Embeddings of polyhedra in $\mathbb{R}^m$ and the deleted product obstruction

J. Segal$^a$,* A. Skopenkov$^b$, S. Spież$^c$

$^a$ Department of Mathematics, University of Washington, Seattle, WA 98195, USA
$^b$ Chair of Differential Geometry, Department of Mechanics and Mathematics, Moscow State University, Moscow 119899, Russia
$^c$ Institute of Mathematics, Polish Academy of Sciences, ul. Śniadeckich 8, 00-950 Warsaw, Poland

Received 14 November 1996; revised 16 April 1997

Abstract

Weber has proved that if $2m \geq 3(n + 1)$ then an $n$-dimensional polyhedron $K$ embeds in $\mathbb{R}^m$ if and only if there exists an equivariant map from the deleted product $K^*$ into the sphere $S^{m-1}$. As a consequence he has obtained that in the same range of dimensions an $n$-dimensional polyhedron embeds in $\mathbb{R}^m$ if and only if it quasi embeds in $\mathbb{R}^m$. We show that for $m \geq \max(4, n)$ the dimension restrictions in Weber's results are necessary in all cases. This leaves only two open cases remaining (namely $m = 3$ and $n = 2$ or 3) in related questions about embeddings. © 1998 Elsevier Science B.V.

Keywords: Embedding; Quasi embedding; Deleted product; Finger move; Smith index; Whitehead product; Hilton's theorem

AMS classification: Primary 57Q35; 54C25, Secondary 55S15; 55Q15

0. Introduction

The well-known Menger Embedding Theorem [22] says that any $n$-dimensional separable metric space $X$ can be embedded in $(2n + 1)$-dimensional Euclidean space $\mathbb{R}^{2n+1}$. If $X$ is a $n$-dimensional (compact) polyhedron then simply any PL-map in general position of $X$ into $\mathbb{R}^{2n+1}$ is an embedding. In general, the dimension of the Euclidean space cannot be decreased.

In 1930, Kuratowski [17] showed that a graph can be embedded in $\mathbb{R}^2$ if and only if it does not contain an embedded copy of one of the two graphs $K_5$ or $K_{3,3}$. In 1933,
van Kampen [37] generalized these graphs to \( n \)-dimensional polyhedra which are not embeddable in \( \mathbb{R}^{2n} \) (see also [8]). In the same paper, he also gave a rough description of a certain \( \mathbb{Z}/2\mathbb{Z} \)-equivariant 2\( n \)-dimensional cohomology class of the deleted product of an \( n \)-dimensional polyhedron \( K \), which vanishes if and only if \( K \) is PL-embeddable in \( \mathbb{R}^{2n} \), provided \( n \geq 3 \). (For a given topological space \( X \), by the deleted product of \( X \) we understand the space \( X^* := \{(x, y) \in X \times X: x \neq y\} \) and we assume that \( \mathbb{Z}/2\mathbb{Z} \) acts on \( X^* \) by switching the coordinates.) By Kuratowski’s theorem and the naturality of the obstruction under embeddings, van Kampen’s result is also true for \( n = 1 \). Many details were clarified by Wu [41] and Shapiro [31].

Observe that if \( f: X \to \mathbb{R}^m \) is an embedding then the map \( f^* \) from the deleted product \( X^* \) into the sphere \( S^{m-1} \) defined by

\[
    f^*(x, y) = \frac{f(x) - f(y)}{||f(x) - f(y)||}
\]

for each \((x, y) \in X^*\), is an equivariant map. (We assume that \( \mathbb{Z}/2\mathbb{Z} \) acts on \( S^{m-1} \) by switching the antipodes, thus \( F: X^* \to S^{m-1} \) is an equivariant map if \( F(x, y) = -F(y, x) \) for each \((x, y) \in X^*\).)

Weber [38] (see also [13]) proved a converse to this when \( X \) is a polyhedron and the dimensions are in the so-called metastable range. His result extends van Kampen’s result to a wider range of dimensions and also generalizes an earlier theorem of Haefliger [12], on embedding (in the same range of dimensions) of compact differentiable manifolds into Euclidean spaces.

**Theorem 1** [38]. Suppose \( K \) is an \( n \)-dimensional compact polyhedron and \( m \) an integer such that \( 2m \geq 3(n + 1) \). Then the following holds:

(W) If there exist an equivariant map \( F: K^* \to S^{m-1} \) then there exists a PL-embedding \( f: K \to \mathbb{R}^m \) such that \( f^* \) is equivariantly homotopic to \( F \).

(A homotopy \( H: X^* \times [0, 1] \to S^{m-1} \) is equivariant if the map \( H_t: X^* \to S^{m-1} \), defined by \( H_t(z) = H(z, t) \), is equivariant for each \( t \in [0, 1] \). For a controlled version of Theorem 1 see [27].)

The dimension restrictions in Weber’s theorem are due to the use of the Freudenthal Suspension Isomorphism Theorem and general position arguments. The first step of the proof is that from the existence of an equivariant map \( K^* \to S^{m-1} \) follows the existence of an almost embedding \( K \to \mathbb{R}^m \). (An almost embedding (see [10]) is a PL-map in general position such that the pairs of disjoint simplexes in a triangulation of \( K \) do not intersect in the image.) The second, more difficult step is that from the existence of an almost embedding \( K \to \mathbb{R}^m \) follows the existence of a PL-embedding \( K \to \mathbb{R}^m \) (for a short proof see [32]).

In 1966, using [5], Mardešić and Segal [19] proved that a polyhedron is not embeddable in \( \mathbb{R}^2 \) if and only if it contains a copy of \( K_5 \), \( K_{3,3} \) or of a “spiked disc” \( \perp \); i.e., a space which consists of a disc and of an arc which have only one point in common and this point is an interior point of the disc and the end point of the arc (see also [16]). One can compute that the Smith index of each of the deleted products \( K_5^*, K_{3,3}^* \) and \( \perp^* \) is
equal 2. (The properties of the Smith index, which are needed in this paper, can be found in [34] or in [42].) Thus if a polyhedron \( K \) is nonplanar then the Smith index of the deleted product \( K^* \) is \( \geq 2 \) and consequently there is no equivariant map of \( K^* \) into \( S^1 \). Therefore from the existence of an equivariant map \( F : K^* \to S^1 \) follows the existence of an embedding \( f : K \to \mathbb{R}^2 \); additionally one can prove that \( f \) can be chosen so that \( f^* \) is equivariantly homotopic to \( F \). So statement (W) is true if \( m = 2 \). One can also prove that (W) is true if \( m = 0 \) or \( 1 \).

The Smith index of the deleted product \( K^* \) of an \( n \)-dimensional polyhedron \( K \) is \( 3n - 1 \). Consequently there is no equivariant map \( K^* \to S^{m-1} \) and so the statement (W) is true if \( m < n \).

A continuous map \( f \) from a metric space \( X \) onto a space \( Y \) is an \( \varepsilon \)-map, for \( \varepsilon > 0 \), if for each \( y \in Y \), the diameter of \( f^{-1}(y) \) is less the \( \varepsilon \). \( X \) quasi embeds in \( \mathbb{R}^m \) if for each \( \varepsilon > 0 \) there exists an \( \varepsilon \)-map from \( X \) onto some subspace in \( \mathbb{R}^m \). For polyhedra this condition is equivalent to the following: A polyhedron \( X \) quasi embeds in \( \mathbb{R}^m \) if for each triangulation of \( X \) there exists an almost-embedding of \( X \) in \( \mathbb{R}^m \).

One can consider the following:

**Problem** [20]. For which pairs \((m, n)\) of positive integers is the statement (S) below true?

(S) If a \( n \)-dimensional polyhedron quasi embeds in \( \mathbb{R}^m \), then it embeds in \( \mathbb{R}^m \).

Among the reasons for considering such a problem is that its negative solution (such as in this paper) implies that even for the deleted product cube (or the \( p \)-fold deleted product) the necessary condition is not sufficient for the embeddability of polyhedra in \( \mathbb{R}^m \).

By the Menger Embedding Theorem one has that for \((2n+1, n)\), \( n \geq 0 \), the statement is true (stable range). In 1959, Ganea [11] showed, by using van Kampen’s and Kuratowski’s theorems that the statement is true for \((2n, n)\), \( n \neq 2 \). Mardešić and Segal [19], have shown (by applying their characterization above of nonplanar polyhedra) that the statement (S) is true if \( m = 2 \). Also the statement (S) is true if \( m = 0 \) or \( 1 \).

Weber ([39], also cf. [15]) observed that if a compact polyhedron \( X \) quasi embeds in \( \mathbb{R}^m \) then there exists an equivariant map \( X^* \to S^{m-1} \). Consequently, by Theorem 1, the statement (S) is true if \( 2m \geq 3(n+1) \), which generalizes Ganea’s result for \( n \geq 3 \).

**Corollary 2** [39]. If \( X \) is a compact \( n \)-dimensional polyhedron and \( m \) is an integer such that \( 2m \geq 3(n+1) \) then \( X \) embeds in \( \mathbb{R}^m \) if and only if \( X \) quasi embeds in \( \mathbb{R}^m \).

By Borsuk's version of the theorem on the invariance of domain for \( \varepsilon \)-maps [1], one has that the \( n \)-dimensional ball \( B^n \) is not quasi embeddable in \( \mathbb{R}^m \) if \( m < n \). (It also follows from that in this case there is no equivariant map \( (B^n)^* \to S^{m-1} \).) Therefore the statement (S) is true if \( m < n \).

On the other hand, Mardešić and Segal [20] have shown that \((n, n)\) and \((n, n - 1)\) are false for \( n \geq 4 \). It is known from the work of Curtis [6] and Mazur [21] that for \( n \geq 4 \) there are contractible combinatorial manifolds \( M^n \) such that \( \pi_1(\partial M^n) \neq 0 \) and
The cone over \((\partial M^n)\) is, for each \(n \geq 4\), an \(n\)-dimensional polyhedron \(P^n\) which can be quasi embedded in \(\mathbb{R}^n\), but nevertheless cannot be embedded in \(\mathbb{R}^n\).

Segal and Spie\'e [29], have shown that Weber's results are sharp in all but a finite number of cases. More precisely, the statement (S), for \((m, k)\) such that \(2m < 3(k + 1)\) and \(m \geq \max(7, k)\), is false except for the six cases \((10, 6), (11, 7), (12, 8), (22, 14), (23, 15)\) and \((24, 16)\). This is done by constructing first a \(k\)-dimensional polyhedron \(Q\) in \(\mathbb{R}^m\), containing two disjoint spheres of dimensions \(k\) and \(l\), \(0 < l < k\), such that for any embedding of \(Q\) into \(\mathbb{R}^m\), where \(m = k + l + 1\), the spheres link with an odd linking number. Then they use the higher dimensional finger moves (analogue of the classical Casson finger moves). These were first described by Whitehead to prove the hard part of the Freudenthal Suspension theorem [9, Section 10] and first used by Šepin to move compacta in \(\mathbb{R}^m\) apart, cf. [7,28,35]. They obtain (by modifying \(Q\)) \(k\)-dimensional polyhedra \(Y \subset \mathbb{R}^m\) and \(R\) such that \(R\) is not embeddable in \(\mathbb{R}^m\) but for any \(\varepsilon\) there exists an \(\varepsilon\)-map from \(R\) onto \(Y\).

In 1994 Freedman et al. [10] completed the line of investigation begun by van Kampen in 1933. They gave an example which shows that van Kampen's obstruction is not sufficient for embeddability of 2-complexes in \(\mathbb{R}^4\). As for many codimension 2 problems, the fundamental group of the complement plays an additional role (in contrast to higher codimensions).

Using the work of Segal and Spie\'e [29] and Freedman et al. [10], and finger moves we establish that the dimension restrictions in the above results of Weber, for \(m \geq \max(4, n)\), are necessary in all cases (for \(m = n + 2\) this was earlier done by the second named author [30], but the proof was not published before it was generalized to the result of the present paper). Therefore the only open cases are \((3, 2)\) and \((3, 3)\).

**Example.** For any pair of integers \((m, n)\) such that \(m \geq \max(4, n)\) and \(2m < 3(n + 1)\) there exists a \(n\)-dimensional polyhedron \(R\) which is quasi embeddable in \(\mathbb{R}^m\) but which is not embeddable in \(\mathbb{R}^m\).

The construction of the example is similar to the one in [29], but as in [10] we take a union of two copies of the polyhedron \(Q\). The proof of nonembeddability of \(R\) is based, for \(m > n + 2\), on Hilton's theorem on homotopy groups of wedges (or on an elementary argument using the homotopy exact sequence) instead (cf. [29]) of Adams' theorem on the Hopf invariant, and for \(m = n + 2\), on Stallings' theorem on central series of groups [36] (cf. [10]). This example shows that already the first step of Weber's proof fails beyond the metastable range of dimensions. Note that for **manifolds** the deleted product criterion is true beyond the metastable range [33].

We use the notation of [25] and [14]. For a survey on embeddings see [26].

**1. An auxiliary construction**

Consider integers \(m\) and \(n\) such that \(4 \leq n + 2 \leq m \leq 3n/2 + 1\). Let \(l = m - n - 1\). The upper index of a polyhedron always indicates its dimension.
Suppose that an \( l \)-sphere \( S^l \) is embedded in \( S^m \) and that \( f \) is a map of an oriented \( n \)-sphere \( S^n \) into \( S^m \setminus S^l \). By a homological linking number we understand the image under \( H_n(f) \) of the generator of \( H_n(S^n) \) in \( H_n(S^m \setminus S^l) \cong \mathbb{Z} \); here \( H_n(X) \) denotes \( H_n(X; \mathbb{Z}) \).

**Auxiliary Lemma 1.1** (cf. [10, Lemma 6]; [29, Lemma 1.4]). For each \( n \geq m/2 \) there exist an \( n \)-polyhedron \( K \) containing two disjoint wedges of spheres \( \Sigma^n \setminus \tilde{\Sigma}^n \) and \( \Sigma^l \setminus \tilde{\Sigma}^l \) such that

(a) for each PL-embedding \( K \hookrightarrow \mathbb{R}^m \) the pairs \( \Sigma^n, \tilde{\Sigma}^n \) and \( \Sigma^l, \tilde{\Sigma}^l \) are not linked and the homological linking numbers of the pairs \( \Sigma^n, \Sigma^l \) and \( \tilde{\Sigma}^n, \tilde{\Sigma}^l \) are odd;

(b) there exists a PL-embedding \( K \hookrightarrow \mathbb{R}^m \) for which \( \Sigma^n \setminus \tilde{\Sigma}^n \) is unknotted in \( \mathbb{R}^m \).

**Proof.** Denote by \( \Delta^k_{a_0\ldots a_s} \) the \( k \)-skeleton of an \( s \)-simplex with vertices \( a_0 \cdots a_s \). Let

\[
P = \Delta^n_{a_0\ldots a_{m+2}} \cup \text{Con}(\Delta^l_{a_0\ldots a_{m+2}}, 0) \quad \text{and} \quad Q = \Delta^n_{a_0\ldots a_{m+2}} \cup \text{Con}(\Delta^l_{a_0\ldots a_{m+2}} \setminus \Delta^l_{a_0\ldots a_{m+3}}, 0).
\]

Let \( \Sigma^l = \partial \Delta^{l+1}_{a_0\ldots a_{l+1}} \) and \( \tilde{\Sigma}^n = \partial \Delta^{n+1}_{a_0\ldots a_{m+2}} \) be spheres in \( Q \). Let \( \tilde{Q} \) be a copy of \( Q \). For a subset \( A \subset Q \) denote its copy by \( \tilde{A} \subset \tilde{Q} \). Set \( K = Q \cup_{\tilde{0}=0, m=m+2} \tilde{Q} \).

The unlinkedness in Lemma 1.1(a) follows since \( \Sigma^l \) (respectively \( \tilde{\Sigma}^l \)) bounds a disk \( \Delta^{l+1}_{a_0\ldots a_{l+1}} \) (respectively \( \tilde{\Delta}^{l+1}_{a_0\ldots a_{m+3}} \)) in \( K \setminus \Sigma^n \) (respectively in \( K \setminus \tilde{\Sigma}^n \)). The second part of Lemma 1.1(a) follows from [29, Lemma 1.4].

By [29, Lemma 1.1], \( Q \) PL-embeds in \( \mathbb{R}^m \). For our embedding \( \tilde{Q} \hookrightarrow \mathbb{R}^m \) the sphere \( \Sigma^n \) is unknotted in \( \mathbb{R}^m \) [29, the first two paragraphs of the proof of Lemma 1.1]. Then we can embed into \( \mathbb{R}^m \) two copies of \( Q \) which are far apart. Join two points of \( \Sigma^n \) and \( \tilde{\Sigma}^n \) by an arc and pull two points of the spheres together along this arc. Making the same construction for \( \Sigma^l \) and \( \tilde{\Sigma}^l \) we obtain the required PL-embedding. (For \( m > n + 2 \), Lemma 1.1(b) is true for every PL-embedding \( K \hookrightarrow \mathbb{R}^m \) [18, Theorem 8].) \( \square \)

**Remark 1.2** (cf. [10, Lemma 8]; [29, Lemma 1.4]). In Lemma 1.1(a) the assumption that \( K \hookrightarrow \mathbb{R}^m \) is a PL-embedding can be relaxed to a topological embedding. (In the sequel we use this stronger version of Lemma 1.1(a) in the case \( m = n + 2 \).)

**Proof.** There are arbitrarily close PL-approximations \( f : K \hookrightarrow \mathbb{R}^m \) of an embedding \( K \hookrightarrow \mathbb{R}^m \) such that \( f\sigma \cap f\tau = \emptyset \) for each two disjoint simplexes \( \sigma, \tau \) of some triangulation of \( K \). By [2], since \( l \leq m - 3 \) (or by general position if \( m = n + 2 \)), we may assume that \( f|_{\Sigma^l \setminus \tilde{\Sigma}^l} \) is a PL-embedding. The second part of the stronger version of Lemma 1.1(a) follows by [29, Lemma 1.4]. The unlinkedness of pairs \( \Sigma^n, \tilde{\Sigma}^n \) and \( \Sigma^l, \tilde{\Sigma}^l \) can be proved analogously to the PL-case. \( \square \)

**Remark.** In the sequel, for \( m > n + 2 \), we only use a weaker version of Lemma 1.1(a); namely we only need that the linking numbers of the pairs are nonzero. To prove this version it suffices to prove that \( P \) is not embeddable in \( \mathbb{R}^m \) (cf. [29, proof of Lemma 1.4]).
Hence it suffices to prove that there are no equivariant maps $\widetilde{P} \to S^{m-1}$. This follows from [29, Lemma 1.2] and [42].

Observe that if $n = l + 1 = 1$, then $P$ is one of Kuratowski’s nonplanar graphs, namely $K_5$, and if $n = l + 1 > 1$ then $P$ is an $n$-dimensional polyhedron which is not embeddable in $\mathbb{R}^{2n}$ (cf. [8,37]).

2. Construction of the example

**Finger Move Lemma 2.1** (cf. [29, Section 2]) Let $K$ be the polyhedron and $K \hookrightarrow \mathbb{R}^m$ be the PL-embedding from Lemma 1.1. Let $D^n \subset \Sigma^n$ and $\tilde{D}^n \subset \tilde{\Sigma}^n$ be PL-disks in the interiors of some $n$-simplices, of some triangulation of $K$, adjacent to the unique common point of $\Sigma^n$ and $\tilde{\Sigma}^n$. Then there is a PL-map $g : K \to \mathbb{R}^m$ such that

(a) $g|_{K \setminus \tilde{D}^n}$ is the inclusion and $g|_{K \setminus D^n}$ is an embedding but $g(D^n) \cap g(\tilde{D}^n) \neq \emptyset$;

(b) the Whitehead product of the maps $\Sigma^l \subset \Sigma^l \vee \tilde{\Sigma}^l \hookrightarrow \mathbb{R}^m \setminus g(\Sigma^n \vee \tilde{\Sigma}^n)$ and $\tilde{\Sigma}^l \subset \Sigma^l \vee \tilde{\Sigma}^l \hookrightarrow \mathbb{R}^m \setminus g(\Sigma^n \vee \tilde{\Sigma}^n)$ is null-homotopic.

**Remark.** In Fig. 1 the $l$- and $n$-dimensional spheres $\Sigma^l$ and $\tilde{\Sigma}^l$, $\Sigma^n$ and $\tilde{\Sigma}^n$ are shown as 1-dimensional. Also the $2l$-dimensional distinguished torus is shown as 0-dimensional.

![Fig. 1](image-url)
Proof. Take points \( a \in \hat{D}^n \) and \( \hat{a} \in \hat{D}^n \). Take a small arc \( s \subset \mathbb{R}^m \) joining \( a \) to \( \hat{a} \). By general position we may assume that \( s \cap K = \{a, \hat{a}\} \). Make a finger move of \( D^n \) along \( s \) to get a new PL-map \( g: K \to \mathbb{R}^m \) such that Lemma 2.1(a) is true. By general position we may assume that \( \dim(g(D^n) \cap \hat{D}^n) \leq 2n - m \) and \( g(D^n) \) intersects \( \hat{D}^n \) transversally. We can represent a regular neighborhood \( B^m \) of an arbitrary point \( x \) of this intersection as the product \( B^{2n-m} \times B^{l+1} \times B^{l+1} \) of balls with \( B^{2n-m} \times 0 \times 0 \) corresponding to the intersection, \( B^{2n-m} \times B^{l+1} \times 0 \) and \( B^{2n-m} \times 0 \times B^{l+1} \) to \( g(D^n) \) and \( D^n \), respectively (we denote by 0 the center of \( B^k \)). In a neighborhood of \( x \) we have the ‘distinguished’ torus \( 0 \times \partial B^{l+1} \times \partial B^{l+1} \). By Lemma 1.1(b) \( \mathbb{R}^m \setminus \Sigma^n \cup \tilde{\Sigma}^n \) has the homotopy type of \( S^l \vee S^l \). Denote by \( \alpha \) and \( \hat{\alpha} \) the elements of \( \pi_1(\mathbb{R}^m \setminus \Sigma^n \cup \tilde{\Sigma}^n) \) represented by homeomorphisms \( S^l \to y \vee S^l \) and \( S^l \to S^l \vee y \ (y \in S^l) \), respectively (with chosen orientations). With appropriate orientations the inclusions of \( 0 \times \partial B^{l+1} \times y \) and \( 0 \times y \times \partial B^{l+1} \) into \( \mathbb{R}^m \setminus \Sigma^n \cup \tilde{\Sigma}^n \) are homotopic in \( \mathbb{R}^m \setminus \Sigma^n \cup \tilde{\Sigma}^n \) to \( \alpha \) and \( \hat{\alpha} \), respectively. Since the map

\[ [\alpha, \hat{\alpha}]: S^{2l-1} \to S^l \vee S^l \cong (0 \times y \times \partial B^{l+1}) \vee (0 \times \partial B^{l+1} \times y) \]

extends to a map \( B^{2l} \to 0 \times \partial B^{l+1} \times \partial B^{l+1} \) [3], it follows that \( [\alpha, \hat{\alpha}] \) is null-homotopic in \( \mathbb{R}^m \setminus g(\Sigma^n \cup \tilde{\Sigma}^n) \). Let \( p = \text{link}(\Sigma^n, \tilde{\Sigma}^n) \) and \( \tilde{p} = \text{link}(\tilde{\Sigma}^l, \Sigma^n) \). The inclusions of \( \Sigma^l \) and \( \tilde{\Sigma}^l \) into \( \mathbb{R}^m \setminus \Sigma^n \cup \tilde{\Sigma}^n \) represent the elements \( p\alpha \) and \( \tilde{p}\hat{\alpha} \) of \( \pi_1(\mathbb{R}^m \setminus \Sigma^n \cup \tilde{\Sigma}^n) \), respectively. Therefore Lemma 2.1(b) follows since \( [p\alpha, \tilde{p}\hat{\alpha}] = p\tilde{p}[\alpha, \hat{\alpha}] \) is null-homotopic in \( \mathbb{R}^m \setminus g(\Sigma^n \cup \tilde{\Sigma}^n) \). □

Lemma 2.2 (cf. [29, Lemmas 2.1, 2.2]; [10, Sections 3.2, 4]). Let \( g: K \to \mathbb{R}^m \) be the map from Lemma 2.1. Let \( r: B^{2l} \to \mathbb{R}^m \setminus g(\Sigma^n \cup \tilde{\Sigma}^n) \) be a PL-map such that \( r|_{\partial B^{2l}}: \partial B^{2l} \to \Sigma^l \vee \tilde{\Sigma}^l \) represents the Whitehead product of inclusions \( \Sigma^l \subset \Sigma^l \vee \tilde{\Sigma}^l \) and \( \tilde{\Sigma}^l \subset \Sigma^l \vee \tilde{\Sigma}^l \). Let

\[
Y = (K \setminus \hat{D}^n) \cup r(B^{2l}) \cup g(D^n) \subset \mathbb{R}^m, \quad \text{and} \\
R = (K \setminus \hat{D}^n) \cup r(B^{2l}) \cup \partial D^n \cup \partial D^n \cup B^n.
\]

Then \( \dim R = n \), \( R \) is quasi homeomorphic to \( Y \) but is not topologically embeddable in \( \mathbb{R}^m \).

Proof. Since \( m \leq 3n/2 + 1 \), it follows that \( 2l \leq n \) and hence \( \dim Y = \dim R = n \). We have \( R \supset (K \setminus \hat{D}^n) \cup B^n \cong K \). Therefore by the definition of \( D^n \) and \( \hat{D}^n \), \( B^n \) and \( \hat{D}^n \) are contained in the interiors of some adjacent \( n \)-simplices of some triangulation of \( R \). Then there is an obvious map \( R \to Y \) such that its singular set is contained in \( B^n \cup \hat{D}^n \) and so in the interiors of the two adjacent simplices of some triangulation of \( R \). Therefore analogously to [29, Lemma 2.1], \( R \) is quasi homeomorphic to \( Y \), hence \( R \) quasi embeddable in \( \mathbb{R}^m \).

Suppose to the contrary that there is an embedding \( h: R \hookrightarrow S^m \). Let

\[ \Sigma^n = (\Sigma^n \setminus \hat{D}^n) \cup_{\partial B^n = \partial D^n} B^n \subset R. \]
The map $h \circ r|_{\partial B^2}$ can be extended to the map

$$h \circ r: B^2 \to S^m \setminus h(\Sigma_1^n \vee \Sigma^n).$$

Hence $h \circ r|_{\partial B^2}$ is homotopically trivial in $S^m \setminus h(\Sigma_1^n \vee \Sigma^n)$. Now we shall show the contrary and get a contradiction.

Case $m > n+2$. By [2] we may assume that $h$ is a PL-embedding. By [18, Theorem 8] $h(\Sigma_1^n \vee \Sigma^n)$ is unknotted, thus $S^m \setminus h(\Sigma_1^n \vee \Sigma^n)$ has the homotopy type of $S^l \vee S^l$. Denote by $\beta$ and $\tilde{\beta}$ the elements of $\pi_1(S^m \setminus h(\Sigma_1^n \vee \Sigma^n))$ represented by homeomorphisms $S^l \to y \vee S^l$ and $S^l \to S^l \vee y$ ($y \in S^l$), respectively (with chosen orientations). By [40] the homotopy class of the map

$$h \circ r|_{\partial B^2}: \partial B^2 \to S^m \setminus h(\Sigma_1^n \vee \Sigma^n)$$

can be considered as an element $qq[\beta, \tilde{\beta}]$ of $\pi_{2l-1}(S^m \setminus h(\Sigma_1^n \vee \Sigma^n))$, where $q := \text{link}(h, \Sigma_1^n, h(\Sigma_1^n))$ and $\tilde{q} := \text{link}(h, \tilde{\Sigma}_1^n, h(\Sigma_1^n))$. By the Hilton theorem ([24, pp. 231, 257] or [40, p. 511]), or an elementary argument using the homotopy exact sequence (cf. [14, V.3]), the map $\varphi: \pi_{2l-1}(S^{2l-1}) \to \pi_{2l-1}(S^l \vee S^l)$ defined by $\varphi(\gamma) = [\beta, \tilde{\beta}] \circ \gamma$ is an injection. Hence $[\beta, \tilde{\beta}]$ has infinite order. This implies that the element $qq[\beta, \tilde{\beta}]$ is nontrivial since $q$ and $\tilde{q}$ are both nonzero by Lemma 1.1(a). □

Case $m = n+2$. Consider the maps $\Sigma_l \subset \Sigma^l \vee \tilde{\Sigma}^l \hookrightarrow S^m \setminus h(\Sigma^n \vee \Sigma^n)$ and $\tilde{\Sigma}^l \subset \Sigma^l \vee \tilde{\Sigma}^l \hookrightarrow S^m \setminus h(\Sigma^n \vee \Sigma^n)$. By the stronger version of Lemma 1.1(a) (cf. Remark 1.2) and the following Lemma 2.3 (which is a modification of [10, Lemma 7], and is a consequence of a result of Stallings [36]) we obtain that the commutator of the homotopy classes of the above maps is nonzero. □

Lemma 2.3. Let $f_1: S_1 \vee S'_1 \to S^m$ and $f_2: S_2 \vee S'_2 \to S^m$ be embeddings of the wedges of two spheres of dimensions $d_1 = m-2$ and $d_2 = 1$, respectively such that $f_1(S_1 \vee S'_1) \cap f_2(S_2 \vee S'_2) = \emptyset$. If the homological linking numbers of the maps

$$f_1|_{S_1}: S_1 \to S^m \setminus f_2(S_2) \quad \text{and} \quad f_1|_{S'_1}: S'_1 \to S^m \setminus f_2(S'_2)$$

are odd and the homological linking numbers of the maps

$$f_1|_{S_1}: S_1 \to S^m \setminus f_2(S'_2) \quad \text{and} \quad f_1|_{S'_1}: S'_1 \to S^m \setminus f_2(S_2)$$

are even, then the inclusion map

$$f_2(S_2 \vee S'_2) \to S^m \setminus f_1(S_1 \vee S'_1)$$

induces a monomorphism of the fundamental groups.

Acknowledgments

The second author would like to acknowledge P.M. Akhmetiev and M.M. Postnikov for useful discussions.
References