G. In this paper, we determine the fractional metric dimension of G where G is a tree or G is a unicyclic graph.

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1. Introduction

Throughout this paper, all graphs G are finite, connected, and simple. We denote by V the vertex set of G and by E the edge set of G. The distance between two vertices $u, v \in V(G)$, denoted by $d(u,v)$, is the length of a shortest path from u to v in G. For $u, v \in V(G)$, we define $R\{u,v\} = \{z \in V(G) \mid d(u,z) \neq d(v,z)\}$. A vertex set $W \subseteq V(G)$ is called a *resolving set* of G if $W \cap R\{u,v\} \neq \emptyset$ for any two distinct vertices $u, v \in V(G)$. The minimum cardinality of all resolving sets of G is called the *metric dimension* of G.

The metric dimension concept has been applied to many various areas including coin weighing problem[8], robot navigation[7], and strategies for the mastermind game[3]. Determining the metric dimension of a graph was formulated as an integer programming problem by Chartrand *et al.*[2]. Furthermore, Currie and Oellermann[4] defined fractional metric dimension as the optimal solution of the linear relaxation of the integer programming problem.

Let f be a function assigning each vertex v of G a real number $f(u) \in [0,1]$. For $W \subseteq V(G)$, denote $f(W) = \sum_{v \in W} f(v)$. The function f is called a *resolving function* of G if $f(R\{u,v\}) \geq 1$ for any two distinct vertices u and v in G. The minimum value of $f(V(G))$ for any resolving function f of G is called the *fractional metric dimension* of G, denoted by $dim_f(G)$.

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The fractional metric dimension problem was initiated by Arumugam and Mathew[1] in 2012. They provided a sufficient condition for a connected graph G whose fractional metric dimension is $\frac{|V(G)|}{2}$. They also determined $dim_f(G)$ where G is Petersen graph, cycles, hypercubes, stars, wheels, friendship graph, and grids.

Some authors applied this topic to some product graphs. Feng *et al.*[5] have determined the fractional metric dimension of Cartesian product of two graphs. In other paper[6], Feng *et al.* also studied the fractional metric dimension of corona and lexicographic product graphs.

In this paper we consider tree and unicyclic graph. Unicyclic graph is a graph obtained from a tree by adding one more edge. Note that unicyclic graph contains one cycle. The study of fractional metric dimension for a tree has been initiated by Arumugam and Mathew[1]. They have proved that $dim_f(P_n) = 1$. They also provided an exact value for the fractional metric dimenison of star and bistar. In this paper, we are interested to determine the fractional metric dimension of any trees. We also investigate the fractional metric dimension of unicyclic graph.

2. Fractional metric dimension of tree

Let G be a tree and v be a vertex of G. A *branch* of G at v is defined as a maximal subtree of G containing v as an end point. So, if degree of v is k, then v has at most k different branches. A branch of v which is isomorphic to a path is called a *leaf* of v. If there is a leaf from v, then v is called a *stem* of G. A stem vertex v is called a *node* if v has more than one leaves. Those definitions are firstly introduced by Slater[9].

For $1 \leq i \leq n$ and $m_i \geq 1$, we define a generalized star graph $S_n(m_1, m_2, \ldots, m_n)$ as a tree graph having $m_1 + m_2 + \ldots + m_n + 1$ vertices and containing a vertex of degree n which has n leaves where the length of the i-th leaf is m_i. Note that if v is a node in a tree graph G, then an induced subgraph of G by v and all of its leaves is isomorphic to a generalized star graph.

Lemma 1. *Let G be a generalized star graph and w be a node in G. Let $u \in V(G)$ where degree of u is 1. If two distinct vertices $x, y \in V(G)$ satisfy $w \in R\{x, y\}$, then $u \in R\{x, y\}$.*

Proof. Suppose that x and y are two distinct vertices in G such that $w \in R\{x, y\}$ but $u \notin R\{x, y\}$. Let L_1 and L_2 are leaves of w which are containing x and y respectively. Since $u \notin R\{x, y\}$, we have $d(x, u) = d(y, u)$. So, $u \notin V(L_1)$ and $u \notin V(L_2)$. Therefore, we obtain $d(x, u) = d(x, w) + d(w, u)$ and $d(y, u) = d(y, w) + d(w, u)$. It follows that $d(x, w) = d(y, w)$ which implies $w \notin R\{x, y\}$, a contradiction. □

Lemma 2. *For $n \geq 3$, let G be a generalized star graph $S_n(m_1, m_2, \cdots, m_n)$. Then $dim_f(G) = \frac{n}{2}$.*

Proof. First, we will prove that $dim_f(G) \leq \frac{n}{2}$. We define the function $f : V(G) \to [0, 1]$ where

$$f(v) = \begin{cases} \frac{1}{2}, & \text{if degree of } v \text{ is } 1, \\ 0, & \text{otherwise.} \end{cases}$$

For any two different vertices $u, v \in V(G)$ and also considering Lemma 1, $R\{u, v\}$ contains two distinct vertices of degree 1. So, we have $f(R\{u, v\}) \geq 1$. Therefore, f is a resolving function of G. Hence, $dim_f(G) \leq f(V(G)) = \frac{n}{2}$.

Now, we will prove $dim_f(G) \geq \frac{n}{2}$. Let g be a minimum resolving function of G. Let x and y be two different vertices in G. Then there exist two different vertices $u, v \in V(G)$ from two different leaves of G such that $u, v \in R\{x, y\}$. Let L_u and L_v be two leaves of G containing u and v respectively. Note that $V(L_u) \cup V(L_v) \subseteq R\{x, y\}$. So, $g(V(L_u) \cup V(L_v)) \geq 1$. It follows that for every leaf L of G, we have $g(V(L)) \geq \frac{1}{2}$. Hence, $dim_f(G) \geq \frac{n}{2}$. □

Using both Lemmas 1 and 2, we can obtain the following theorem.

Theorem 3. *Let G be a tree graph with k nodes $w_1, w_2, \ldots, w_k$. Let $L(w)$ be the number of leaf of node w of G. Then*

$$dim_f(G) = \frac{1}{2} \sum_{i=1}^{k} L(w_i)$$

Proof. To proof Theorem 3 above, we use the following definitions. For $1 \leq i \leq k$, we define S_i as an induced subgraph of G by w_i and all vertices of its leaves. Note that S_i is isomorphic to a generalized star graph. Let $S = V(S_1) \cup V(S_2) \cup \ldots \cup V(S_k)$.

First, we define a function $f : V(G) \to [0, 1]$ as

$$f(v) = \begin{cases} \frac{1}{2}, & \text{if } v \in S \text{ and degree of } v \text{ is } 1, \\ 0, & \text{otherwise.} \end{cases}$$

For any two different vertices $u, v \in V(G)$, $R\{u, v\}$ contains two distinct vertices of S of degree 1. So, we have $f(R\{u, v\}) \geq 1$. Therefore, f is a resolving function of G. Hence, $dim_f(G) \leq f(V(G)) = \frac{1}{2} \sum_{i=1}^{k} L(w_i)$.

Now, let g be a minimal resolving function of G. Since S_i is a generalized star graph for $1 \leq i \leq k$, by Lemma 2 we obtain $dim_f(G) \geq g(S) = \frac{1}{2} \sum_{i=1}^{k} L(w_i)$. □

Remark 4. *Theorem 3 above strengthen Arumugam and Mathew's result of star graph.*

3. Fractional metric dimension of unicyclic graph

In this section, let G be a unicyclic graph. Let R be an induced subgraph of G which is isomorphic to a cycle. A vertex $v \in V(R)$ is called a *root* if degree of v is more than 2. If a root v has degree 3 and $G - v$ has a path component, then v is called a *grass root*. We also consider a vertex set $Q \subseteq V(R)$ which is a set of all vertices of degree at least 3 in R and vertex set $A \subseteq Q$ which is a maximal subset of Q such that for every $u, v \in A$, $d(u, v) \neq \left\lfloor \frac{|V(R)|}{2} \right\rfloor$. For vertex $v \notin V(R)$, we still use the definitions of branch, leaf, stem, and node as stated in Section 2.

Let $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}$ be collections of unicyclic graphs G where:

1. $\mathcal{A}$ is a set of G which is isomorphic to a cycle graph.
2. $\mathcal{B}$ is a set of G having only grass root(s).
   Furthermore, we decompose $\mathcal{B}$ into 5 partitions $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_5$ where:
   (a) $G \in \mathcal{B}_1$ if $G \in \mathcal{B}$ and R of G is isomorphic to C_4.
   (b) $G \in \mathcal{B}_2$ if $G \in \mathcal{B}$ and R of G is an even cycle of order at least 6 and $|A| \leq 3$.
   (c) $G \in \mathcal{B}_3$ if $G \in \mathcal{B}$ and R of G is an even cycle of order at least 6 and $|A| \geq 4$.
   (d) $G \in \mathcal{B}_4$ if $G \in \mathcal{B}$ and R is isomorphic to C_3.
   (e) $G \in \mathcal{B}_5$ if $G \in \mathcal{B}$ and R of G is an odd cycle of order at least 5.
3. $\mathcal{C}$ is a set of G having exactly one root which is not grass root.
4. $\mathcal{D}$ is a set of G having exactly two roots which are not grass root.
   Furthermore, we decompose $\mathcal{D}$ into 2 partitions $\mathcal{D}_1$ and $\mathcal{D}_2$ where:
   (a) $G \in \mathcal{D}_1$ if $G \in \mathcal{D}$, R is isomorphic to even cycle, and for two distinct roots $u, v \in V(G)$, $d(u, v) = \frac{|V(R)|}{2}$.
   (b) $G \in \mathcal{D}_2$ if $G \in \mathcal{D}$ but $G \notin \mathcal{D}_1$.
5. $\mathcal{E}$ is the set of G which is not in class $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, or $\mathcal{D}$.

In theorem below, we give an exact values of fractional metric dimension of a unicyclic graph G in some collections.

Theorem 5. *Let G be a unicyclic graph of order $n \geq 3$ having k nodes $w_1, w_2, \ldots, w_k$. Let $L(w)$ be the number of leaf of node w of G. Let R be an induced subgraph of G which is isomorphic to a cycle. Let $Q \subseteq V(R)$ be a set of all vertices of degree at least 3 in R and $A \subseteq Q$ be a maximal subset of Q such that for every $u, v \in A$, $d(u, v) \neq \left\lfloor \frac{|V(R)|}{2} \right\rfloor$. If $G \notin \mathcal{B}_5$, then*

$$dim_f(G) = \begin{cases} \frac{|V(R)|}{|V(R)|-1}, & \text{if } G \in \mathcal{A} \text{ and } R \text{ is even cycle}, \\ \frac{|V(R)|}{|V(R)|-2}, & \text{if } G \in \mathcal{A} \text{ and } R \text{ is odd cycle}, \\ 2, & \text{if } G \in \mathcal{B}_1, \\ \frac{3}{2}, & \text{if } G \in \mathcal{B}_2 \cup \mathcal{B}_4, \\ \frac{2|A|}{|A|+1}, & \text{if } G \in \mathcal{B}_3, \\ 1 + \frac{1}{2} \sum_{i=1}^{k} L(w_i), & \text{if } G \in \mathcal{C} \cup \mathcal{D}_1, \\ \frac{1}{2} \sum_{i=1}^{k} L(w_i), & \text{if } G \in \mathcal{D}_2 \cup \mathcal{E}. \end{cases}$$

Proof. Note that all unicyclic graphs in $\mathcal{A}$ are isomorphic to cycle graph. The fractional metric dimension of cycle graph has been determined by Arumugam and Mathew[1]. Now, we assume that $G \notin \mathcal{A} \cup \mathcal{B}_5$.

Case 5.1. $G \in \mathcal{B}_1$

First, we define a function $f_1 : V(G) \to [0, 1]$ such that for $v \in V(G)$, we have

$$f_1(v) = \begin{cases} \frac{1}{2}, \text{ if degree of } v \text{ is } 1, \\ \frac{1}{2}, \text{ if degree of } v \text{ is 2 and } v \in V(R), \\ 0, \text{ otherwise.} \end{cases}$$

Let X_1 be vertex set of G containing all vertices of degree 1. Let X_2 be vertex set of R containing all vertices of degree 2. Note that for any two different vertices $u, v \in V(G)$, there exist two distinct vertices $x, y \in X_1 \cup X_2$ such that $x, y \in R\{u, v\}$. So, we have $f_1(R\{u, v\}) \geq 1$. Therefore, f_1 is a resolving function of G. Hence, $dim_f(G) \leq f_1(V(G)) = 2$.

Now, let g_1 be a minimal resolving function of G. Note that there exist two different pair of vertices (a, b) and (c, d) in $V(G)$ such that $R\{a, b\} \cap R\{c, d\} = \emptyset$. It follows that $dim_f(G) \geq g_1(R\{a, b\} \cup R\{c, d\}) \geq 2$.

Case 5.2. $G \in \mathcal{B}_2 \cup \mathcal{B}_3$

We define vertex set X as the set of all vertices of R where for every $x \in X$ there exists $a \in A$ such that $d(x, a) = \frac{|V(R)|}{2}$. If $|A \cup X| \leq 6$, we define vertex set $C \subseteq V(R) \setminus (A \cup X)$ such that $|C| = 6 - |A \cup X|$ and for every $x \in C$ there exists $y \in C$ such that $d(x, y) = \frac{|V(R)|}{2}$. We also define B as the subset of X where degree of every vertex of B is 2. Note that A, B, and C are disjoint. Let V_1 be a subset of $V(G)$ containing all vertices of degree 1, and $S = V_1 \cup B \cup C$.

First, we define a function $f_2 : V(G) \to [0, 1]$ such that for $v \in V(G)$, we have

$$f_2(v) = \begin{cases} \frac{1}{4}, & \text{if } v \in S \text{ and } G \in \mathcal{B}_2, \\ \frac{1}{|A|+1}, & \text{if } v \in S \text{ and } G \in \mathcal{B}_3, \\ 0, & \text{otherwise.} \end{cases}$$

Note that for any two different vertices $u, v \in V(G)$, if $G \in \mathcal{B}_2$, then $|R\{u, v\} \cap S| \geq 4$, otherwise $|R\{u, v\} \cap S| \geq |A| + 1$. So, we have $f_2(R\{u, v\}) \geq 1$. Therefore, f_2 is a resolving function of G. Hence, $dim_f(G) \leq f_2(V(G)) = \frac{3}{2}$ for $G \in \mathcal{B}_2$ and $dim_f(G) \leq f_2(V(G)) = \frac{2|A|}{|A|+1}$ for $G \in \mathcal{B}_3$.

For $1 \leq i \leq |A|$, let $w_i \in A$, x_i and y_i be two different vertices in $V(R)$ such that $d(x_i, w_i) = d(y_i, w_i) = 1$, and $z_i \notin V(R)$ such that $d(w_i, z_i) = 1$. Let $H_i = R\{x_i, z_i\} \cap R\{y_i, z_i\}$. We distinguish two cases.

1. $G \in \mathcal{B}_2$
   Let g_2 be a minimal resolving function of G. Since $V(G) = (R\{x_i, z_i\} \cup R\{y_i, z_i\}) \setminus H_i$, we obtain $g_2(V(G)) + g_2(H_i) \geq 2$. Since $R\{x_i, y_i\} \subseteq V(G) \setminus H_i$, we obtain $g_2(V(G)) - g_2(H_i) \geq 1$. Therefore, we have $dim_f(G) \geq g_2(V(G)) \geq \frac{3}{2}$.
2. $G \in \mathcal{B}_3$
   Suppose that g_2 be a minimal resolving function of G where $g_2(V(G)) < \frac{2|A|}{|A|+1}$. Since $V(G) = (R\{x_i, z_i\} \cup R\{y_i, z_i\}) \setminus H_i$, we obtain $g_2(V(G)) + g_2(H_i) \geq 2$. Since $R\{x_i, y_i\} \subseteq V(G) \setminus H_i$, we obtain $g_2(H_i) \geq \frac{2}{|A|+1}$. Note that $V(G) = V(H_1) \cup V(H_2) \cup \ldots \cup V(H_{|A|})$. Therefore,

$$g_2(V(G)) = \sum_{i=1}^{|A|} g_2(H_i) \geq \frac{2|A|}{|A| + 1}.$$

   So, it is impossible to have a minimal resolving function g_2 of G where $g_2(V(G)) < \frac{2|A|}{|A|+1}$.

Case 5.3. $G \in \mathcal{B}_4$

Let S be a vertex set of G containing all vertices of G of degree 1 and all vertices of R of degree 2. First, we define a function $f_3 : V(G) \to [0, 1]$ such that for $v \in V(G)$, we have

$$f_3(v) = \begin{cases} \frac{1}{2}, \text{ if } v \in S, \\ 0, \text{ otherwise.} \end{cases}$$

Note that for any two different vertices $u, v \in V(G)$, $|R\{u, v\} \cap S| \geq 2$. So, we have $f_3(R\{u, v\}) \geq 1$. Therefore, f_3 is a resolving function of G. Hence, $dim_f(G) \leq f_3(V(G)) = \frac{3}{2}$.

Now, let g_3 be a minimal resolving function of G. Let $V(R) = \{x_1, x_2, x_3\}$ and L_i be a leaf of x_i for $1 \le i \le 3$. Note that for distinct $i, j \in \{1, 2, 3\}$, $V(L_i) \cup V(L_j) \in R\{x_i, x_j\}$. Without loss of generality, let $g_3(V(L_1)) = t$ for $t \in [0, 1]$. So, we obtain $g_3(V(L_2)) = 1 - t$ and $g_3(V(L_3)) = \max\{t, 1 - t\}$. It follows that $t = \frac{1}{2}$. So, we have that $dim_f(G) \ge g_3(V(G)) = \frac{3}{2}$.

Case 5.4. $G \in \mathcal{C} \cup \mathcal{D}_1$

For $1 \le i \le k$, we define a vertex set S_i as the set of all vertices of degree one in leaf of node w_i. Choose a pair of vertices $x, y \in V(R)$ such that $d(x, y) = \left\lfloor \frac{|V(R)|}{2} \right\rfloor$ and for $a \in \{x, y\}$, either degree of a is 2 or a is a grass root of G. We define $S = S_1 \cup S_2 \cup \ldots \cup S_k$ and $T = S \cup \{x, y\}$.

First, we define a function $f_4 : V(G) \to [0, 1]$ such that for $v \in V(G)$, we have

$$f_4(v) = \begin{cases} \frac{1}{2}, \text{ if } v \in T, \\ 0, \text{ otherwise.} \end{cases}$$

Note that for any two different vertices $u, v \in V(G)$, $|R\{u, v\} \cap T| \ge 2$. So, we have $f_4(R\{u, v\}) \ge 1$. Therefore, f_4 is a resolving function of G. Hence, $dim_f(G) \le f_4(V(G)) = 1 + \frac{1}{2}\sum_{i=1}^{k} L(w_i)$.

For the lower bound, let g_4 be a minimal resolving function of G. For $1 \le i \le k$, let F_i be an induced subgraphs of G by S_i. Note that F_i is a generalized star graph. By Lemma 2, we obtain $dim_f(F_i) = \frac{L(w_i)}{2}$. So, we can say that $dim_f(S) = \frac{1}{2}\sum_{i=1}^{k} L(w_i)$. However, we also always can find a pair of vertices $u, v \in V(G)$ such that $R\{u, v\} \cap S = \emptyset$. Therefore, we obtain that $dim_f(G) \ge g_4(V(G)) \ge 1 + \frac{1}{2}\sum_{i=1}^{k} L(w_i)$.

Case 5.5. $G \in \mathcal{D}_1 \cup \mathcal{E}$

For $1 \le i \le k$, we define a vertex set S_i as the set of all vertices of degree one in leaf of node w_i. We define $S = S_1 \cup S_2 \cup \ldots \cup S_k$. First, we define a function $f_5 : V(G) \to [0, 1]$ such that for $v \in V(G)$, we have

$$f_5(v) = \begin{cases} \frac{1}{2}, \text{ if } v \in S, \\ 0, \text{ otherwise.} \end{cases}$$

Note that for any two different vertices $u, v \in V(G)$, $|R\{u, v\} \cap S| \ge 2$. So, we have $f_5(R\{u, v\}) \ge 1$. Therefore, f_5 is a resolving function of G. Hence, $dim_f(G) \le f_5(V(G)) = \frac{1}{2}\sum_{i=1}^{k} L(w_i)$.

For the lower bound, let g_5 be a minimal resolving function of G. For $1 \le i \le k$, let F_i be an induced subgraphs of G by S_i. Note that F_i is a generalized star graph. By Lemma 2, we obtain $dim_f(F_i) = \frac{L(w_i)}{2}$. So, we can say that $dim_f(S) = \frac{1}{2}\sum_{i=1}^{k} L(w_i)$. Therefore, we obtain that $dim_f(G) \ge g_5(V(G)) \ge \frac{1}{2}\sum_{i=1}^{k} L(w_i)$. □

In Theorem 6, we give the upper bound of $dim_f(G)$ for $G \in \mathcal{B}_5$. We also show that the bound is sharp. In Theorem 7, we provide an existence of unicyclic graph $G \in \mathcal{B}_5$ such that $dim_f(G)$ is not equal to the upper bound in Theorem 6.

Theorem 6. *Let G be a unicyclic graph in class $\mathcal{B}_5$. Let R be an induced subgraph of G which is isomorphic to cycle graph. Then*

$$dim_f(G) \le \frac{2|V(R)|}{|V(R)| + 3}.$$

The bounds is sharp.

Proof. Let $S = \{v \in V(G) | \text{degree of } v \text{ is } 1\} \cup \{v \in V(R) | \text{degree of } v \text{ is } 2\}$. We define a function $f : V(G) \to [0, 1]$ such that for $v \in V(G)$, we have

$$f(v) = \begin{cases} \frac{2}{|V(R)|+3}, & \text{if } v \in S, \\ 0, & \text{otherwise.} \end{cases}$$

Note that for any two different vertices $u, v \in V(G)$, $|R\{u, v\} \cap S| \ge \frac{|V(R)|+3}{2}$. So, we have $f(R\{u, v\}) \ge 1$. Therefore, f is a resolving function of G. Hence, $dim_f(G) \le f(V(G)) = \frac{2|V(R)|}{|V(R)|+3}$.

Now, we consider a unicyclic G which is obtained from C_n where $n \ge 5$ is odd and every vertex of C_n has one leaf. We will prove that $dim_f(G) = \frac{2|V(R)|}{|V(R)|+3}$. Now, we need to show that $dim_f(G) \le \frac{2|V(R)|}{|V(R)|+3}$.

Let g be a resolving function of G. Let $V(R) = \{x_0, x_1, \ldots, x_{n-1}\}$ where $R = x_0x_1 \ldots x_{n-1}x_0$. For $0 \le i \le n-1$, let $y_i \notin V(R)$ but $y_ix_i \in E(G)$ and $L(x_i)$ be a leaf of x_i. Note that $R\{x_{i \mod n}, y_{(i-1) \mod n}\} = V(G) \setminus (V(L(x_{(\frac{n+1}{2}+i) \mod n})) \cup \ldots \cup V(L(x_{(n-1+i) \mod n})) \cup \{x_{i-1}\})$. Since $g(R\{a,b\}) \ge 1$ for every two distinct vertices $a, b \in V(G)$, we obtain that

$$\sum_{i=0}^{n-1} g(R\{x_{i \mod n}, y_{(i-1) \mod n}\}) = n \cdot g(V(G)) - \frac{n-3}{2} \cdot g(V(G)) - \sum_{i=0}^{n-1} g(x_i) \ge n$$

Since $g(x_i) \ge 0$ for $1 \le i \le n-1$, we will obtain that $g(V(G)) \ge \frac{2n}{n+3}$. □

Theorem 7. *There exists a unicyclic graph $G \in \mathcal{B}_5$ such that for an induced subgraph R of G which is isomorphic to cycle graph, $dim_f(G) < \frac{2|V(R)|}{|V(R)|+3}$.*

Proof. Let G be a unicyclic graph having an odd cycle R with $|V(R)| \ge 7$ and 3 grass roots such that 2 grass roots $a, b \in V(G)$ are adjacent and another grass root $c \in V(G)$ satisfies $d(c,a) = d(c,b) = \frac{|V(R)|-1}{2}$. We will prove that $dim_f(G) = \frac{5}{4}$. Note that for $|V(R)| \ge 7$, we have

$$\frac{5}{4} = \frac{5(|V(R)|+3)}{4(|V(R)|+3)} < \frac{8|V(R)|}{4(|V(R)|+3)} = \frac{2|V(R)|}{|V(R)|+3}.$$

First, let $S = \{u \in V(G)|$ degree of u is $1\} \cup \{v \in V(R)|vc \in E(G)\}$. Now, we define a function $f : V(G) \to [0,1]$ such that for $v \in V(G)$, we have

$$f(v) = \begin{cases} \frac{1}{4}, & \text{if } v \in S, \\ 0, & \text{otherwise.} \end{cases}$$

Note that for any two different vertices $u, v \in V(G)$, $|R\{u,v\} \cap S| \ge 4$. So, we have $f(R\{u,v\}) \ge 1$. Therefore, f is a resolving function of G. Hence, $dim_f(G) \le f(V(G)) = \frac{5}{4}$.

For the lower bound, suppose that $dim_f(G) < \frac{5}{4}$. Let g be a minimal resolving function of G such that $g(V(G)) < \frac{5}{4}$. Let $y_a, y_b, y_c \notin V(R)$ such that for $t \in \{a,b,c\}$, $ty_t \in E(G)$. Let $v_1, v_2 \in V(R)$ be two distinct vertices which are adjacent to c and L_w be a leaf of node w in R.

Since $R\{v_1, v_2\} = V(G) \setminus V(L_c)$, we obtain $g(V(L_c)) = \frac{1}{4} - k$ for positive $k \in \mathbb{R}$. Now, we consider $R\{y_a, b\}$ and $R\{y_b, a\}$. We can say that $R\{y_a, b\} = V(L_c) \cup X$ and $R\{y_b, a\} = V(L_c) \cup Y$. So, we obtain that $g(X) = \frac{3}{4} + k = g(Y)$. Note that $X \cap Y = \emptyset$ and $R\{v_1, v_2\} = V(G) \setminus V(L_c) = X \cup Y$. It follows that $g(V(G)) = g(V(L_c)) + g(X) + g(y) = \frac{7}{4} + k$, a contradiction. □

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