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Eigenfunctions of transfer operators and cohomology

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ABSTRACT

The eigenfunctions with eigenvalues 1 or −1 of the transfer operator of Mayer are in bijective correspondence with the eigenfunctions with eigenvalue 1 of a transfer operator connected to the nearest integer continued fraction algorithm. This is shown by relating these eigenspaces of these operators to cohomology groups for the modular group with coefficients in certain principal series representations.

1. Introduction

D. Mayer [7], defined the operator

$$\mathcal{L}_s^{\mathrm{Ma}} f(z) = \sum_{n=1}^{\infty} (z+n)^{-2s} f\left(\frac{1}{z+n}\right) \tag{1}$$

on the Banach space of continuous functions on the disk $|z-1| \leqslant \frac{3}{2}$, holomorphic on $|z-1| < \frac{3}{2}$, with the supremum norm. The series converges absolutely if $\mathrm{Re}(s) > \frac{1}{2}$. There is a meromorphic con-

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tinuation in s, with a pole at $\frac{1}{2}$ as the sole singularity in the region $\mathrm{Re}(s) > 0$. The operator $\mathcal{L}_s^{\mathrm{Ma}}$ is a transfer operator for the Artin billiard dynamical system [1]. It is connected to the Gauss map $x \mapsto \frac{1}{x} - [\frac{1}{x}]$ on $[0,1]$. Ultimately, this dynamical system comes from closed billiard flows on the quotient of the upper half plane by PGL(2,)Z. (Here $\pm\left(\begin{smallmatrix}-1 & 0\\ 0 & 1\end{smallmatrix}\right)$ acts by $z \mapsto -\bar{z}$.) An introductory lecture on Mayer's transfer operator is [6].

The DFG research project *Transfer operators and nonarithmetic quantum chaos* (Ma 633/16-1) involves finding transfer operators connected to the dynamical systems of closed geodesic flows on the hyperbolic surfaces represented by the quotient of the upper half plane by arbitrary Hecke triangle groups. For the modular group, it leads to another transfer operator $\tilde{\mathcal{L}}_s$. This operator acts in the space of vectors of two holomorphic functions on the open unit disk which are continuous on the closed unit disk. With the supremum norm these vectors form a Banach space. The operator is given by

$$\tilde{\mathcal{L}}_s \vec{f} = \begin{pmatrix} \tilde{\mathcal{L}}_s^{1,1} & \tilde{\mathcal{L}}_s^{2,1} \\ \tilde{\mathcal{L}}_s^{1,2} & \tilde{\mathcal{L}}_s^{2,2} \end{pmatrix} \vec{f} \quad \text{with}$$

$$\tilde{\mathcal{L}}_s^{1,1} f_1(z) = \sum_{n=3}^{\infty} (z+n)^{-2s} f_1\left(\frac{-1}{z+n}\right),$$

$$\tilde{\mathcal{L}}_s^{1,2} f_2(z) = \sum_{n=2}^{\infty} (n-z)^{-2s} f_2\left(\frac{1}{-z+n}\right),$$

$$\tilde{\mathcal{L}}_s^{2,1} f_1(z) = \sum_{n=2}^{\infty} (z+n)^{-2s} f_1\left(\frac{-1}{z+n}\right), \quad \text{and}$$

$$\tilde{\mathcal{L}}_s^{2,2} f_2(z) = \sum_{n=3}^{\infty} (n-z)^{-2s} f_2\left(\frac{1}{-z+n}\right). \tag{2}$$

This converges absolutely for $\mathrm{Re}(s) > \frac{1}{2}$, and has a meromorphic continuation in s with a first order pole at $s = \frac{1}{2}$ as sole singularity in the region $\mathrm{Re}(s) > 0$.

Our main result is:

Theorem 1.1. *Let $s \in \mathbb{C}$, $0 < \mathrm{Re}(s) < 1$, $s \neq \frac{1}{2}$. There is a bijective correspondence between the spaces* $\ker(\mathcal{L}_s^{\mathrm{Ma}} - 1) \oplus \ker(\mathcal{L}_s^{\mathrm{Ma}} + 1)$ *and* $\ker(\tilde{\mathcal{L}}_s - 1)$.

We will establish this correspondence by a number of steps. The main steps are indicated in Proposition 2.1, Theorem 2.3, Proposition 2.4 and Theorem 2.5.

The eigenfunctions of both transfer operators satisfy finite linear identities. Lewis and Zagier [5, Proposition in Section 3, Chapter IV] show that if $\mathcal{L}_s^{\mathrm{Ma}} f = \pm f$, then $P(z) = f(z-1)$ satisfies the three term equation

$$P(z) = P(z+1) + (z+1)^{-2s} P\left(\frac{z}{z+1}\right) \tag{3}$$

and the parity condition

$$z^{-2s} P(1/z) = \pm P(z). \tag{4}$$

These functions extend holomorphically to $\mathbb{C}' = \mathbb{C} \setminus (-\infty, 0]$. So $\ker(\mathcal{L}_s^{\mathrm{Ma}} - 1) \oplus \ker(\mathcal{L}_s^{\mathrm{Ma}} + 1)$ corresponds to a subspace of the space of all holomorphic solutions of (3) on $\mathbb{C}'$. This subspace is characterized by the asymptotic behavior $P(x) = c_\infty x^{1-2s} + O(x^{-2\,\mathrm{Re}(s)})$ as $x \uparrow \infty$, and $P(x) = c_0 x^{-1} + O(1)$ as $x \downarrow 0$.

For both transfer operators, we relate, in Section 3, the eigenfunctions on disks to eigenfunctions in the real analytic functions on an interval. This allows a cohomological interpretation, to be discussed in Sections 3.2 and 4. We will show that solutions of $\tilde{\mathcal{L}}_s \vec{f} = \vec{f}$ correspond to solutions of the four term equation

$$g(z) + (z+2)^{-2s} g\left(\frac{-1}{z+2}\right) = g(z-1) + (2-z)^{-2s} g\left(\frac{1-z}{z-2}\right), \tag{5}$$

on a suitable domain containing $(-1, 1)$.

Not all solutions of (3), respectively (5), correspond to eigenfunctions of $\mathcal{L}_s^{\mathrm{Ma}}$, respectively eigenfunctions of $\tilde{\mathcal{L}}_s$. We will see in Theorem 2.3 and Proposition 2.4 that the space of all real analytic solutions of (5) on the interval

$$\left(\frac{-3-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right)$$

is isomorphic to the first cohomology group of the modular group with coefficients in the principal series representation with spectral parameter s. Theorem 2.5 shows that the eigenspace of $\tilde{\mathcal{L}}_s$ for the eigenvalue 1 corresponds to a well defined subspace of this cohomology group. This same subspace is also isomorphic to the sum of the eigenspaces of $\mathcal{L}_s^{\mathrm{Ma}}$ for the eigenvalues 1 and -1. This can be shown by methods in [5] and [2].

We will take care to indicate the various steps in the correspondence between eigenspaces of $\mathcal{L}_s^{\mathrm{Ma}}$ and $\tilde{\mathcal{L}}_s$ as explicitly as possible, even for steps where we might refer to [5] or [2]. The least explicit step is an application of Proposition 3.5, where a function with two singularities is written as a difference of two functions which each have a singularity in only one point. Fig. 1 at the end of Section 2.5 gives a detailed overview of the steps by which we prove Theorem 1.1.

As background information, we discuss in Section 1.1 how $\tilde{\mathcal{L}}_s$ arises from the nearest integer continued fraction transformation. Our proof of the correspondence does not use that both transfer operators arise from the geodesic flow on related quotients of the upper half plane. It would be interesting to go directly from the geodesic flow to the relevant cohomology groups.

The Selberg trace formula relates the recurrent points of the geodesic flow to spectral data. So both transfer operators have a relation to Maass forms.

Our cohomological approach to the correspondence is based on ideas in [5] and [2]. The leading idea in [2] is the relation between certain cohomology groups and spaces of Maass forms, which we discuss in Section 2.6. This relates eigenfunctions of $\mathcal{L}_s^{\mathrm{Ma}}$ and $\tilde{\mathcal{L}}_s$ to Maass forms without use of the Selberg trace formula.

1.1. The transfer operator for the nearest integer continuous fraction algorithm

Although the origin of $\tilde{\mathcal{L}}_s$ from a dynamical system is not used in this paper, it seems right to explain why $\tilde{\mathcal{L}}_s$ deserves to be called a *transfer operator*. It is derived by the Ruelle transfer operator method applied to a dynamical system based on the nearest integer map and associated continued fractions. Such nearest integer continued fractions have already been discussed in [4].

Consider the interval map

$$f_3 : I_3 \to I_3; \qquad x \mapsto \frac{-1}{x} - \left\lfloor \frac{-1}{x} + \frac{1}{2} \right\rfloor \tag{6}$$

with $I_3 = [-\frac{1}{2}, \frac{1}{2}]$ and $\lfloor x \rfloor$ the element $n \in \mathbb{Z}$ such that $n \leqslant x < n+1$ if $x > 0$, and $n < x \leqslant n+1$ if $x \leqslant 0$. The function f_3 is closely related to the nearest integer continuous fractions. (We keep the subscript 3 since it is a specialization of an interval map associated to Hecke triangle groups.) Basically

f_3 acts as the "left-shift" on the space of configurations $(a_1, a_2, \ldots)$ for the nearest integer continued fraction expansion

$$[0; a_1, a_2, \ldots] := \frac{-1}{a_1 + \frac{-1}{a_2 + \frac{-1}{\cdots}}} \in I_3.$$

The map f_3 generates a discrete dynamical system of finite type. The transfer operator associated to f_3 is

$$T_s f(x) = \sum_{y \in f_3^{-1}(x)} \left| \frac{\mathrm{d} f_3^{-1}(x)}{\mathrm{d}x} \right|^s f(y)$$

defined on a suitable function space. The expression $\frac{\mathrm{d} f_3^{-1}(x)}{\mathrm{d}x}$ denotes the derivative of the appropriate invertible branch of f_3^{-1} in x. We find $\tilde{\mathcal{L}}_s = T_s$ on the Banach space on which $\tilde{\mathcal{L}}_s$ is defined. Forthcoming work in the project Ma633/16-1 of the Deutsche Forschungsgemeinschaft will give more details.

The dynamical system is related to the geodesic flow on the hyperbolic surface $\mathrm{PSL}(2, \mathbb{Z}) \setminus \mathbb{H}$. One can show that the Fredholm-determinant $\det(1 - \tilde{\mathcal{L}}_s)$ is essentially equal to the Selberg zeta-function $Z(s)$ for the full modular group Γ. Here, essentially equal means that

$$Z(s) = \frac{\det(1 - \tilde{\mathcal{L}}_s)}{\det(1 - \mathcal{K}_s)} \tag{7}$$

with $\mathcal{K}_s$ defined on the same space of pairs of function by

$$\mathcal{K}_s \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} g_1|_{2s} S T^3 \\ g_1|_{2s} S T^3 \end{pmatrix}, \tag{8}$$

where $g_1|_{2s} S T^3(x) = (x + 3)^{-2s} g(\frac{-1}{x+3})$. It is known that $\mathcal{K}_s$ only admits the eigenvalue 1 if $s \in \{-n + \pi i k/\alpha;\ n \in \mathbb{Z}_{\geqslant 0},\ k \in \mathbb{Z}\}$, with $\alpha = 2 \log \frac{1+\sqrt{5}}{2}$. These values of s are not in the domain under consideration in this note.

The zeros of $Z(s)$ on the line $\mathrm{Re}(s) = \frac{1}{2}$ correspond to eigenvalues $s(1-s)$ of the hyperbolic Laplace operator. Hence it is not surprising that eigenfunctions of $\tilde{\mathcal{L}}_s$ with eigenvalue 1 can be related to cohomology classes that are themselves related to Maass forms. See Section 2.6.

The advantages of $\tilde{\mathcal{L}}_s$ compared to Mayer's transfer operator $\mathcal{L}_s^{\mathrm{Ma}}$ are that its construction is directly related to the geodesic flow and that the same construction works for all Hecke triangle groups. The disadvantage of $\tilde{\mathcal{L}}_s$ compared to $\mathcal{L}_s^{\mathrm{Ma}}$ is its more complicated structure.

2. Definitions and results

In this preliminary section we define or recall various concepts to be used in this paper. Among them are the principal series representations in Section 2.2, and the definition of parabolic cohomology groups in Section 2.3. We state in Sections 2.4 and 2.5 the results implying the main result Theorem 1.1. Finally we give in Section 2.6 background information on the relation between certain spaces of Maass forms and cohomology groups.

2.1. Modular group

We use $\left[\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right]$ to denote $\{\left(\begin{smallmatrix} ta & tc \\ tb & td \end{smallmatrix}\right):\ t \neq 0\}$ in

$$\mathrm{PGL}(2,\mathbb{R}) = \mathrm{GL}(2,\mathbb{R})\Big/\left\{\begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}:\ t \in \mathbb{R}\setminus\{0\}\right\}.$$

We work with the full modular group $\Gamma = \mathrm{PSL}(2,\mathbb{Z})$, which is discrete in $\mathrm{PSL}(2,\mathbb{R})$, and is generated by $S = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$ and $T = \left[\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right]$, with the relations

$$S^2 = (ST)^3 = 1. \tag{9}$$

We denote also

$$T' := TST = ST^{-1}S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in \Gamma \quad \text{and} \quad C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in \mathrm{PGL}(2,\mathbb{Z}). \tag{10}$$

We have $\mathrm{PGL}(2,\mathbb{Z}) = \Gamma \cup C\Gamma$.

2.2. Principal series representations

We describe the standard realization of the principal series representations of the Lie group $\mathrm{PGL}(2,\mathbb{R})$ in the functions on $\mathbb{R}$.

The group $\mathrm{PGL}(2,\mathbb{R})$ has a family of actions, parametrized by $s \in \mathbb{C}$, on functions defined on a subset of $\mathbb{R}$, given by

$$h|_{2s}M(x) = |ad-bc|^s|cx+d|^{-2s}h\left(\frac{ax+b}{cx+d}\right). \tag{11}$$

This is a right action; the natural place for the symbol $|_{2s}$ is after the function. We call s the *spectral parameter*.

For each value of s, this action preserves the spaces $\mathcal{V}_s^{\omega}$ and $\mathcal{V}_s^{\infty}$ of real-analytic and smooth vectors in the discrete series representation with spectral parameter s. The space $\mathcal{V}_s^{\omega}$ consists of the $h:\mathbb{R}\to\mathbb{C}$ that are real-analytic on $\mathbb{R}$ and for which $x \mapsto |x|^{-2s}h(-1/x)$ extends as a real-analytic function on a neighborhood of 0. Replace 'real-analytic' by 'smooth' to obtain the characterization of $h \in \mathcal{V}_s^{\infty}$.

We refer to the first chapter of [2] for a discussion of other models of the principal series representations. Here it suffices to note that elements of $\mathcal{V}_s^{\omega}$ and $\mathcal{V}_s^{\infty}$ can be viewed as functions on the projective line $\mathbb{P}^1_{\mathbb{R}} = \mathbb{R}\cup\{\infty\}$, and that the required behavior of

$$x \mapsto (h|_{2s}S)(x) = |x|^{-2s}h(-1/x) \tag{12}$$

may be viewed as the description of analyticity or smoothness at ∞.

The real-analytic functions in $\mathcal{V}_s^{\omega}$ are the restriction of a holomorphic function on some neighborhood of $\mathbb{R}$, that depends on the functions. On such holomorphic functions the slash operator takes the form

$$h|_{2s}M(z) := |ad-bc|^s\big((cz+d)^2\big)^{-s}h\left(\frac{az+b}{cz+d}\right). \tag{13}$$

The factor $((cz+d)^2)^{-s}$ is holomorphic in z for $\mathrm{Re}(z) \neq -\frac{d}{c}$. In some cases, for instance for period functions satisfying (3), we may prefer to choose the factor differently, such that it is holomorphic on the domain of the function and positive on the real points in the domain.

We need more spaces related to $\mathcal{V}_s^{\omega}$ and $\mathcal{V}_s^{\infty}$. If $I \subset \mathbb{P}^1_{\mathbb{R}}$ is an open subset, we define $\mathcal{V}_s^{\omega}(I)$ as the space of $h : I \cap \mathbb{R} \to \mathbb{C}$ that are real-analytic on $I \cap \mathbb{R}$, and for which in the case $\infty \in I$ the function in (12) is real-analytic at 0. In particular, if $I = \mathbb{P}^1_{\mathbb{R}} \setminus E$ for some finite set E, then $\mathcal{V}_s^{\omega}(I)$ consists of analytic vectors with finitely many singularities on $\mathbb{P}^1_{\mathbb{R}}$.

The space $\mathcal{V}_s^{\omega^*}$ is defined as the inductive limit

$$\mathcal{V}_s^{\omega^*} = \varinjlim \mathcal{V}_s^{\omega}\big(\mathbb{P}^1_{\mathbb{R}} \setminus E\big),$$

where E runs through the finite subsets of $\mathbb{P}^1_{\mathbb{R}}$. If $h \in \mathcal{V}_s^{\omega^*}$, then there is a minimal finite set $E \subset \mathbb{P}^1_{\mathbb{R}}$ such that $h \in \mathcal{V}_s^{\omega}(\mathbb{P}^1_{\mathbb{R}} \setminus E)$. We call the elements of E the *singularities* of h and denote this set by $\mathrm{Sing}(h)$.

By imposing conditions at the singularities, we define subspaces of $\mathcal{V}_s^{\omega^*}$. For instance

$$\mathcal{V}_s^{\omega^*,\infty} := \mathcal{V}_s^{\omega^*} \cap \mathcal{V}_s^{\infty} \tag{14}$$

is the space of smooth vectors that are real-analytic outside finitely many points. A slightly larger space is $\mathcal{V}_s^{\omega^*,\mathrm{simple}}$ consisting of the $h \in \mathcal{V}_s^{\omega^*}$ for which we allow simple pole at finitely many points. Its elements f have to satisfy at their finitely many singularities x_0:

$$\begin{aligned} x \mapsto (x - x_0) f(x) \quad &\text{is smooth at } x_0 \quad \text{if } x_0 \neq \infty, \quad \text{and} \\ y \mapsto y|y|^{-2s} f(-1/y) = y(f \mid S)(y) \quad &\text{is smooth at } 0 \quad \text{if } x_0 = \infty. \end{aligned} \tag{15}$$

Thus we have various spaces, all invariant under the action $|_{2s}$ of $\mathrm{PGL}(2, \mathbb{R})$, that satisfy the following inclusions:

$$\begin{array}{ccccccc} \mathcal{V}_s^{\omega} & \hookrightarrow & \mathcal{V}_s^{\omega^*,\infty} & \hookrightarrow & \mathcal{V}_s^{\omega^*,\mathrm{simple}} & \hookrightarrow & \mathcal{V}_s^{\omega^*} \\ & & \big\downarrow & & & & \\ & & \mathcal{V}_s^{\infty} & & & & \end{array}$$

Throughout this note we use the assumption

$$0 < \mathrm{Re}(s) < 1, \quad s \neq \frac{1}{2}. \tag{16}$$

Mostly, we work with a fixed value of the spectral parameter s. Then we shall write $h \mid M$ instead of $h|_{2s}M$.

For vector valued functions we write $\binom{f_1}{f_2}|_{2s}M$ for $\binom{f_1|_{2s}M}{f_2|_{2s}M}$. The slash operator extends to elements of the group ring $\mathbb{C}[\mathrm{PGL}(2, \mathbb{R})]$ by

$$f|_{2s}(M_1 + aM_2) = f|_{2s}M_1 + af|_{2s}M_2 \quad \text{for all } g_1, g_2 \in \Gamma \text{ and } a \in \mathbb{C}.$$

In some circumstances one can make sense of $f|_{2s}\Xi$ where Ξ is an infinite sum of elements of Γ.

2.3. Cohomology groups

As usual, the *first cohomology group* of Γ with values in a right Γ-module V can be described by

$$H^1(\Gamma;V)=Z^1(\Gamma;V)/B^1(\Gamma;V),$$
$$Z^1(\Gamma;V)=\{\psi\colon\ \Gamma\to V;\ \psi_{\gamma\delta}=\psi_\gamma\mid\delta+\psi_\delta\ \text{for all}\ \gamma,\delta\in\Gamma\}\quad\text{and}$$
$$B^1(\Gamma;V)=\bigl\{\psi\in Z^1(\Gamma;V);\ \exists v\in V\ \text{such that}\ \psi_\gamma=v\mid(1-\gamma)\bigr\}. \tag{17}$$

We give the arguments of the *cocycles* $\psi\in\mathbb{Z}^1(\Gamma;V)$ by a subscript. Furthermore we denote the right action of Γ on V as $v\mapsto v\mid\gamma$ for $\gamma\in\Gamma$ and $v\in V$. If the cohomology group is clear we use the notation $[\psi]$ for the cohomology class of the cocycle ψ.

The *first parabolic cohomology group* is the subgroup of $H^1(\Gamma;V)$ given by

$$H^1_{\text{par}}(\Gamma;V)=Z^1_{\text{par}}(\Gamma;V)/B^1(\Gamma;V),$$
$$Z^1_{\text{par}}(\Gamma;V)=\bigl\{\psi\in Z^1(\Gamma;V);\ \exists v\in V\ \text{such that}\ \psi_T=v\mid(1-T)\bigr\}. \tag{18}$$

If $W\supset V$ is a larger Γ-module, then the *first mixed parabolic cohomology group* is given by

$$H^1_{\text{par}}(\Gamma;V,W)=Z^1_{\text{par}}(\Gamma;V,W)/B^1(\Gamma;V),$$
$$Z^1_{\text{par}}(\Gamma;V,W)=\bigl\{\psi\in Z^1(\Gamma;V);\ \exists v\in W\ \text{such that}\ \psi_T=v\mid(1-T)\bigr\}. \tag{19}$$

Note that $H^1_{\text{par}}(\Gamma;V,W)$ is a subspace of $H^1(\Gamma;V)$, and that there is a natural map $H^1_{\text{par}}(\Gamma;V,W)\to H^1_{\text{par}}(\Gamma;W)$.

Remark 1. In the definitions above we use that $\Gamma\backslash\mathbb{H}$ has only one Γ-class of cusps represented by ∞, and that the subgroup $\Gamma_\infty\subset\Gamma$ fixing ∞ is generated by T. We refer to the section "Cohomology and parabolic cohomology for groups with cusps" in [2] for a discussion of parabolic cohomology for more general discrete subgroups of $\mathrm{PSL}(2,\mathbb{R})$.

Remark 2. Any cocycle is determined by its values on a set of generators, so by ψ_S and ψ_T for Γ. The relations (9) determine the relations

$$\psi_S\mid(1+S)=0,\qquad \psi_{T^{-1}S}\bigm|\bigl(1+T^{-1}S+ST\bigr)=0. \tag{20}$$

For a parabolic cocycle ψ we can arrange $\psi_T=0$ without changing the cohomology class. The resulting cocycle is determined by its value on S, subject to the relations

$$\psi_S|_{2s}(1+S)=0,\qquad \psi_S=\psi_S|_{2s}(T+T'). \tag{21}$$

2.4. Eigenfunctions of the Mayer operator and parabolic cohomology

In the Introduction we have already mentioned that eigenfunctions of $\mathcal{L}_s^{\text{Ma}}$ with eigenvalue 1 or -1 give rise to elements of ${}^3\mathrm{FE}_s(\mathbb{C}')_\omega$, where for $X\subset\mathbb{C}$:

$${}^3\mathrm{FE}_s(X)_\omega=\bigl\{\text{analytic}\ P:X\to\mathbb{C}:\ P=P|_{2s}T+P|_{2s}T'\ \text{on}\ X\cap T^{-1}X\cap(T')^{-1}X\bigr\}. \tag{22}$$

By analytic on $X\subset\mathbb{R}$ we mean *real analytic*. For open $X\subset\mathbb{C}$, analytic means *holomorphic*.

The cocycle condition $\psi_S = \psi_S \mid (T + T')$ in (21) is similar to the three term equation (3). In the section "Parabolic cohomology and mixed parabolic cohomology" in [2], various aspects of the relation between ${}^3\mathrm{FE}_s(\mathbb{C}')_\omega$ and cohomology are discussed. For the present paper it is important that under assumption (16):

$$H^1_{\mathrm{par}}\bigl(\Gamma; \mathcal{V}_s^\omega, \mathcal{V}_s^{\omega^*,\mathrm{simple}}\bigr) \cong H^1_{\mathrm{par}}\bigl(\Gamma; \mathcal{V}_s^{\omega^*,\mathrm{simple}}\bigr) \cong {}^3\mathrm{FE}_s(0,\infty)_\omega^{\mathrm{simple}}. \tag{23}$$

The third superscript *simple* indicates that we impose on $P \in {}^3\mathrm{FE}_s(0,\infty)_\omega$ an asymptotic behavior at the end points of $(0,\infty)$:

$$P(x) \sim \sum_{m=-1}^{\infty} c_m^\infty x^{-2s-m} \quad (x \uparrow \infty), \qquad P(x) \sim \sum_{m=-1}^{\infty} c_m^0 x^m \quad (x \downarrow 0). \tag{24}$$

This is a one-sided version of the behavior at singularities of elements of $\mathcal{V}_s^{\omega^*,\mathrm{simple}}$ defined in Section 2.2.

In Section 3.2 we shall show:

Proposition 2.1. *Let $s \in \mathbb{C}$, $0 < \mathrm{Re}(s) < 1$, $s \neq \frac{1}{2}$. The space $\ker(\mathcal{L}_s^{\mathrm{Ma}} - 1) \oplus \ker(\mathcal{L}_s^{\mathrm{Ma}} + 1)$ is in bijective correspondence to $H^1_{\mathrm{par}}(\Gamma; \mathcal{V}_s^\omega, \mathcal{V}_s^{\omega^*,\mathrm{simple}})$.*

2.5. Eigenfunctions of the nearest integer transfer operator and hyperbolic cohomology

The definitions (18) and (19) are related to the Γ-orbit $\mathbb{P}^1_{\mathbb{Q}}$ of cusps in $\mathbb{P}^1_{\mathbb{R}}$. The element $T \in \Gamma$ generates the subgroup Γ_∞ of Γ fixing the element ∞ in this orbit. Let us now work with what we would like to call the *Fibonacci orbit* $\mathrm{Fib} = \Gamma(-\phi) \subset \mathbb{P}^1_{\mathbb{R}}$, where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio. The hyperbolic element $TST^2 \in \Gamma$ generates the subgroup $\Gamma_{-\phi}$ of Γ fixing $-\phi$.

Definition 2.2. For Γ-modules $W \supset V$:

$$\begin{aligned} H^1_{\mathrm{Fib}}(\Gamma; V, W) &= Z^1_{\mathrm{Fib}}(\Gamma; V, W)/B^1(\Gamma; V), \\ Z^1_{\mathrm{Fib}}(\Gamma; V, W) &= \bigl\{\psi \in Z^1(\Gamma; V)\colon\ \exists v \in W\colon\ \psi_{TST^2} = v \bigm| \bigl(1 - TST^2\bigr)\bigr\}, \end{aligned} \tag{25}$$

and $H^1_{\mathrm{Fib}}(\Gamma; V) := H^1_{\mathrm{Fib}}(\Gamma; V, V)$.

In particular, $H^1_{\mathrm{Fib}}(\Gamma; \mathcal{V}_s^{\omega^*})$ is a subspace of $H^1(\Gamma; \mathcal{V}_s^{\omega^*})$. The inclusion $\mathcal{V}_s^\omega \hookrightarrow \mathcal{V}_s^{\omega^*}$ induces a linear map $H^1(\Gamma; \mathcal{V}_s^\omega) \to H^1(\Gamma; \mathcal{V}_s^{\omega^*})$.

Let us also define for $X \subset \mathbb{C}$:

$$\begin{aligned} {}^4\mathrm{FE}_s(X)_\omega = \bigl\{&\text{analytic } g : X \to \mathbb{C}\colon\ g + g|_{2s}ST^2 = g|_{2s}T^{-1} + g|_{2s}T^{-1}ST^{-2} \\ &\text{on } X \cap T^{-2}SX \cap TX \cap T^2STX\bigr\}, \end{aligned} \tag{26}$$

with the same convention concerning analyticity as in Section 2.4. We shall prove in Section 4.2:

Theorem 2.3. *Let $s \in \mathbb{C}$, $0 < \mathrm{Re}(s) < 1$, $s \neq \frac{1}{2}$. There is an injective map $\vartheta : {}^4\mathrm{FE}_s(-\phi^2, \phi) \to H^1_{\mathrm{Fib}}(\Gamma; \mathcal{V}_s^{\omega^*})$. The image $\vartheta({}^4\mathrm{FE}_s(-\phi^2, \phi)) \subset H^1(\Gamma; \mathcal{V}_s^{\omega^*})$ is equal to the image of $H^1(\Gamma; \mathcal{V}_s^\omega)$ in $H^1(\Gamma; \mathcal{V}_s^{\omega^*})$.*

Proposition 2.4. *The natural map $H^1(\Gamma; \mathcal{V}_s^\omega) \to H^1(\Gamma; \mathcal{V}_s^{\omega^*})$ is injective.*

Proof. Let $\psi \in Z^1_{\mathrm{par}}(\Gamma; \mathcal{V}^\omega_s)$ such that $\psi_\gamma = f \mid (1-\gamma)$ for all $\gamma \in \Gamma$ for some $f \in \mathcal{V}^{\omega^*}_s$. From $f \mid (1-T) = \psi_T \in \mathcal{V}^\omega_s$ it follows that the set of singularities $\mathrm{Sing}(f)$ can contain at most the point ∞; otherwise $\mathrm{Sing}(f)$ would be infinite. Hence $\mathrm{Sing}(f \mid S) \subset \{0\}$. From $f - f \mid S = \psi_S \in \mathcal{V}^\omega_s$ we conclude that f has no singularities at all, i.e., $f \in \mathcal{V}^\omega_s$. Hence $[\psi] = 0$ in $H^1(\Gamma; \mathcal{V}^\omega_s)$. □

We now have the following system of injective maps:

$$\begin{array}{ccccc} {}^4\mathrm{FE}_s(-\phi^2, \phi) & \xhookrightarrow{\vartheta} & H^1_{\mathrm{Fib}}(\Gamma; \mathcal{V}^{\omega^*}_s) & \hookrightarrow & H^1(\Gamma; \mathcal{V}^{\omega^*}_s) \\ & & & & \uparrow \\ & & H^1_{\mathrm{par}}(\Gamma; \mathcal{V}^\omega_s, \mathcal{V}^{\omega^*,\mathrm{simple}}_s) & \hookrightarrow & H^1(\Gamma; \mathcal{V}^\omega_s) \end{array}$$

In Section 4.3 we will prove:

Theorem 2.5. *Let $s \in \mathbb{C}$, $0 < \mathrm{Re}(s) < 1$, $s \neq \frac{1}{2}$. The kernel of $\tilde{\mathcal{L}}_s - 1$ determines a subspace of ${}^4\mathrm{FE}_s(-\phi^2, \phi)_\omega$ that is mapped by ϑ onto the image of the mixed parabolic cohomology group $H^1_{\mathrm{par}}(\Gamma; \mathcal{V}^\omega_s, \mathcal{V}^{\omega^*,\mathrm{simple}}_s)$ in $H^1(\Gamma; \mathcal{V}^{\omega^*}_s)$.*

This establishes a bijective map between $\ker(\tilde{\mathcal{L}}_s - 1)$ and the cohomology group $H^1_{\mathrm{par}}(\Gamma; \mathcal{V}^\omega_s, \mathcal{V}^{\omega^*,\mathrm{simple}}_s)$. We prove this theorem in Section 4.3.

The results in Proposition 2.1, Theorem 2.3, Proposition 2.4 and Theorem 2.5 imply Theorem 1.1. See also Fig. 1.

2.6. *Automorphic forms and cohomology groups*

In this note we work with transfer operators and cohomology groups. In [5] and [2] the main theme is the relation between period functions, automorphic forms and cohomology. We mention the relevant facts as background material.

We denote by ${}^3\mathrm{FE}_s(\mathbb{C}')^0_\omega$ the subspace of $P \in {}^3\,\mathrm{FE}_s(\mathbb{C}')_\omega$, as defined in (22), with $\mathbb{C}' = \mathbb{C} \setminus (-\infty, 0]$, that satisfy $P(x) = O(1)$ as $x \downarrow 0$ and $P(x) = O(x^{-2s})$ as $x \to \infty$. The main theorem in [5] states that the space ${}^3\mathrm{FE}_s(\mathbb{C}')^0_\omega \cong H^1_{\mathrm{par}}(\Gamma; \mathcal{V}^{\omega^*,\infty}_s)$ is in bijective correspondence with the space of Maass cusp forms with spectral parameter s. A Maass cusp form is a function $u : \mathbb{H} \to \mathbb{C}$ satisfying $u(\gamma z) = u(z)$ for all $\gamma \in \Gamma$ that is given by a convergent Fourier expansion

$$u(x+iy) = \sum_{n \neq 0} A_n e^{2\pi i n x} \sqrt{y} K_{s-1/2}(2\pi \mid n \mid y). \tag{27}$$

The space M^0_s of such Maass cusp forms is known to be non-zero only for a discrete set of values of s satisfying $\mathrm{Re}(s) = \frac{1}{2}$, $s \neq \frac{1}{2}$.

A slightly larger space of Γ-invariant functions is M^1_s, consisting of the Γ-invariant u on $\mathbb{H}$ with a converging Fourier expansion

$$u(x+iy) = A_0 y^{1-s} + \sum_{n \neq 0} A_n e^{2\pi i n x} \sqrt{y} K_{s-1/2}(2\pi \mid n \mid y). \tag{28}$$

This space is equal to M^0_s for $\mathrm{Re}(s) = \frac{1}{2}$, $s \neq \frac{1}{2}$. For values of s with $0 < \mathrm{Re}(s) < \frac{1}{2}$ such that $\zeta(2s) = 0$ the residue of the Eisenstein series is an element of M^1_s. The results in the section “Maass forms and cohomology” in [2] show that for $0 < \mathrm{Re}(s) < 1$, $s \neq \frac{1}{2}$ there is a bijective correspondence between M^1_s and $H^1_{\mathrm{par}}(\Gamma; \mathcal{V}^{\omega^*,\mathrm{simple}}_s)$. These spaces are finite dimensional, and zero for most values of s. All elements of M^1_s are eigenfunctions of the hyperbolic Laplace operator: $-y^2(\partial^2_y + \partial^2_x)u = s(1-s)u$.

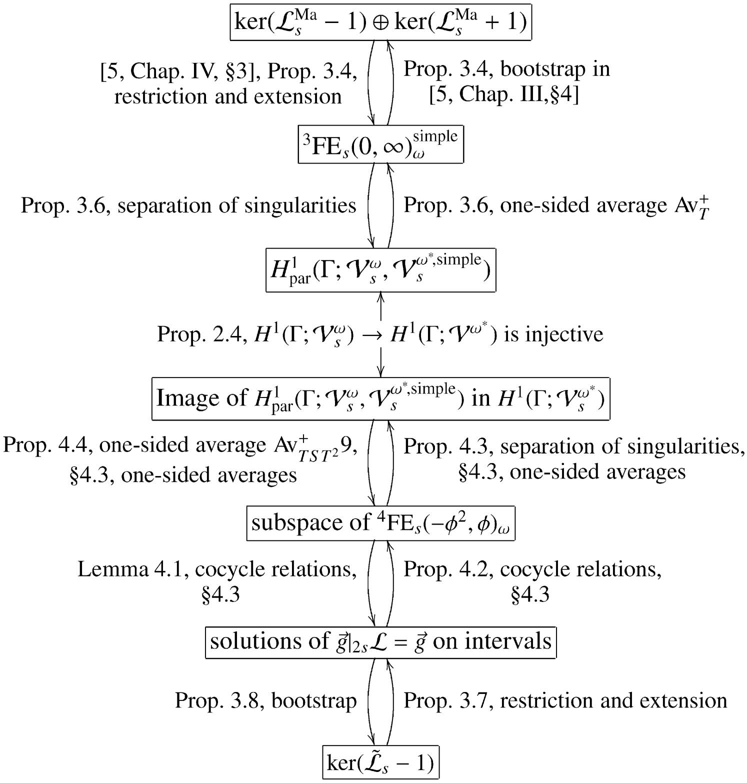

Fig. 1. Overview of the steps in the proof of Theorem 1.1.

The conclusion is that the eigenfunctions of $\mathcal{L}_s^{\mathrm{Ma}}$ with eigenvalues 1 and -1, and the eigenfunctions of $\tilde{\mathcal{L}}_s$ with eigenvalue 1 are in bijective correspondence to elements of the space M_s^1. This gives a confirmation of the relation between eigenfunctions of transfer operators and automorphic forms that we know already from the relation via the Selberg zeta function. (See [6] and (7).)

3. The transfer operators on disks and intervals

In the context of dynamical systems one usually considers transfer operators in Banach spaces of holomorphic functions on a disk. For the relation to cohomology groups with values in principal series spaces, it is more natural to consider the corresponding operators on functions on intervals in $\mathbb{R}$ or $\mathbb{P}^1_{\mathbb{R}}$. We discuss this relation in Section 3.2 for the Mayer operator and in Section 3.3 for $\tilde{\mathcal{L}}_s$. In Section 3.4 we derive the four term equation (5) from the transfer operator $\tilde{\mathcal{L}}_s$.

We start with a discussion of one-sided averages.

3.1. One-sided averages

Both in the definition of $\mathcal{L}_s^{\mathrm{Ma}}$ in (1) and in that of $\tilde{\mathcal{L}}_s$ in (2), one recognizes infinite sums of the type $f \mid \gamma T^n$ over infinitely many $n \in \mathbb{Z}$ for a fixed $\gamma \in \mathrm{PGL}(2, \mathbb{Z})$. The one-sided averages

$$\mathrm{Av}_T^+ = \sum_{n=0}^{\infty} T^n \quad \text{and} \quad \mathrm{Av}_T^- = -\sum_{n=-\infty}^{-1} T^n \tag{29}$$

play also an important role in [2]. In this subsection we recall the relevant results.

Consider a function of the form $f = h \mid S$, where h is holomorphic on a neighborhood of 0. Then

$$\mathrm{Av}_T^+(f) = f \mid \mathrm{Av}_T^+(z) = \sum_{n=0}^{\infty}\bigl((z+n)^2\bigr)^{-s} h\left(\frac{-1}{z+n}\right) \quad \text{and}$$
$$\mathrm{Av}_T^-(f) = f \mid \mathrm{Av}_T^-(z) = -\sum_{n=1}^{\infty}\bigl((z-n)^2\bigr)^{-s} h\left(\frac{-1}{z-n}\right) \tag{30}$$

converge absolutely if $\mathrm{Re}(s) > \frac{1}{2}$, and define $f \mid \mathrm{Av}_T^+$ as a holomorphic function on a right half-plane, and $f \mid \mathrm{Av}_T^-$ on a left half-plane. If $h(0) = 0$, the convergence is absolute for $\mathrm{Re}(s) > 0$. Using the Hurwitz zeta function for the contribution of the constant term of h at 0, we obtain in general a meromorphic continuation, with at most a first order singularity at $s = \frac{1}{2}$ on $\mathrm{Re}(s) > 0$. In this note we will understand $f \mid \mathrm{Av}_T^+$ and $f \mid \mathrm{Av}_T^-$ always in this regularized sense. We have given two notations in (30). With $f \mid \mathrm{Av}_T^+$ we stress that Av_T^+ is an element of the completion of the group ring of Γ, for which we have made sense of the action on certain functions by regularization. With $\mathrm{Av}_T^+(f)$ we emphasize that this one-sided average defines an operator on suitable spaces of functions. In this note we will use $f \mid \mathrm{Av}_T^+$ and $f \mid \mathrm{Av}_T^-$.

These one-sided averages satisfy

$$f \mid \mathrm{Av}_T^+ \mid (1-T) = f, \qquad f \mid \mathrm{Av}_T^- \mid (1-T) = f, \tag{31}$$
$$f \mid (1-T) \mid \mathrm{Av}_T^+ = f, \qquad f \mid (1-T) \mid \mathrm{Av}_T^- = f, \tag{32}$$
$$f \mid T \mid \mathrm{Av}_T^+ = f \mid \mathrm{Av}_T^+ \mid T, \qquad f \mid T \mid \mathrm{Av}_T^- = f \mid \mathrm{Av}_T^- \mid T, \tag{33}$$
$$f + f \mid T \mid \mathrm{Av}_T^+ = f \mid \mathrm{Av}_T^+, \qquad -f \mid T^{-1} + f \mid T^{-1} \mid \mathrm{Av}_T^- = f \mid \mathrm{Av}_T^-, \tag{34}$$

on suitable right half-planes, respectively left half-planes. These relations hold trivially in the domain $\mathrm{Re}(s) > \frac{1}{2}$ of absolute convergence, and survive under meromorphic continuation.

In particular, we consider these one-sided averages for $f \in \mathcal{V}^{\omega}(I)$ where $I \subset \mathbb{P}^1_{\mathbb{R}}$ is a neighborhood of ∞. Then f has the form indicated above. Let us consider a cyclic interval $I = (a,b)_c$ in $\mathbb{P}^1_{\mathbb{R}}$ containing ∞. (This means that $a > b$ in $\mathbb{R}$ and $(a,b)_c = (a,\infty) \cup \{\infty\} \cup (-\infty, b)$.) As in the section "Averages" in [2] we have:

Lemma 3.1. *Let $I \supset (a,b)_c \ni \infty$. If $f \in \mathcal{V}_s^{\omega}(I)$, then $f \mid \mathrm{Av}_T^+ \in \mathcal{V}_s^{\omega}(a,\infty)$ and is represented by a function holomorphic on a neighborhood of (a,∞) containing a right half-plane, and $f \mid \mathrm{Av}_T^- \in \mathcal{V}_s^{\omega}(\infty, b+1)$ is represented by a function holomorphic on a neighborhood of $(\infty, b+1)$ containing a left half-plane.*

There are constants C_m^* for $m = -1, 0, 1, \ldots$ such that

$$f|_{2s}\, \mathrm{Av}_T^+(x) \sim \sum_{m=-1}^{\infty} C_m^* x^{-m-2s} \quad (x \uparrow \infty), \tag{35}$$
$$f|_{2s}\, \mathrm{Av}_T^-(x) \sim \sum_{m=-1}^{\infty} C_m^* x^{-m} |x|^{-2s} \quad (x \downarrow -\infty). \tag{36}$$

In particular, if $f \in \mathcal{V}_s^{\omega} = \mathcal{V}_s^{\omega}(\mathbb{P}^1_{\mathbb{R}})$, then $f \mid \mathrm{Av}_T^{\pm} \in \mathcal{V}_s^{\omega}(\mathbb{R})$.

The asymptotic behavior in (35) and (36) is related to the singularity behavior (15) in the definition of $\mathcal{V}_s^{\omega^*,\mathrm{simple}}$:

Lemma 3.2. *(See [2].) For $f \in \mathcal{V}_s^{\omega}$ the following statements are equivalent:*

(i) $f \in \mathcal{V}_s^{\omega^*,\mathrm{simple}} \mid (1-T)$.
(ii) $f \mid \mathrm{Av}_T^+ = f \mid \mathrm{Av}_T^-$.

Proof. If $f \mid \mathrm{Av}_T^+ = f \mid \mathrm{Av}_T^-$, then $f \mid \mathrm{Av}^\pm \mid S(z) \sim \sum_{m=-1}^\infty C_m^* x^m$ as $\mp x \downarrow 0$. Hence $f \in \mathcal{V}_s^{\omega^*,\mathrm{simple}}$.

Conversely, suppose that $f = h \mid (1 - T)$ with $h \in \mathcal{V}_s^{\omega^*,\mathrm{simple}}$. Then $p_+ = h - f \mid \mathrm{Av}_T^+$ and $p_- = h - f \mid \mathrm{Av}_T^-$ satisfy $p_\pm \mid T = p_\pm$. Moreover, $p_+(x)$ has an asymptotic expansion of the form (35) as $x \uparrow \infty$. Hence the periodic function p_+ vanishes. For p_- let $x \downarrow -\infty$. □

Lemma 3.3. *Suppose that $b, c \in \mathcal{V}_s^\omega$ satisfy $b \mid \mathrm{Av}_T^+ + c \mid \mathrm{Av}_T^- \in \mathcal{V}_s^\omega$. Then $b \mid \mathrm{Av}_T^+ = b \mid \mathrm{Av}_T^- \in \mathcal{V}_s^{\omega^*,\mathrm{simple}}$, and $c \mid \mathrm{Av}_T^+ = c \mid \mathrm{Av}_T^- \in \mathcal{V}_s^{\omega^*,\mathrm{simple}}$.*

Proof. Relations (31) and (32) imply that $p = c \mid \mathrm{Av}_T^+ - c \mid \mathrm{Av}_T^-$ satisfies $p|_{2s}T = p$, hence p is a periodic function on $\mathbb{R}$. Put $a = b \mid \mathrm{Av}_T^+ + c \mid \mathrm{Av}_T^-$. As $x \uparrow \infty$, the term $c \mid \mathrm{Av}_T^+$ has an asymptotic expansion as in (35), and $c \mid \mathrm{Av}_T^- = a - b \mid \mathrm{Av}_T^+$ also has an expansion of this type. (For a we know that $a(x) = |x|^{-2s}$ (analytic in $-1/x$).) Hence $p(x) \sim x^{-2s}(p_{-1}x + p_0 + \cdots)$ as $x \to \infty$. The periodicity implies that p is bounded, hence $p_{-1} = 0$. Next $p(x) = \mathcal{O}(x^{-2\,\mathrm{Re}(s)})$ implies $p = 0$, hence $c \mid \mathrm{Av}_T^+ = c \mid \mathrm{Av}_T^-$. Now $c \mid \mathrm{Av}_T^\pm \in \mathcal{V}_s^{\omega^*,\mathrm{simple}}$, and also $b \mid \mathrm{Av}_T^\pm \in \mathcal{V}_s^{\omega^*,\mathrm{simple}}$. □

We can build one-sided averages for other elements of Γ. If $\eta \in \Gamma$ is hyperbolic, for instance $\eta = TST^2$, then the averages $\mathrm{Av}_\eta^+ = \sum_{n\geqslant 0} \eta^n$ and $\mathrm{Av}_\eta^- = -\sum_{n\leqslant -1} \eta^n$ have the properties corresponding to (31)–(34), if they converge. See [2]. If the attracting fixed point $\omega(\eta)$ of η is in the cyclic interval $I \subset \mathbb{P}_\mathbb{R}^1$, then $f|_{2s}\mathrm{Av}_\eta^+$ converges without regularization for $\mathrm{Re}(s) > 0$ for all $f \in \mathcal{V}_s^\omega(I)$, and provides us with $f|_{2s}\mathrm{Av}_\eta^+ \in \mathcal{V}_s^\omega(I \setminus \{\alpha(\eta)\})$, where $\alpha(\eta)$ is the repelling fixed point of η. In particular, $\mathrm{Av}_\eta^+ : \mathcal{V}_s^\omega \to \mathcal{V}_s^\omega(\mathbb{P}_\mathbb{R}^1 \setminus \{\alpha(\eta)\})$, and similarly $\mathrm{Av}_\eta^- : \mathcal{V}_s^\omega \to \mathcal{V}_s^\omega(\mathbb{P}_\mathbb{R}^1 \setminus \{\omega(\eta)\})$.

3.2. Eigenfunctions of the Mayer operator

Eigenfunctions of $\mathcal{L}_s^{\mathrm{Ma}}$ with eigenvalue ± 1, or briefly (± 1)-eigenfunctions of $\mathcal{L}_s^{\mathrm{Ma}}$, can be related to eigenfunctions of a similar operator on $\mathcal{V}_s^\omega(0,\infty)$. This statement is almost contained in the results in [5]. Nevertheless, we recall the main steps in the proof, since we want to describe the correspondence between ± 1-eigenfunctions of $\mathcal{L}_s^{\mathrm{Ma}}$ and 1-eigenfunctions of $\tilde{\mathcal{L}}_s$ explicitly.

In this subsection we also indicate how to prove Proposition 2.1, on the basis of results in [2].

Under the step $P = f \mid T$, the function $f \in \ker(\mathcal{L}_s^{\mathrm{Ma}} \mp 1)$ corresponds to P holomorphic on $|z-2| < \frac{3}{2}$, and continuous on $|z-2| \leqslant \frac{3}{2}$ satisfying $P \mid CT' \mathrm{Av}_T^+ = \pm P$, with use of the notation introduced in (10). This implies on suitable non-empty domains:

$$P = P \mid T \pm P \mid TC, \qquad P \mid C = \pm P,$$
$$P = P \mid T + P \mid T'.$$

The last equality is the three term equation (3). The proposition in [5, Chapter IV, Section 3] shows that these functions extend to $\mathbb{C}' = \mathbb{C} \setminus (-\infty, 0]$ and satisfy $P(x) \sim c_{-1}x^{1-2s} + O(x^{-2\,\mathrm{Re}(s)})$ as $x \uparrow \infty$ for some c_{-1}. Since P satisfies $P \mid T' \mathrm{Av}_T^+ = P$ on $\mathbb{C}'$ and $P \mid C = \pm P$, we have the asymptotic behavior (24) near both end points of $(0,\infty)$, with $c_m^0 = \pm c_m^\infty$. Thus we have $P \in {}^3\mathrm{FE}_s^\pm(\mathbb{C}')_\omega^{\mathrm{simple}}$, where the upper index $\pm$ indicates the (± 1)-eigenspace of C in ${}^3\mathrm{FE}_s(\mathbb{C}')_\omega$, and where the superscript *simple* indicates the subspace satisfying (24).

Conversely, starting with $P \in {}^3\mathrm{FE}_s^\pm(\mathbb{C}')_\omega^{\mathrm{simple}}$, we have $P \mid T' \in \mathcal{V}_s^\omega((0,-1)_c)$, where $(0,-1)_c = (0,\infty) \cup \{\infty\} \cup (-\infty, 0)$ denotes a cyclic interval in $\mathbb{P}_\mathbb{R}^1$. Hence $P \mid T' \mathrm{Av}_T^+ \in \mathcal{V}_s^\omega(0,\infty)$. By (31):

$$\bigl(P - P \bigm| T' \mathrm{Av}_T^+\bigr) \bigm| (1-T) = P - P \mid T - P \mid T' = 0.$$

The asymptotic behavior of $P - P \mid T' \mathrm{Av}_T^+$ near ∞ shows that this periodic function vanishes. So P satisfies $P \mid T' \mathrm{Av}_T^+ = P$, or with use of the parity condition, $P \mid CT' \mathrm{Av}_T^+ = \pm P$. The function $P \mid T' \mathrm{Av}_T^+$

is holomorphic on a right half-plane. With the parity condition $P \mid C = \pm P$ this implies that P is holomorphic on a wedge of the form $|\arg z| < \varepsilon$. This suffices as the point of departure for the second stage of the bootstrap procedure in [5, Chapter III, Section 4], which gives a holomorphic extension of P to $\mathbb{C}'$, still satisfying $P \mid CT' \operatorname{Av}_T^+ = \pm P$. This leads to $f = P \mid T \in \ker(\mathcal{L}_s^{\mathrm{Ma}} \mp 1)$.

Thus, we have an explicit bijective correspondence between the following spaces:

$$\ker\bigl(\mathcal{L}_s^{\mathrm{Ma}} \mp 1\bigr),$$
$$\ker\bigl(T' \operatorname{Av}_T^+ - 1 : \mathcal{V}_s^\omega(0,\infty) \to \mathcal{V}_s^\omega(0,\infty)\bigr) \cap \ker\bigl(C \mp 1 : \mathcal{V}_s^\omega(0,\infty) \to \mathcal{V}_s^\omega(0,\infty)\bigr),$$
$${}^3\mathrm{FE}_s^\pm(\mathbb{C}')_\omega^{\mathrm{simple}} \quad \text{and} \quad {}^3\mathrm{FE}_s^\pm(0,\infty)_\omega^{\mathrm{simple}}.$$

As in [5, Chapter I, Section 3], we have ${}^3\mathrm{FE}_s(\mathbb{C}')_\omega^{\mathrm{simple}} = {}^3\mathrm{FE}_s^+(\mathbb{C}')_\omega^{\mathrm{simple}} \oplus {}^3\mathrm{FE}_s^-(\mathbb{C}')_\omega^{\mathrm{simple}}$. This gives the following result:

Proposition 3.4. *Let* $0 < \mathrm{Re}(s) < 1$, $s \neq \frac{1}{2}$. *There is an explicit bijection between the following two spaces*:

$$\ker\bigl(\mathcal{L}_s^{\mathrm{Ma}} - 1\bigr) \oplus \ker\bigl(\mathcal{L}_s^{\mathrm{Ma}} + 1\bigr),$$
$${}^3\mathrm{FE}_s(0,\infty)_\omega^{\mathrm{simple}}.$$

To complete the proof of Proposition 2.1 we have to establish a relation between ${}^3\mathrm{FE}_s(0,\infty)_\omega^{\mathrm{simple}}$ and $H^1_{\mathrm{par}}(\Gamma; \mathcal{V}_s^\omega, \mathcal{V}_s^{\omega^*,\mathrm{simple}})$. The least explicit step in the proof is provided by the following result:

Proposition 3.5. *If* $f \in \mathcal{V}_s^{\omega^*}$ *satisfies* $\mathrm{Sing}(f) \subset \{\xi, \eta\}$ *for two different points* ξ *and* η *in* $\mathbb{P}^1_{\mathbb{R}}$, *then there are* $f_\xi, f_\eta \in \mathcal{V}_s^{\omega^*}$ *such that* $f = f_\eta - f_\xi$ *and* $\mathrm{Sing}(f_\xi) \subset \{\xi\}$, $\mathrm{Sing}(f_\eta) \subset \{\eta\}$. *The functions* f_ξ *and* f_η *are not unique. The freedom consists of adding the same element of* $\mathcal{V}_s^\omega$ *to both functions.*

Sketch of a proof. This follows from, e.g., [3, Theorem 1.4.5]. See the section "Parabolic cohomology and mixed parabolic cohomology" in [2] for the application to elements of $\mathcal{V}_s^{\omega^*}$.

The idea is to use another model of the principal series, in which the elements of $\mathcal{V}_s^\omega$ correspond to functions holomorphic on an annulus in $\mathbb{C}$ containing the unit circle. The function f in the proposition is represented by a holomorphic function on an open set $\Omega \subset \mathbb{C}$ containing the unit circle minus the points $\tilde{\xi}$ and $\tilde{\eta}$ corresponding to ξ and η. Write $\Omega = \Omega_1 \cap \Omega_2$ with $\tilde{\eta} \in \Omega_1$, $\tilde{\xi} \in \Omega_2$. Apply [3, Theorem 1.4.5] with $g_{1,2} = f$ to obtain $f = g_1 - g_2$ on Ω with g_j holomorphic on Ω_j. □

Proposition 3.6. *Let* $0 < \mathrm{Re}(s) < 1$, $s \neq \frac{1}{2}$. *There is an explicit bijection between* ${}^3\mathrm{FE}_s(0,\infty)_\omega^{\mathrm{simple}}$ *and* $H^1_{\mathrm{par}}(\Gamma; \mathcal{V}_s^\omega, \mathcal{V}_s^{\omega^*,\mathrm{simple}})$.

Proof. Suppose that $P \in {}^3\mathrm{FE}_s(0,\infty)_\omega^{\mathrm{simple}}$. We extend it to $\tilde{P} \in \mathcal{V}_s^{\omega^*}$ by

$$\tilde{P} = P \quad \text{on } (0,\infty), \quad \text{and} \quad \tilde{P} = -P \mid S \quad \text{on } (-\infty,0). \tag{37}$$

So $\mathrm{Sing}(\tilde{P}) \subset \{0,\infty\}$. By separate computations on $(-\infty,-1)$ and $(-1,0)$ we conclude that $\tilde{P}$ satisfies (3) on $\mathbb{P}^1_{\mathbb{R}} \setminus \{\infty,-1,0\}$. Thus, we obtain $\psi \in Z^1_{\mathrm{par}}(\Gamma; \mathcal{V}_s^{\omega^*})$, determined by

$$\psi_T = 0, \qquad \psi_S = \tilde{P}.$$

Since $\tilde{P} \mid T' \in \mathcal{V}_s^\omega(\mathbb{P}^1_{\mathbb{R}} \setminus \{-1,0\})$, we have $\tilde{P} \mid T' \operatorname{Av}_T^+ = P \mid T' \operatorname{Av}_T^+ \in \mathcal{V}_s^\omega(0,\infty)$ and $\tilde{P} \mid T' \operatorname{Av}_T^- \in \mathcal{V}_s^\omega(-\infty,0)$. We have indicated earlier in this subsection that the asymptotic behavior of $P(x)$ as

$x \uparrow \infty$ implies that $\tilde{P} \mid T' \operatorname{Av}_T^+ = \tilde{P}$ on $(0, \infty)$. The asymptotic behavior of $P(x)$ as $x \downarrow 0$ implies a similar asymptotic behavior of $\tilde{P}(x)$ as $x \downarrow -\infty$, which in turn implies analogously that $\tilde{P} \mid T' \operatorname{Av}_T^- = \tilde{P}$ on $(-\infty, 0)$. Consulting (35) and (36) we conclude that $\tilde{P}$ has the same coefficients in its expansions for both directions of approach to $\infty \in \mathbb{P}^1_{\mathbb{R}}$. The fact that $\tilde{P} \mid S = -\tilde{P}$ implies the same statement at 0. Hence $\tilde{P} \in \mathcal{V}_s^{\omega^*,\mathrm{simple}}$ and $\psi \in Z^1_{\mathrm{par}}(\Gamma; \mathcal{V}_s^{\omega^*,\mathrm{simple}})$. It is the unique cocycle in its cohomology class in $H^1_{\mathrm{par}}(\Gamma; \mathcal{V}_s^{\omega^*,\mathrm{simple}})$, since $(\mathcal{V}_s^{\omega^*,\mathrm{simple}})^T = \{0\}$, as is shown in [2], section “Invariants.”

Proposition 3.5 implies that there are $F_\infty, F_0 \in \mathcal{V}_s^{\omega^*}$ such that $\tilde{P} = F_\infty - F_0$ with $\operatorname{Sing}(F_\infty) \subset \{\infty\}$, $\operatorname{Sing}(F_0) \subset \{0\}$. Since F_∞ has the same type of asymptotic behavior at ∞ as $\tilde{P}$, we conclude that $F_\infty, F_0 \in \mathcal{V}_s^{\omega^*,\mathrm{simple}}$. Moreover, $\tilde{P} \mid S = -\tilde{P}$ implies that $F_0 = F_\infty + \alpha$ for some $\alpha \in \mathcal{V}_s^{\omega}$. From $\operatorname{Sing}(F_\infty \mid T') \subset \{-1\}$ and $\operatorname{Sing}(F_0 \mid ST') \subset \{0\}$, it follows that

$$\begin{aligned}
\operatorname{Sing}\bigl(\tilde{P} \bigm| (1-T)\bigr) &= \operatorname{Sing}(F_\infty \mid T' - F_\infty \mid ST' - \alpha \mid T') \subset \{-1, 0\},\\
\operatorname{Sing}\bigl(F_\infty \bigm| (1-T)\bigr) &= \operatorname{Sing}\bigl(\tilde{P} \bigm| (1-T) + F_0 \bigm| (1-T)\bigr) \subset \{-1, 0\}.
\end{aligned}$$

On the other hand $\operatorname{Sing}(F_\infty) \subset \{\infty\}$ implies that $\operatorname{Sing}(F_\infty \mid (1-T)) \subset \{\infty\}$. The conclusion is that $F_\infty \mid (1-T) \in \mathcal{V}_s^{\omega}$. We conclude that the cocycle

$$\tilde{\psi} : \gamma \mapsto \psi_\gamma - F_\infty \mid (1-\gamma)$$

takes values in $\mathcal{V}_s^{\omega}$. The freedom in the choice of F_∞ and F_0 amounts to the freedom of choosing $\tilde{\psi}$ in its class in $H^1_{\mathrm{par}}(\Gamma; \mathcal{V}_s^{\omega}, \mathcal{V}_s^{\omega^*,\mathrm{simple}})$.

Conversely, we start with a cocycle $\tilde{\psi} \in Z^1_{\mathrm{par}}(\Gamma; \mathcal{V}_s^{\omega}, \mathcal{V}_s^{\omega^*,\mathrm{simple}})$. By the definition of mixed parabolic cohomology in (19), there exists $v \in \mathcal{V}_s^{\omega^*,\mathrm{simple}}$ such that $\tilde{\psi}_T = v \mid (1-T)$. A possible choice is $\tilde{\psi}_T \mid \operatorname{Av}_T^+$, which coincides with $\tilde{\psi}_T \mid \operatorname{Av}_T^-$ according to Lemma 3.2, since $\psi_T \in \mathcal{V}_s^{\omega^*,\mathrm{simple}}$. Lemma 3.3 shows that $v = \tilde{\psi}_T \mid \operatorname{Av}_T^+$ is the sole possibility.

Now $\psi_\gamma = \tilde{\psi}_\gamma - \tilde{\psi}_T \mid \operatorname{Av}_T^+ \mid (1-\gamma)$ determines the unique cocycle in the cohomology class of $\tilde{\psi}$ in $H^1_{\mathrm{par}}(\Gamma; \mathcal{V}_s^{\omega^*,\mathrm{simple}})$ vanishing on T. Note that ψ does not change if we change $\tilde{\psi}$ in its cohomology class in $H^1_{\mathrm{par}}(\Gamma; \mathcal{V}_s^{\omega}, \mathcal{V}_s^{\omega^*,\mathrm{simple}})$. We have $\psi_T = 0$ and $\psi_S = \tilde{\psi}_S - \tilde{\psi}_T \mid \operatorname{Av}_T^+ \mid (1-S) \in \mathcal{V}_s^{\omega^*,\mathrm{simple}}$ with singularities contained in $\{0, \infty\}$. Restriction of ψ_S to $(0, \infty)$ gives an element of ${}^3\mathrm{FE}_s(0, \infty)_s^{\mathrm{simple}}$.

Noting that $F_\infty = -\tilde{\psi}_T \mid \operatorname{Av}_T^+$, we check that both procedures are inverse to each other. □

There might be cohomology classes in $H^1_{\mathrm{par}}(\Gamma; \mathcal{V}_s^{\omega^*,\mathrm{simple}})$ that are not represented by a cocycle ψ such that $\operatorname{Sing}(\psi_S) \subset \{0, \infty\}$. In [2] it takes work to show that such classes do not exist.

Proof of Proposition 2.1. The desired bijective correspondence between the direct sum of eigenspaces $\ker(\mathcal{L}_s^{\mathrm{Ma}} - 1)) \oplus \ker(\mathcal{L}_s^{\mathrm{Ma}} + 1))$ and $H^1_{\mathrm{par}}(\Gamma; \mathcal{V}_s^{\omega}, \mathcal{V}_s^{\omega^*,\mathrm{simple}})$ is established by Propositions 3.4 and 3.6 via the space ${}^3\mathrm{FE}_s(0, \infty)_\omega^{\mathrm{simple}}$. □

3.3. Eigenfunctions of the nearest integer transfer operator

In this subsection we relate 1-eigenfunctions of $\tilde{\mathcal{L}}_s$ in (2) to vectors of real analytic functions on intervals. We note that $\tilde{\mathcal{L}}_s$ is given by

$$\vec{g} \mapsto \vec{g} \mid \mathcal{L}, \quad \text{where } \mathcal{L} = \begin{pmatrix} ST^3 \operatorname{Av}_T^+ & -ST^{-1} \operatorname{Av}_T^- \\ ST^2 \operatorname{Av}_T^+ & -ST^{-2} \operatorname{Av}_T^- \end{pmatrix}, \tag{38}$$

for $\vec{g} = (g_1, g_2)$ continuous on $D = \{z \in \mathbb{C}\colon |z| \leqslant 1\}$ and holomorphic on the interior $\mathring{D}$. The transition from the operator notation in (2) to the right module notation here causes a transition from column vectors to row vectors and a transposition of the matrix.

For the first component g_1 of $\vec{g} = (g_1, g_2)$ in the domain of $\mathcal{L}$ the translate $g_1 \mid ST^3$ is holomorphic on the region $|z+3| > 1$ in $\mathbb{P}^1_{\mathbb{C}}$, hence $g_1 \mid ST^3 \operatorname{Av}_T^+$ is at least defined as a holomorphic function on the right half-plane $\operatorname{Re}(z) > -2$ in $\mathbb{C}$. Proceeding similarly with the other components we find that $\vec{g} \mid \mathcal{L}$ is holomorphic on $-1 < \operatorname{Re}(z) < 1$, which contains the interior of the unit disk D. Considering two terms in the infinite sum separately, with (34), we see that $\vec{g} \mid \mathcal{L}$ is continuous on the unit disk D, its boundary included. Restriction of $\vec{g}$ to a neighborhood of $(-1, 1)$ provides us with a solution of $\vec{g} \mid \mathcal{L} = \vec{g}$ in the vector valued analytic functions on $(-1, 1)$.

We shall see that the components of this restriction to $(-1, 1)$ can be extended to a larger interval, and that conversely solutions of $\vec{g} = \vec{g} \mid \mathcal{L}$ on the resulting interval come from 1-eigenfunctions of $\tilde{\mathcal{L}}_s$.

It turns out that a crucial role is played by the intervals $(-\phi^2, \phi)$ and $(-\phi, \phi^2)$, where ϕ denotes the golden ratio $\phi = \frac{1+\sqrt{5}}{2}$.

Proposition 3.7. *Let $a_1, b_2 \in (\phi^{-2}, \phi^2)$ and $a_2, b_1 \in (0, 1]$. Suppose that the analytic functions $f_1 \in \mathcal{V}_s^\omega(-a_1, b_1)$ and $f_2 \in \mathcal{V}_s^\omega(-a_2, b_2)$ satisfy $(f_1, f_2)|_{2s}\mathcal{L} = (f_1, f_2)$. Then f_1 is the restriction of $h_1 \in \mathcal{V}_s^\omega(-\phi^2, \phi)$ and f_2 of $h_2 \in \mathcal{V}_s^\omega(-\phi, \phi^2)$ such that $(h_1, h_2) \mid \mathcal{L} = (h_1, h_2)$, and $h_1 = h_2 \mid T$. The values of $h_1(x)$, respectively $h_2(x)$ for each given $x \in (-\phi^2, \phi)$, respectively $x \in (-\phi, \phi^2)$, can be expressed in values of f_1 and f_2 by a finite number of applications of the relation $(f_1, f_2) = (f_1, f_2)|_{2s}\mathcal{L}$.*

Proof. By analyticity the eigenfunction equation extends from given open intervals to larger ones, and relation (33) implies $h_1 = h_2 \mid T$. The statement that requires work is the extension of the domains.

Denote $(f_1, f_2) \mid \mathcal{L}$ by $(\tilde{f}_1, \tilde{f}_2)$. We have

$$f_1 \mid ST^3 \in \mathcal{V}_s^\omega\left(\left(\frac{1}{a_1} - 3, \frac{-1}{b_1} - 3\right)_c\right), \qquad f_2 \mid ST^{-1} \in \mathcal{V}_s^\omega\left(\left(\frac{1}{a_2} + 1, \frac{-1}{b_2} + 1\right)_c\right),$$
$$f_1 \mid ST^2 \in \mathcal{V}_s^\omega\left(\left(\frac{1}{a_1} - 2, \frac{-1}{b_1} - 2\right)_c\right), \qquad f_2 \mid ST^{-2} \in \mathcal{V}_s^\omega\left(\left(\frac{1}{a_2} + 2, \frac{-1}{b_2} + 2\right)_c\right),$$

where $(x, y)_c = (x, \infty) \cup \{\infty\} \cup (-\infty, y)$ is the notation for cyclic intervals in $\mathbb{P}^1_{\mathbb{R}}$. With Lemma 3.1:

$$f_1 \mid ST^3 \operatorname{Av}_T^+ \in \mathcal{V}_s^\omega\left(\frac{1}{a_1} - 3, \infty\right), \qquad f_2 \mid ST^{-1} \operatorname{Av}_T^- \in \mathcal{V}_s^\omega\left(-\infty, \frac{-1}{b_2} + 2\right),$$
$$f_1 \mid ST^2 \operatorname{Av}_T^+ \in \mathcal{V}_s^\omega\left(\frac{1}{a_1} - 2, \infty\right), \qquad f_2 \mid ST^{-2} \operatorname{Av}_T^- \in \mathcal{V}_s^\omega\left(-\infty, \frac{-1}{b_2} + 3\right),$$

which implies

$$\tilde{f}_1 \in \mathcal{V}_s^\omega\left(\frac{1}{a_1} - 3, \frac{-1}{b_2} + 2\right), \qquad \tilde{f}_2 \in \mathcal{V}_s^\omega\left(\frac{1}{a_1} - 2, \frac{-1}{b_2} + 3\right).$$

The end points of the domains are transformed according to $-\tilde{a}_1 = \frac{1}{a_1} - 3$, $\tilde{b}_1 = \tilde{b}_2 - 1$, $-\tilde{a}_2 = -\tilde{a}_1 + 1$, $\tilde{b}_2 = 3 - \frac{1}{b_2}$. Iterating this, the a_1 starting in (ϕ^{-2}, ϕ^2) form a sequence increasing to ϕ^2, and similarly for the b_2. This leads to the extension indicated in the proposition. □

This proposition shows that 1-eigenfunctions of $\tilde{\mathcal{L}}_s$ extend to vectors of the form $(g, g \mid T)$ with g holomorphic on a neighborhood of $(-\phi^2, \phi)$, such that the relation $(g, g \mid T) \mid \mathcal{L} = (g, g \mid T)$ is valid on a neighborhood of $(-\phi, \phi)$.

Proposition 3.8. *Suppose that* $g \in \mathcal{V}_s^\omega$ *satisfies*

$$\bigl(g, g \mid T^{-1}\bigr) \mid \mathcal{L} = \bigl(g, g \mid T^{-1}\bigr) \tag{39}$$

on $(-\phi, \phi)$. *Then* g *extends holomorphically to a neighborhood of the closed unit disk* D *and* (39) *holds on that neighborhood.*

Proof. Since g is real analytic on the interval $(-\phi^2, \phi) \supset [-1, 1]$, there is a complex ε-neighborhood U of $[-1, 1]$ to which g extends as a holomorphic function. Relation (39) stays valid on this neighborhood.

Denote $\vec{g} = (g, g \mid T^{-1})$. We have

$$\vec{g} \mid \mathcal{L} = \sum_{n \geqslant 0} \vec{g} \mid A_n, \quad A_n = \begin{pmatrix} ST^{3+n} & ST^{-2-n} \\ ST^{2+n} & ST^{-3-n} \end{pmatrix}. \tag{40}$$

For sufficiently large n the four images $(A_n)_{i,j}D$, $i, j \in \{1, 2\}$, are contained in the given neighborhood U of $[-1, 1]$. Lemma 3.1 and repeated application of relation (34) show that there is a tail of the series in (40) representing a holomorphic function on a neighborhood of D. Thus we have on an open neighborhood Ω of $(-\phi, \phi)$:

$$\vec{g} \mid \mathcal{L} = \sum_{n=0}^{N} g \mid A_n + (\text{holomorphic on a neighborhood of } D).$$

For g in the remaining terms $g \mid A_n$ we substitute (40) again. Repeating this process, we obtain for each $k \geqslant 1$ on an open neighborhood Ω_k of $(-\phi, \phi)$:

$$\vec{g}(z) = \sum_{n_1=0}^{N_1} \cdots \sum_{n_k=0}^{N_k} g \mid A_{n_k} \cdots A_{n_1}(z) + (\text{holomorphic on a neighborhood of } D). \tag{41}$$

The neighborhood Ω_k is increasing in k.

The matrix elements of $A_{n_k} \cdots A_{n_1}$ are of the form $ST^{a_k} ST^{a_{k-1}} \cdots ST^{a_1}$, where $a_j \in \mathbb{Z}$, $|a_j| \geqslant 2$, and $a_j a_{j+1} = -4$ if $|a_j| = |a_{j+1}| = 2$. (The transfer operator $\tilde{\mathcal{L}}_s$ is designed to reflect this condition on the a_j in nearest integer continuous fraction expansions.) Each ST^{a_j} maps the unit disk D into itself. For the imaginary part y of $z \in D$ we have

$$\operatorname{Im}\bigl(ST^{a_j} z\bigr) = \frac{y}{|z + a_j|^2} \leqslant \frac{y}{(|a_j| - 1)^2}.$$

So each ST^{a_j} with $|a_j| \geqslant 3$ decreases the imaginary part with at least a factor 4. If $a_j = \pm 2$ the imaginary part does not increase and if moreover $j < k$, we know that $a_{j+1} \neq a_j$. One checks for $a_j a_{j+1} \geqslant 6$ and $a_j a_{j+1} \leqslant -4$ separately that the imaginary part decreases at least with a factor 4 under $ST^{a_{j+1}} ST^{a_j}$. So if k is sufficiently large, all $ST^{a_k} \cdots ST^{a_1} z$ with $z \in D$ are contained in the ε-neighborhood U of $[-1, 1]$ we started with. For such k, all explicit terms in (41) can be absorbed in the last term. □

These propositions imply:

Corollary 3.9. *Let* $0 < \operatorname{Re}(s) < 1$, $s \neq \frac{1}{2}$. *Restriction and analytic extension give a bijective correspondence between* $\ker(\tilde{\mathcal{L}}_s - 1)$ *and the space of solutions in* $\mathcal{V}_s^\omega(-\phi^2, \phi) \times \mathcal{V}_s^\omega(-\phi, \phi^2)$ *of*

$$(g_1, g_2)|_{2s}\mathcal{L} = (g_1, g_2). \tag{42}$$

3.4. Four term equation

Proposition 3.10. *The solutions in $\mathcal{V}_s^\omega(-\phi^2,\phi)\times\mathcal{V}_s^\omega(-\phi,\phi^2)$ of $\vec{g}=\vec{g}|_{2s}\mathcal{L}$ are of the form $(g, g \mid T^{-1})$ with $g \in^4 \mathrm{FE}_s(-\phi^2,\phi)_\omega$.*

Proof. The eigenfunction relations

$$h_1 = h_1 \mid ST^3 \operatorname{Av}_T^+ - h_2 \mid ST^{-1} \operatorname{Av}_T^-,$$
$$h_2 = h_1 \mid ST^2 \operatorname{Av}_T^+ - h_2 \mid ST^{-2} \operatorname{Av}_T^-,$$

imply $h_2 = h_1 \mid T^{-1}$. So they are equivalent to

$$h = h \mid ST^3 \operatorname{Av}_T^+ - h \mid T^{-1}ST^{-1} \operatorname{Av}_T^- \quad \text{on } \left(-\phi^2,\phi\right). \tag{43}$$

Applying $|(1-T)$ to (43), and using (31), we get

$$h \mid (1-T) = h \Bigm| \left(ST^3 - T^{-1}ST^{-1}\right),$$

as an identity in $\mathcal{V}_s^\omega(-\phi^2,\phi^{-1})$. Apply $|_{2s}T^{-1}$ to obtain as an equality in $\mathcal{V}_s^\omega(-\phi,\phi)$

$$h \Bigm| \left(1+ST^2\right) = h \Bigm| \left(T^{-1}+T^{-1}ST^{-2}\right), \tag{44}$$

which is (5). □

4. Four term equation and cohomology

In the case of the three term equation (3) the step from solutions on $(0,\infty)$ to cohomology has been discussed in Section 3.2. In the case of the four term equation (5), the situation is more complicated, but the approach is essentially based on the same ideas. The main steps are indicated in Section 2.5. In this section we prove Theorems 2.3 and 2.5.

4.1. A model for parabolic and hyperbolic cohomology

Instead of the inhomogeneous cocycles used in Section 2.3 to describe the first cohomology group, one can also employ homogeneous ones. To an inhomogeneous cocycle ψ corresponds the *homogeneous cocycle* $\tilde{c}:\Gamma^2\to V$ given by

$$\tilde{c}_{\gamma,\delta} = \psi_{\gamma\delta^{-1}} \mid \delta \quad (\gamma,\delta\in\Gamma). \tag{45}$$

satisfying for $\gamma,\delta,\varepsilon\in\Gamma$

$$\tilde{c}_{\gamma\varepsilon,\delta\varepsilon} = \tilde{c}_{\gamma,\delta} \mid \varepsilon \quad \text{and} \quad \tilde{c}_{\gamma,\delta}+\tilde{c}_{\delta,\varepsilon} = \tilde{c}_{\gamma,\varepsilon}. \tag{46}$$

(This implies that $\tilde{c}_{\gamma,\gamma}=0$ and $\tilde{c}_{\delta,\gamma}=-\tilde{c}_{\gamma,\delta}$.) Coboundaries in $B^1(\Gamma;V)$ correspond to functions

$$(\gamma,\delta)\mapsto v \mid \delta - v \mid \gamma \tag{47}$$

with $v\in V$.

Homogeneous cocycles are parabolic if and only if by adding a coboundary, we can arrange $\tilde{c}_{T,1}=0$.

A homogeneous parabolic cocycle satisfies

$$\tilde{c}_{T\gamma,\delta} = \tilde{c}_{T\gamma,\gamma} + \tilde{c}_{\gamma,\delta} = 0 \mid \gamma + \tilde{c}_{\gamma,\delta} \quad \text{and} \tag{48}$$

$$\tilde{c}_{\gamma,T\delta} = \tilde{c}_{\gamma,\delta} + \tilde{c}_{1,T} \mid \delta = \tilde{c}_{\gamma,\delta} - 0 \mid \delta. \tag{49}$$

So $\tilde{c}$ induces a function $(\Gamma_\infty \setminus \Gamma)^2 \to V$, where Γ_∞ is the subgroup of Γ generated by T, which is the subgroup fixing ∞. Conversely, every such function satisfying (46) induces a homogeneous parabolic cocycle. Taking into account that the set $\Gamma_\infty \setminus \Gamma$ can be identified with the set of cusps $\mathbb{P}^1_{\mathbb{Q}} = \mathbb{Q} \cup \infty \subset \mathbb{P}^1_{\mathbb{R}}$ via $\gamma \mapsto \gamma^{-1}\infty$, we obtain the description of $H^1_{\text{par}}(\Gamma; V)$ as the quotient $Z^1_{\mathbb{P}^1_{\mathbb{Q}}}(\Gamma; V)/B^1_{\mathbb{P}^1_{\mathbb{Q}}}(\Gamma; V)$, where $Z^1_{\mathbb{P}^1_{\mathbb{Q}}}(\Gamma; V)$ is the space of maps $c : \mathbb{P}^1_{\mathbb{Q}} \times \mathbb{P}^1_{\mathbb{Q}} \to V$ such that $c_{\xi,\eta} + c_{\eta,\zeta} = c_{\xi,\zeta}$ and $c_{\gamma^{-1}\xi,\gamma^{-1}\eta} = c_{\xi,\eta} \mid \gamma$ for $\gamma \in \Gamma$ and $\xi, \eta, \zeta \in \mathbb{P}^1_{\mathbb{Q}}$, and where $B^1_{\mathbb{P}^1_{\mathbb{Q}}}(\Gamma; V)$ consists of the subset of elements of the form $c_{\xi,\eta} = f_\eta - f_\gamma$ with $f : \mathbb{P}^1_{\mathbb{Q}} \to V$ satisfying $f_{\gamma^{-1}\xi} = f_\xi \mid \gamma$.

If we take another base point $\xi \in \mathbb{P}^1_{\mathbb{R}}$, we can work with cocycles of the same type on other Γ-orbits in $\mathbb{P}^1_{\mathbb{R}}$. If the subgroup Γ_ξ of Γ leaving ξ fixed is trivial, we get a description of $H^1(\Gamma; V)$. In fact a homogeneous cocycle $\tilde{c}$ on Γ^2 corresponds to a cocycle c on $\Gamma\xi \subset \mathbb{P}^1_{\mathbb{R}}$ by

$$\tilde{c}_{\gamma,\delta} = c_{\gamma^{-1}\xi,\delta^{-1}\xi}. \tag{50}$$

The situation is different if ξ is a hyperbolic fixed point of Γ. Then the procedure indicated above leads to the hyperbolic cohomology group discussed in Section 2.5. A corresponding homogeneous group cocycle $\psi : \gamma \mapsto \psi_\gamma = c_{\gamma^{-1}\xi,\xi}$ satisfies

$$\psi_H \in V \mid (1 - H)$$

for a generator H of the stabilizer Γ_ξ of the point ξ. We will apply this for the stabilizer $\Gamma_{-\phi}$ for the golden ratio $\phi = \frac{1+\sqrt{5}}{2}$. A generator of $\Gamma_{-\phi}$ is TST^2. Thus, we have a model of H^1_{Fib}. A group cocycle corresponding ψ to the cocycle c on Fib is given by $\psi_\gamma = c_{\gamma^{-1}(-\phi),-\phi}$. In particular, c is determined by $c_{-\phi,\phi^{-1}} = -\psi_S$ and $c_{-\phi,\phi} = -\psi_{T^{-1}S}$, subject to the relations

$$c_{-\phi,\phi^{-1}} \mid (1 + S) = 0, \qquad c_{-\phi,\phi} \bigm| \left(1 + T^{-1}S + ST\right) = 0; \tag{51}$$

see (20). Such a cocycle is a coboundary if there is $v \in V$ such that $v \mid TST^2 = v$ and $c_{\gamma^{-1}(-\phi),\delta^{-1}(-\phi)} = v \mid \delta - v \mid \gamma$ for $\gamma, \delta \in \Gamma$.

4.2. Solutions of the four term equation and hyperbolic cohomology

In this subsection we prove Theorem 2.3 of Section 2.5. We work with cocycles on $\text{Fib} = \Gamma(-\phi)$ as the model of hyperbolic cohomology. For the computations we found the graph in Fig. 2 useful.

Lemma 4.1. *Suppose that c is a 1-cocycle on* Fib *with values in $\mathcal{V}_s^{\omega^*}$ such that* $\text{Sing}(c_{-\phi^2,\phi}) \cap (-\phi^2, \phi) = \emptyset$. *Then the restriction of $c_{-\phi^2,\phi}$ to the interval $(-\phi^2, \phi)$ is in ${}^4\text{FE}_s(-\phi^2, \phi)_\omega$.*

Proof. Cocycles on Fib satisfy the relations in (46). Hence

$$\begin{aligned} c_{-\phi^2,\phi} \bigm| \left(1 - T^{-1}ST^{-2}\right) &= c_{-\phi^2,\phi} - c_{\phi^2,\phi} = c_{-\phi^2,\phi^2} \\ &= c_{-\phi^2,-\phi} + c_{-\phi,\phi^2} = c_{-\phi^2,\phi} \bigm| \left(T^{-1} - ST^2\right). \end{aligned} \tag{52}$$

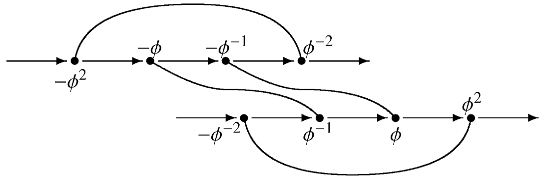

Fig. 2. Part of the Γ-orbit Fib. Arrows denote the action of T, curves the action of S. The horizontal coordinates of the points correspond to their positions in $\mathbb{R}$. Note that $\phi^2 = \phi + 1$, $\phi^{-1} = \phi - 1$ and $\phi^{-2} = 2 - \phi$.

The fact that the restriction g of $c_{-\phi^2,\phi}$ to $(-\phi^2, \phi)$ has no singularities in $(-\phi^2, \phi)$ implies that relation (52) holds on $(-\phi, \phi)$ as an identity of real analytic functions. Hence $g \in {}^4\mathrm{FE}_s(-\phi^2, \phi)_\omega$. $\square$

Proposition 4.2. *Each element $g \in {}^4\mathrm{FE}_s(-\phi^2, \phi)_\omega$ is the restriction*

$$c_{-\phi^2,\phi}|_{(-\phi^2,\phi)}$$

for a unique Γ-cocycle c on Fib *with values in $\mathcal{V}_s^{\omega^*}$. This cocycle satisfies*

$$\mathrm{Sing}(c_{-\phi^2,\phi}) \subset \{-\phi^2, \phi\}, \qquad \mathrm{Sing}(c_{-\phi,\phi^{-1}}) \subset \{-\phi, \phi^{-1}\}, \qquad \mathrm{Sing}(c_{-\phi,\phi}) \subset \{-\phi, \phi\}.$$

The induced map $\vartheta : {}^4\mathrm{FE}_s(-\phi^2, \phi)_\omega) \to H^1_{\mathrm{Fib}}(\Gamma; \mathcal{V}_s^{\omega^*})$ is injective.

Proof. First we assume that a cocycle c exists such that g is equal to the restriction of $c_{-\phi^2,\phi}$ to $(-\phi^2, \phi)$. The cocycle relations imply

$$\begin{aligned}
g \mid ST^2 &= c_{-\phi,-\phi^2} \quad \text{on } (-\phi, -\phi^2)_c,\\
g \mid (1 + ST^2) &= c_{-\phi^2,\phi} + c_{-\phi,-\phi^2} = c_{-\phi,\phi} \quad \text{on } (-\phi, \phi),\\
g \mid (1 + ST^2)STS &= c_{\phi^2,\phi^{-1}} \quad \text{on } (\phi^2, \phi^{-1})_c,\\
g \mid T^{-1} &= c_{-\phi,\phi^2} \quad \text{on } (-\phi, \phi^2),\\
g \mid (T^{-1} + (1 + ST^2)STS) &= c_{-\phi,\phi^2} + c_{\phi^2,\phi^{-1}} = c_{-\phi,\phi^{-1}} \quad \text{on } (-\phi, \phi^{-1}).
\end{aligned}$$

This shows that the restriction of $c_{-\phi,\phi^{-1}}$ to $(-\phi, \phi^{-1})$ is determined by g. From (51) we know that $c_{-\phi,\phi^{-1}} \mid S = -c_{-\phi,\phi^{-1}}$. Hence g determines the restriction of $c_{-\phi,\phi^{-1}}$ to $(\phi^{-1}, -\phi)_c$ as well. So g determines $c_{-\phi,\phi^{-1}}$ as an element of $\mathcal{V}_s^{\omega^*}$.

The situation for the other generator $c_{-\phi,\phi}$ is slightly more complicated.

$$\begin{aligned}
c_{-\phi,\phi} &= g \mid (1 + ST^2) \quad \text{on } (-\phi, \phi),\\
c_{\phi,\phi^{-2}} = c_{-\phi,\phi} \mid T^{-1}S &= g \mid (1 + ST^2) \mid T^{-1}S \quad \text{on } (\phi, \phi^{-2})_c,\\
c_{\phi^{-2},-\phi} = c_{-\phi,\phi} \mid ST &= g \mid (1 + ST^2) \mid ST \quad \text{on } (\phi^{-2}, -\phi)_c.
\end{aligned}$$

The relation $c_{-\phi,\phi} + c_{\phi,\phi^{-2}} + c_{\phi^{-2},-\phi} = 0$ implies that $c_{-\phi,\phi}$ is determined by g on each of the cyclic intervals $(-\phi, \phi^{-2})$, (ϕ^{-2}, ϕ) and $(\phi, -\phi)_c$. So any $\mathcal{V}_s^{\omega^*}$-valued cocycle c on Fib is determined by the restriction of $c_{-\phi^2,\phi}$ to $(-\phi^2, \phi)$.

This reasoning also shows how to construct c from g. We put for a given function $g \in {}^4\mathrm{FE}_s(-\phi^2, \phi)_\omega$:

$$h = g \bigm| \left(1 + ST^2\right) \in \mathcal{V}_s^\omega(-\phi, \phi),$$
$$k = g \mid T^{-1} + h \mid STS \in \mathcal{V}_s^\omega\left(-\phi, \phi^{-1}\right). \tag{53}$$

By the reasoning given above we should have:

$$c_{-\phi,\phi^{-1}} := \begin{cases} k & \text{on } (-\phi, \phi^{-1}), \\ -k \mid S & \text{on } (\phi^{-1}, -\phi)_c, \end{cases}$$
$$c_{-\phi,\phi} := \begin{cases} h & \text{on } (-\phi, \phi), \\ -h \mid (T^{-1}S + ST) & \text{on } (\phi, -\phi)_c. \end{cases} \tag{54}$$

To see that this indeed defines a cocycle, we choose the base point $-\phi$ and consider not the potential cocycle c on Fib, but the corresponding cocycle ψ on Γ:

$$\psi_S = -c_{-\phi,\phi^{-1}}, \qquad \psi_{T^{-1}S} = -c_{-\phi,\phi}.$$

The relations (20) turn out to be satisfied. So indeed there exists a cocycle c as desired.

For the singularities, we note first that the expressions above for $c_{-\phi,\phi}$ and $c_{-\phi,\phi^{-1}}$ in terms of g imply that

$$\mathrm{Sing}(c_{-\phi,\phi^{-1}}) \subset \left\{-\phi, \phi^{-1}\right\}, \qquad \mathrm{Sing}(c_{-\phi,\phi}) \subset \{-\phi, \phi\}.$$

The cocycle c is determined by $c_{-\phi,\phi^{-1}}$ and $c_{-\phi,\phi}$. We now check that the restriction of $c_{-\phi^2,\phi}$ to $(-\phi^2, \phi)$ is equal to g, as desired, and cannot have singularities in $(\phi, -\phi^2)_c$. The cocycle relations imply that

$$c_{-\phi^2,\phi} = c_{-\phi,\phi^{-1}} \mid T + c_{-\phi,\phi} + c_{-\phi,\phi} \mid ST. \tag{55}$$

On various intervals we have:

	$(-\phi^2, -\phi)$	$(-\phi, -\phi^{-2})$	$(-\phi^{-2}, \phi)$	$(\phi, -\phi^2)_c$
$c_{-\phi,\phi^{-1}} \mid T$	$k \mid T$	$k \mid T$	$-k \mid ST$	$-k \mid ST$
$c_{-\phi,\phi}$	$-h \mid (T^{-1}S + ST)$	h	h	$-h \mid (T^{-1} + ST)$
$c_{-\phi,\phi} \mid ST$	$h \mid ST$	$-h \mid (1 + T^{-1}S)$	$h \mid ST$	$h \mid ST$
$c_{-\phi^2,\phi}$	g	g	g	$-k \mid ST - h \mid T^{-1}$

On $(-\phi^{-2}, \phi)$ we use the four term equation:

$$\begin{aligned} c_{-\phi^2,\phi} &= -g \mid T^{-1}ST + h \bigm| \left(-ST^2 + 1 + ST\right) \\ &= g \bigm| \left(-T^{-1}ST + \left(1 + ST^2\right)\left(1 + ST - ST^2\right)\right) \\ &\overset{(44)}{=} -g \bigm| \left(1 + ST^2 - T^{-1}ST^{-2}\right)ST + g \bigm| \left(1 - ST^2ST^2 + ST + ST^2ST\right) \\ &= g \bigm| \left(T^{-1}ST^{-2}ST + 1 - ST^2ST^2\right) \overset{(9)}{=} g. \end{aligned}$$

Hence $c_{-\phi^2,\phi} = g$ on $(-\phi^2, -\phi) \cup (-\phi, -\phi^{-2}) \cup (-\phi^{-2}, \phi)$. Since g is analytic on $(-\phi^2, \phi)$, the points $-\phi$ and $-\phi^{-2}$ are not singularities of $c_{-\phi^2,\phi}$, by the definition of $\mathcal{V}_s^{\omega^*}$ as an inductive limit. Furthermore $c_{-\phi^2,\phi}$ is given by the analytic function $-k \mid ST - h \mid T^{-1}$ on $(\phi, -\phi^2)_c$. This shows that $\mathrm{Sing}(c_{-\phi^2,\phi}) \subset \{-\phi^2, \phi\}$.

To show that the map $\vartheta : {}^4\mathrm{FE}_s(-\phi^2,\phi)_\omega \to H^1_{\mathrm{Fib}}(\Gamma;\mathcal{V}_s^{\omega^*})$ given by $\vartheta : g \mapsto [c]$ is injective, we check that the cocycle c corresponding to g can be a coboundary only if $g=0$. If c is a coboundary, then $c_{\gamma^{-1}(-\phi),\delta^{-1}(-\phi)} = v \mid \delta - v \mid \gamma$. In particular, $v \mid TST^2$ should be equal to v for some $v \in \mathcal{V}_s^{\omega^*}$.

We use that TST^2 is a hyperbolic element of Γ fixing $-\phi$ and ϕ^{-1}. Conjugation in $\mathrm{PSL}(2,\mathbb{R})$ transforms it to $\eta = \left[\begin{smallmatrix}\phi^2 & 0\\ 0 & \phi^{-2}\end{smallmatrix}\right]$, fixing 0 and ∞. Let $w \in \mathcal{V}_s^{\omega^*}$ be invariant under η. The action of η on $(0,\infty)$ and $(-\infty,0)$ is by $x \mapsto \phi^4 x$. So if w were to have singularities in $\mathbb{R}\setminus\{0\}$, then there would be infinitely many, contradicting the definition of $\mathcal{V}_s^{\omega^*}$. On $\mathbb{R}$ we have $w \mid \eta(x) = \phi^{4s} w(\phi^4 x)$. Inserting this into a power series expansion converging on a neighborhood of 0, we see that if w is analytic at 0 it vanishes. The same holds at ∞. So if $w \neq 0$, then $\mathrm{Sing}(w) = \{0,\infty\}$.

Conjugating back, we see that if c is a non-zero coboundary, then it has the form $c_{\gamma^{-1}(-\phi),\delta^{-1}(-\phi)} = v \mid (\delta-\gamma)$, where $v \in \mathcal{V}_s^{\omega^*}$ satisfies $v \mid TST^2 = v$ and $\mathrm{Sing}(v) = \{-\phi,\phi^{-1}\}$. We consider the singularities of $c_{-\phi^2,\phi} = c_{T^{-1}(-\phi),TS(-\phi)} = v \mid ST^{-1} - v \mid T$. Now $\mathrm{Sing}(v \mid ST^{-1}) = \{\phi,-\phi^{-1}\}$, and $\mathrm{Sing}(v \mid T) = \{-\phi^2,-\phi^{-2}\}$. So $c_{-\phi^2,\phi}$ has singularities at $-\phi^{-1}$ and $-\phi^{-2}$, in contradiction to the analyticity of g on $(-\phi^2,\phi)$. □

We note that $H^1_{\mathrm{Fib}}(\Gamma;\mathcal{V}_s^{\omega^*})$ is a subspace of $H^1(\Gamma;\mathcal{V}_s^{\omega^*})$. The inclusion $\mathcal{V}_s^{\omega} \hookrightarrow \mathcal{V}_s^{\omega^*}$ induces a natural map $H^1(\Gamma;\mathcal{V}_s^{\omega}) \to H^1(\Gamma;\mathcal{V}_s^{\omega^*})$.

Proposition 4.3. *The subspace $\vartheta({}^4\mathrm{FE}_s(-\phi^2,\phi)_\omega)$ of $H^1(\Gamma;\mathcal{V}_s^{\omega^*})$ is contained in the image of $H^1(\Gamma;\mathcal{V}_s^{\omega})$ in $H^1(\Gamma;\mathcal{V}_s^{\omega^*})$.*

Proof. Let $g \in^4 \mathrm{FE}_s(-\phi^2,\phi)_\omega$, and let c be the cocycle on Fib representing $\vartheta(\gamma)$. Since $\mathrm{Sing}(c_{-\phi,\phi^{-1}}) \subset \{-\phi,\phi^{-1}\}$, Proposition 3.5 implies that there are $K_{-\phi}$ and $K_{\phi^{-1}}$ in $\mathcal{V}^{\omega^*}$ with $\mathrm{Sing}(K_{-\phi}) \subset \{-\phi\}$ and $\mathrm{Sing}(K_{\phi^{-1}}) \subset \{\phi^{-1}\}$ such that $c_{-\phi,\phi^{-1}} = K_{-\phi} - K_{\phi^{-1}}$. Since $c_{-\phi,\phi^{-1}} \mid (1+S) = 0$, we have $K_{\phi^{-1}} = K_{-\phi} \mid S + \alpha$ for some $\alpha \in \mathcal{V}_s^{\omega}$. Similarly, there are $H_{-\phi}, H_\phi \in \mathcal{V}_s^{\omega^*}$ with $\mathrm{Sing}(H_{-\phi}) \subset \{-\phi\}$ and $\mathrm{Sing}(H_\phi) \subset \{\phi\}$ such that $c_{-\phi,\phi} = H_{-\phi} - H_\phi$. There exists a function $\beta \in \mathcal{V}_s^{\omega}$ such that $H_\phi = H_{-\phi} \mid T^{-1}S + \beta$.

We have, with (55):

$$c_{-\phi^2,\phi} \in c_{-\phi,\phi^{-1}} \mid T + c_{-\phi,\phi} \mid (1+ST) \in K_{-\phi} \mid (T-ST) + H_{-\phi} \,\big|\, \big(ST - T^{-1}S\big) + \mathcal{V}_s^{\omega}.$$

Since $\mathrm{Sing}(c_{-\phi^2,\phi}) \subset \{-\phi^2,\phi\}$, we conclude from the singularities of the various terms that $K_{-\phi} \mid ST \in H_{-\phi} \mid ST + \mathcal{V}_s^{\omega}$. Hence

$$K_{-\phi} - H_{-\phi} \in \mathcal{V}_s^{\omega}. \tag{56}$$

The class $[c] \in H^1_{\mathrm{Fib}}(\Gamma;\mathcal{V}_s^{\omega^*})$ considered as a class in $H^1(\Gamma;\mathcal{V}_s^{\omega^*})$ is given by the group cocycle

$$\psi_S = -c_{-\phi,\phi^{-1}}, \qquad \psi_{T^{-1}S} = -c_{-\phi,\phi}.$$

We add to it the coboundary $\gamma \mapsto K_{-\phi} \mid (1-\gamma)$, obtaining a cocycle $\tilde\psi$ in the same class. It satisfies

$$\begin{aligned}\tilde\psi_S &= -K_{-\phi} + K_{\phi^{-1}} + K_{-\phi} \mid (1-S) = \alpha \in \mathcal{V}_s^{\omega},\\ \tilde\psi_{T^{-1}S} &= -H_{-\phi} + H_\phi + K_{-\phi} \,\big|\, \big(1 - T^{-1}S\big) = (K_{-\phi} - H_{-\phi}) \,\big|\, \big(1 - T^{-1}S\big) + \beta \in \mathcal{V}_s^{\omega}.\end{aligned} \tag{57}$$

Thus the class $[\psi] = [\tilde\psi]$ in $H^1(\Gamma;\mathcal{V}_s^{\omega^*})$ is the image of the class $[\tilde\psi] \in H^1(\Gamma;\mathcal{V}_s^{\omega})$. □

Proposition 4.4. *The subspace $\vartheta({}^4\mathrm{FE}_s(-\phi^2,\phi)_\omega)$ of $H^1(\Gamma;\mathcal{V}_s^{\omega^*})$ is equal to the image of $H^1(\Gamma;\mathcal{V}_s^{\omega})$.*

Proof. We use a hyperbolic one-sided average, as discussed at the end of Section 3.1. Starting with a cocycle $\tilde{\chi} \in Z^1(\Gamma; \mathcal{V}_s^{\omega})$ we define

$$A = \tilde{\chi}_{TST^2} \mid \mathrm{Av}^+_{TST^2} \in \mathcal{V}_s^{\omega^*},$$

with $\mathrm{Sing}(A) \subset \{-\phi\}$ and $A \mid (1 - TST^2) = \tilde{\chi}_{TST^2}$. We have used that $-\phi$ is the repelling fixed point of TST^2.

Now $\chi : \gamma \mapsto \tilde{\chi}_\gamma - A \mid (1-\gamma)$ defines a $\mathcal{V}_s^{\omega^*}$-valued cocycle in the same cohomology class in $H^1(\Gamma; \mathcal{V}_s^{\omega^*})$ as $\tilde{\chi}$. It satisfies $\mathrm{Sing}(\chi_\gamma) \subset \{-\phi, \gamma^{-1}(-\phi)\}$. Moreover, $\chi_{TST^2} = 0$, so χ is hyperbolic for the conjugacy class of TST^2. Hence it corresponds to a cocycle c on Fib, such that $c_{-\phi^2,\phi} = A \mid T - A \mid ST^{-1} - \tilde{\chi}_{ST^{-2}} \mid T$ has singularities at most in $\{-\phi^2, \phi\}$. This means that the class of $\tilde{\chi}$ is in the image of ϑ.

Starting from this cocycle c, one can take $K_{-\phi} = H_{-\phi} = A$, and get back $\tilde{\psi} = \tilde{\chi}$. □

Proof of Theorem 2.3. We have to establish an injection ϑ from ${}^4\mathrm{FE}(-\phi^2,\phi)$ into $H^1_{\mathrm{Fib}}(\Gamma; \mathcal{V}_s^{\omega^*})$, and to show that the image is equal to the image of $H^1(\Gamma; \mathcal{V}_s^{\omega})$ under the natural map to $H^1(\Gamma; \mathcal{V}_2^{\omega^*})$. Proposition 4.2 gives the injective map ϑ. Proposition 4.4 shows the equality of the images. □

4.3. Eigenfunctions of the transfer operator and cohomology

In this final section we prove Theorem 2.5 of Section 2.5.

We use the same notations as in the previous subsection:

$$g \in {}^4\mathrm{FE}_s\big(-\phi^2,\phi\big)_\omega, \qquad c \in Z^1_{\mathrm{Fib}}\big(\Gamma; \mathcal{V}_s^{\omega^*}\big), \qquad \psi \in Z^1\big(\Gamma; \mathcal{V}_s^{\omega^*}\big), \qquad \tilde{\psi} \in Z^1\big(\Gamma; \mathcal{V}_s^{\omega}\big),$$

and $K_{-\phi} \in \mathcal{V}_s^{\omega^*}$ with $\mathrm{Sing}(K_{-\phi}) \subset \{-\phi\}$, related by:

$$g = c_{-\phi^2,\phi} \quad \text{restricted to } \big(-\phi^2,\phi\big), \tag{58}$$

$$c_{\gamma^{-1}(-\phi),\delta^{-1}(-\phi)} = \psi_{\gamma\delta^{-1}} \mid \delta, \quad \psi_\gamma = -c_{-\phi,\gamma^{-1}(-\phi)}, \tag{59}$$

$$\tilde{\psi}_\gamma = \psi_\gamma + K_{-\phi} \mid (1-\gamma), \tag{60}$$

$$K_{-\phi} = \tilde{\psi}_{TST^2} \mid \mathrm{Av}^+_{TST^2}. \tag{61}$$

We consider $P \in \mathcal{V}_s^{\omega}(-\phi^2,\phi)$ given by

$$P = g \mid ST^3 \,\mathrm{Av}^+_T - g \mid T^{-1}ST^{-1}\,\mathrm{Av}^-_T - g. \tag{62}$$

So P measures how much $(g, g \mid T^{-1})$ differs from an 1-eigenfunction of $\mathcal{L}$ in (38). Application of (31) shows that $P \mid T = P$ on $(-\phi^2, \phi^{-1})$. Hence the periodic function P extends as an element of $\mathcal{V}_s^{\omega}(\mathbb{R})^T \subset \mathcal{V}_s^{\omega^*}$. To obtain a representation of P on $\mathbb{R}$, we use (58)–(60) to obtain on $(-\phi^2,\phi)$:

$$\begin{aligned} P &= \psi_{TST} \mid T^{-1}S\big(ST^3\,\mathrm{Av}^+_T - T^{-1}ST^{-1}\,\mathrm{Av}^-_T - 1\big) \\ &= \tilde{\psi}_{TST} \,\big|\, \big(T^2\,\mathrm{Av}^+_T - S\,\mathrm{Av}^-_T - T^{-1}S\big) \\ &\quad + K_{-\phi} \,\big|\, \big(\big(TST^3 - T^2\big)\,\mathrm{Av}^+_T + \big(S - ST^{-1}\big)\,\mathrm{Av}^-_T + T^{-1}S - T\big). \end{aligned}$$

From (61) it follows that $K_{-\phi} \mid TST^2 = K_{-\phi} - \tilde{\psi}_{TST^2}$. With (32) we find for the contribution of $K_{-\phi}$:

$$\begin{aligned}
&K_{-\phi} \mid \left(T-T^2\right)\mathrm{Av}_T^+ - \tilde{\psi}_{TST^2} \mid T\,\mathrm{Av}_T^+ - K_{-\phi} \mid ST^{-1}(1-T)\,\mathrm{Av}_T^- + K_{-\phi} \mid \left(T^{-1}S - T\right)\\
&\quad = K_{-\phi} \mid \left(T - ST^{-1} + T^{-1}S - T\right) - \tilde{\psi}_{TST^2} \mid T\,\mathrm{Av}_T^+\\
&\quad = \left(K_{-\phi} \mid TST^2 + \tilde{\psi}_{TST^2}\right) \mid T^{-1}S - K_{-\phi} \mid ST^{-1} - \tilde{\psi}_{TST^2} \mid T\,\mathrm{Av}_T^+\\
&\quad = \tilde{\psi}_{TST^2} \mid \left(T^{-1}S - T\,\mathrm{Av}_T^+\right).
\end{aligned}$$

This leads to an expression valid on $\mathbb{R}$:

$$\begin{aligned}
P &= \tilde{\psi}_{TST} \mid \left(T^2\,\mathrm{Av}_T^+ - S\,\mathrm{Av}_T^- - T^{-1}S + T\left(T^{-1}S - T\,\mathrm{Av}_T^+\right)\right)\\
&\quad + \tilde{\psi}_T \mid \left(T^{-1}S + 1 - \mathrm{Av}_T^+\right)\\
&= \tilde{\psi}_{TST} \mid \left(S - T^{-1}S\right) + \tilde{\psi}_T \mid \left(T^{-1}S + 1\right) - \tilde{\psi}_T \mid \mathrm{Av}_T^+ - \tilde{\psi}_{TST} \mid S\,\mathrm{Av}_T^-\,. \qquad (63)
\end{aligned}$$

Proof of Theorem 2.5. Suppose that $P = 0$, i.e., $g \in {}^4\mathrm{FE}_s(-\phi^2, \phi)_\omega$ corresponds to an eigenfunction $(g, g \mid T^{-1})$ of $\mathcal{L}$. We note that $\tilde{\psi}_{TST}, \tilde{\psi}_T \in \mathcal{V}_s^\omega$, and apply Lemma 3.3 to conclude that $\tilde{\psi}_T \mid \mathrm{Av}_T^+ = \tilde{\psi}_T \mid \mathrm{Av}_T^- \in \mathcal{V}_s^{\omega^*,\mathrm{simple}}$, hence $\tilde{\psi}_T \in \mathcal{V}_s^{\omega^*,\mathrm{simple}} \mid (1-T)$, which is the condition for a cocycle to be parabolic. Thus we conclude that $[\tilde{\psi}] \in H^1_{\mathrm{par}}(\Gamma; \mathcal{V}_s^\omega, \mathcal{V}_s^{\omega^*,\mathrm{simple}})$.

Conversely, suppose that $\tilde{\psi} \in Z^1_{\mathrm{par}}(\Gamma; \mathcal{V}_s^\omega, \mathcal{V}_s^{\omega^*,\mathrm{simple}})$. Now $\tilde{\psi}_T \in \mathcal{V}_s^{\omega^*,\mathrm{simple}} \mid (1-T)$ by parabolicity, and hence $\tilde{\psi}_T \mid \mathrm{Av}_T^+ = \tilde{\psi}_T \mid \mathrm{Av}_T^-$ by Lemma 3.2. In (63) we see that P is in $\mathcal{V}_s^\omega + \mathcal{V}_s^\omega \mid \mathrm{Av}_T^-$. Hence $P(x)$ has an asymptotic behavior as $x \downarrow -\infty$ of the form indicated in (36). Since P is also periodic, it vanishes. □

Now we have carried out all steps indicated in the diagram in Fig. 1 at the end of Section 2.5. We close the paper with a few remarks on the methods we have used:

- In the context of dynamical systems, eigenfunctions of transfer operators are functions on disks in the complex plane. Restriction gives analytic functions on intervals in $\mathbb{R}$. To go back from analytic eigenfunctions of transfer operators on an interval to functions on a disk or even larger spaces, we need bootstrap methods. In the case of $\tilde{\mathcal{L}}_s$ we have not investigated whether further extension is possible than to the unit disk in Proposition 3.8.
- Transfer operators are given by infinite sums. The eigenfunctions considered here can be characterized by finite linear relations, which may be characterized as cocycle relations on suitable generators of the discrete group under consideration.
- The cocycle relations considered here define a parabolic and a hyperbolic cohomology group with analytic coefficients. To relate these analytic cohomology groups we have used a larger cohomology group with semi-analytic coefficients, where coefficients with a finite number of singularities are allowed.

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