Extensions of regular mappings and the Łojasiewicz exponent at infinity

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Abstract

Let $F : V \to \mathbb{C}^m$ be a regular mapping, where $V \subset \mathbb{C}^n$ is an algebraic set of positive dimension and $m \geq n \geq 2$, and let $\mathcal{L}_\infty (F)$ be the Łojasiewicz exponent at infinity of $F$. We prove that $F$ has a polynomial extension $G : \mathbb{C}^n \to \mathbb{C}^m$ such $\mathcal{L}_\infty (G) = \mathcal{L}_\infty (F)$. Moreover, we give an estimate of the degree of the extension $G$. Additionally, we prove that if $\dim V < n - 2$ then for any $\beta \in \mathbb{Q}$, $\beta \leq \mathcal{L}_\infty (F)$, the mapping $F$ has a polynomial extension $G$ with $\mathcal{L}_\infty (G) = \beta$. We also give an estimate of the degree of this extension.

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1. Introduction

Let $V \subset \mathbb{C}^n$ be an algebraic set and let $F : V \to \mathbb{C}^m$ be a regular mapping, i.e. the restriction of a polynomial mapping to $V$. Let $S \subset V$. By the Łojasiewicz exponent at infinity of $F$ on the set $S$ we mean

$$\mathcal{L}_\infty (F|S) := \sup \{ v \in \mathbb{R} : \exists C, R > 0 \forall z \in S \ (|z| \geq R \Rightarrow |F(z)| \geq C|z|^v) \},$$

where $|\cdot|$ are norms (in $\mathbb{C}^n$ and $\mathbb{C}^m$). For $S = V$ the exponent $\mathcal{L}_\infty (F|V)$ will be called the Łojasiewicz exponent at infinity of $F$ and denoted $\mathcal{L}_\infty (F)$. The Łojasiewicz exponent at infinity does not depend on the chosen norms in $\mathbb{C}^n$ and $\mathbb{C}^m$. In what follows we will use the Euclidean norm.

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The Łojasiewicz exponent at infinity plays an important role in the study of properness and injectivity of polynomial mappings (see e.g. [2,4,6,7,15,28]), triviality of fibres (see e.g. [9,13,14,18,27,30–32,35,37]) and in effective Nullstellensatz (see e.g. [3,4,8,11,19,17,16,21–24,31,34,40]). Estimates of $L_\infty(F)$ play an important role in these considerations. The deepest results in this area were achieved by J. Chądzyński (see [4, Theorem 3.1]), J. Kollár [21, Corollary 1.9 and Proposition 1.10], E. Cygan, T. Krasański, and P. Tworzewski (see [11, Theorem 7.3]), Z. Jelonek (see [17, Theorem 1.4]) and E. Cygan (see [10, Theorem 5.1]). Let us recall these results.

Let $F = (f_1, \ldots, f_m) : \mathbb{C}^n \to \mathbb{C}^m$, $m \geq n \geq 2$, be a polynomial mapping. We put $\deg F := \max\{\deg f_1, \ldots, \deg f_m\}$. Let $d_j = \deg f_j$, $j = 1, \ldots, m$, and let $d_1 \geq \cdots \geq d_m > 0$. Denote $B(d_1, \ldots, d_m; n) = \left\{ \begin{array}{ll} d_1 \cdots d_m & \text{for } m \leq n, \\ d_1 \cdots d_{n-1} d_m & \text{for } m > n. \end{array} \right.$

**Theorem 1.1** (Chądzyński). If $\# F^{-1}(0) < +\infty$ and $n = m = 2$, then

$$L_\infty(F) \geq \min\{d_1, d_2\} - d_1d_2 + \sum_{b \in F^{-1}(0)} \mu_b(F),$$

where $\mu_b(F)$ is the multiplicity of the mapping $F$ at the point $b$ (see [25, V.2.1]).

**Theorem 1.2** (Kollár). If $F^{-1}(0) = \emptyset$, then there exist polynomials $g_1, \ldots, g_m \in \mathbb{C}[z_1, \ldots, z_n]$ such that $\deg g_i \leq B(d_1, \ldots, d_m; n)$ and $\sum_{i=1}^m g_i f_i = 1$. In particular, if $\# F^{-1}(0) < +\infty$, then

$$L_\infty(F) \geq d_m - B(d_1, \ldots, d_m; n).$$

**Theorem 1.3** (Cygan, Krasiński, Tworzewski). If $\# F^{-1}(0) < +\infty$, then

$$L_\infty(F) \geq d_m - B(d_1, \ldots, d_m; n) + \sum_{b \in F^{-1}(0)} \mu_b(F),$$

where $\mu_b(F)$ is the multiplicity of $F$ at $b$ in the sense of R. Achilles, P. Tworzewski and T. Winiarski (i.e. the intersection multiplicity of graph(F) and $\mathbb{C}^n \times \{0\}$ at an isolated point $(b, 0)$, see [1, Definition 5.1]).

**Theorem 1.4** (Jelonek). Let $V \subset \mathbb{C}^n$ be an affine $k$-dimensional variety of degree $D$ and suppose $\nu := \#(F^{-1}(0) \cap V) < +\infty$. Then

$$L_\infty(F|V) \geq d_m - D \cdot B(d_1, \ldots, d_m; k) + \nu.$$

**Theorem 1.5** (Cygan). Suppose $V = F^{-1}(0) \neq \emptyset$. Then there exists $C > 0$ such that

$$|F(z)| \geq C \left( \frac{\varrho(z, V)}{1 + |z|^2} \right)^{B(d_1, \ldots, d_m; n)}$$

for $z \in \mathbb{C}^n$.

where $\varrho(z, V)$ is the distance of the point $z$ to the set $V$ (i.e. $\varrho(z, V) = \inf_{y \in V} |z - y|$).

There are known effective formulas for $L_\infty(F)$ (see e.g. [5,6,9,12,13,28,29,32,33,37]). The main difficulty in this area is to indicate an algebraic set $V \subset \mathbb{C}^n$ such that $L_\infty(F) = L_\infty(F|V)$ (see e.g. [7]). In the context of the above results, the following question seems to be relevant:
Question. Let $V \subset \mathbb{C}^n$ be an algebraic set of positive dimension and let $F : V \to \mathbb{C}^m$ be a regular mapping. Does there exist a polynomial mapping $G : \mathbb{C}^n \to \mathbb{C}^m$ such that $G|_V = F$ and $L_\infty(F) = L_\infty(G)$?

This question was known to participants of the cyclic Workshop on Complex Analytic and Algebraic Geometry Lodz 2002. In the case when $V$ is an affine space it was answered in the affirmative by M. Karaś [20]. The main aim of this paper is to give an affirmative answer in the general case (Theorem 1.6).

Let us start from a definition. By the total degree of an algebraic set $V \subset X$ we mean the number

$$\delta(V) := \deg V_1 + \cdots + \deg V_s,$$

where $V = V_1 \cup \cdots \cup V_s$ is the decomposition of $V$ into irreducible components.

**Theorem 1.6.** Let $\deg F > 0$, let $V \subset \mathbb{C}^n$ be an algebraic set with $\dim V > 0$ and suppose $\#(F^{-1}(0) \cap V) < +\infty$. Then there exists a polynomial mapping $G : \mathbb{C}^n \to \mathbb{C}^m$ such that

(a) $F|_V = G|_V$,
(b) $L_\infty(G) = L_\infty(F|_V)$,
(c) $\deg G \leq -[-(\delta(V))^p(\deg F + 2 - L_\infty(F|_V)) - L_\infty(F|_V)] + \delta(V)$,

where $[x]$ means the integral part of a real number $x$.

In the case $\#(F^{-1}(0) \cap V) = +\infty$ the conditions (a) and (b) in Theorem 1.6 hold true for $G = F$. The proof of Theorem 1.6, given in Section 3, is based on E. Cygan’s inequality (Theorem 1.5) and S. Spodzieja’s method of calculating the Łojasiewicz exponent for overdetermined polynomial mappings (Theorem 2.5 in Section 2).

Under the additional assumption $0 < \dim V < n - 2$ we give a generalization of Theorem 1.6:

**Theorem 1.7.** Let $\deg F > 0$ and let $V \subset \mathbb{C}^n$ be an algebraic set with $0 < \dim V < n - 2$. Let additionally $\beta \in \mathbb{Q}$, $\beta = \frac{k}{l}$, $k \in \mathbb{Z}$, $l \in \mathbb{N}$, be such that $\beta < L_\infty(F|_V)$. Then there exists a polynomial mapping $G : \mathbb{C}^n \to \mathbb{C}^m$ such that

(a) $F|_V = G|_V$,
(b) $L_\infty(G) = \beta$,
(c) $\deg G \leq -[-(\delta(V))^p(D + 2 - L_\infty(F|_V)) - L_\infty(F|_V)] + \delta(V)$,

where $D = (\delta(V))^n + \max\{\deg F, (|k| + l) \cdot l\}$.

The proof of Theorem 1.7, given in Section 4, is based on Kollár’s inequality (Theorem 1.2), Proposition 4.4 and Theorem 1.6.

2. Auxiliary results

Let $X$ be a finite dimensional complex vector space. By $\mathcal{P}(X)$ we denote the set of all complex polynomials on $X$. By $G'_k(X)$, where $0 \leq k \leq n$, we denote the set of all $k$-dimensional complex affine subspaces of $X$. We assume that $G'_k(X)$ is equipped with the topology induced by
the topology of the Grassmann space of \((k + 1)\)-dimensional linear subspaces of \(\mathbb{C} \times X\) (see [25, B.6.11]).

From [25, Corollary VII.11.8], [25, Proposition VII.11.8.10] and [25, Proposition VII.11.8.9] we immediately obtain

**Proposition 2.1.** Let \(n = \dim X\). If \(V \subset X\) is an algebraic set of pure dimension \(k\) and \(p = \deg V\), then the set \(G = \{L \in G_{n-k}(X): \#(L \cap V) = p\}\) is open and dense in \(G_{n-k}(X)\) and has algebraic complement. Moreover, for every \(L \in G_{n-k}(X) \setminus G\) we have either \(\#(L \cap V) < p\) or \(\dim(L \cap V) > 0\).

**Corollary 2.2.** Let \(n = \dim X\). If \(V \subset X\) is an algebraic set of pure dimension \(k\), then \(\deg V = \max\{\#(L \cap V): L \in G_{n-k}(X), \#(L \cap V) < +\infty\}\).

Directly from the definition of the Łojasiewicz exponent at infinity we obtain

**Proposition 2.3.** If \(F: \mathbb{C}^n \to \mathbb{C}^m\) is a polynomial mapping and \(V \subset \mathbb{C}^n\) is an algebraic set such that \(\dim V > 0\), then \(L_{\infty}(F|V) \leq \deg F\).

From the curve selection lemma at infinity we obtain (see [39, Theorem 3.5])

**Proposition 2.4.** Let \(S \subset \mathbb{C}^n\) be a closed and unbounded semi-algebraic set and let \(F: \mathbb{C}^n \to \mathbb{C}^m\) be a polynomial mapping. If \(F^{-1}(0) \cap S\) is a compact set, then \(L_{\infty}(F|S) \in \mathbb{Q}\) and there exist \(C, R > 0\) such that

\[
|F(z)| \geq C|z|^{L_{\infty}(F|S)} \quad \text{for } z \in S, \ |z| \geq R \tag{2.1}
\]

and for some curve \(\varphi: [r, +\infty) \to S\) meromorphic at infinity such that \(\deg \varphi > 0\), we have

\[
|F(\varphi(t))| \leq C'|\varphi(t)|^{L_{\infty}(F|S)}, \quad t \in [r, +\infty), \quad C' > 0. \tag{2.2}
\]

Moreover, inequalities (2.1) and (2.2) determine the number \(L_{\infty}(F|S)\) uniquely.

Let \(m, k \in \mathbb{N}\). By \(L(m, k)\) we denote the set of all nonsingular linear mappings \(A: \mathbb{C}^m \to \mathbb{C}^k\) (i.e. the rank of the matrix of \(A\) is equal to \(\min\{m, k\}\)). If \(k = 0\), we put \(\mathbb{C}^k = \{0\}\). Let \(m \geq k\). By \(\Delta(m, k)\) we denote the set of all linear mappings \(T = (T_1, \ldots, T_k) \in L(m, k)\), where

\[
T_i(y_1, \ldots, y_m) = y_i + \sum_{j=k+1}^m \alpha_{i,j} y_j, \quad \alpha_{i,j} \in \mathbb{C}
\]

for \(i = 1, \ldots, k, \ j = k + 1, \ldots, m\).

The main role in the proof of Proposition 2.6 is played by the following theorem (see [38, Theorem 2.1], cf. [36, Corollary 1]).

**Theorem 2.5 (Spodzieja).** Let \(F: \mathbb{C}^n \to \mathbb{C}^m, m \geq n\), be a polynomial mapping such that \(\#F^{-1}(0) < +\infty\). Then for any \(A \in L(m, n)\) with \(\#(A \circ F)^{-1}(0) < +\infty\) we have

\[
L_{\infty}(F) \geq L_{\infty}(A \circ F).
\]

Moreover, for any mapping \(T \in L(m, n)\) except a proper algebraic subset, we have

\[
L_{\infty}(F) = L_{\infty}(T \circ F).
\]
From Theorem 2.5 we obtain the following proposition, which plays the main role in our considerations.

**Proposition 2.6.** Let \( F : \mathbb{C}^n \to \mathbb{C}^N \), \( N \geq n \), be a polynomial mapping. Then for any \( m \) such that \( N \geq m \geq n \) there exists a mapping \( T \in \Delta(N,m) \) such that
\[
L_\infty(F) = L_\infty(T \circ F).
\]

**Proof.** If \( L_\infty(F) = -\infty \), then the set \( F^{-1}(0) \) is unbounded, so for every \( T \in \Delta(N,m) \) the set \( (T \circ F)^{-1}(0) \) is unbounded. Therefore \( L_\infty(T \circ F) = -\infty \).

Assume that \( L_\infty(F) > -\infty \). By Theorem 2.5 there exists a non-empty set \( U \subset L(N,n) \), open in the Zariski topology, such that for every \( A_1 \in U \)
\[
L_\infty(F) = L_\infty(A_1 \circ F).
\]

Let \( A_1 \in U \), \( A_2 \in L(N,m-n) \) and \( A = (A_1, A_2) : \mathbb{C}^N \to \mathbb{C}^m \). Then
\[
\left| A_1 \circ F(z) \right| \leq \left| A \circ F(z) \right| \leq \|A\| \cdot |F(z)|, \quad z \in \mathbb{C}^n,
\]
where \( \|A\| \) is the norm of \( A \). Therefore
\[
L_\infty(F) \geq L_\infty(A \circ F) \geq L_\infty(A_1 \circ F) = L_\infty(F),
\]
so \( L_\infty(A \circ F) = L_\infty(F) \). This implies that for every \( A \in L(N,m) \) except a proper algebraic subset we have \( L_\infty(F) = L_\infty(A \circ F) \). Hence there exists a non-empty set \( \mathcal{W} \subset L(N,m) \), open in the Zariski topology, such that for all \( A \in \mathcal{W} \) we have \( L_\infty(F) = L_\infty(A \circ F) \). By \( \mathcal{V} \) we denote the set of all linear mappings \( A \in L(N,m) \) with matrix \( \begin{bmatrix} a_{ij} \end{bmatrix}_{1 \leq i \leq N, 1 \leq j \leq m} \) such that \( \det \begin{bmatrix} a_{ij} \end{bmatrix}_{1 \leq i \leq N, 1 \leq j \leq m} \neq 0 \). The set \( \mathcal{V} \) is a proper algebraic subset of \( L(N,m) \). Therefore \( \mathcal{W} \setminus \mathcal{V} \) is non-empty and open in the Zariski topology. Let \( A \in \mathcal{W} \setminus \mathcal{V} \) and let \( \begin{bmatrix} a_{ij} \end{bmatrix}_{1 \leq i \leq N, 1 \leq j \leq m} \) be the matrix of \( A \). Then \( \det \begin{bmatrix} a_{ij} \end{bmatrix}_{1 \leq i \leq N, 1 \leq j \leq m} \neq 0 \) and
\[
B = \left( \begin{bmatrix} a_{ij} \end{bmatrix}_{1 \leq i \leq N, 1 \leq j \leq m} \right)^{-1} \circ \left[ a_{ij} \right]_{1 \leq i \leq N, 1 \leq j \leq m} = \left[ b_{ij} \right]_{1 \leq i \leq N, 1 \leq j \leq m},
\]
where for \( i, j \in \{1, \ldots, m\} \),
\[
b_{ij} = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \neq j. \end{cases}
\]
Denote by \( T_A : \mathbb{C}^N \to \mathbb{C}^m \) the linear mapping with matrix \( B \). Then \( T_A \in \Delta(N,m) \). Therefore,
\[
L_\infty(F) = L_\infty(A \circ F) = L_\infty(T_A \circ F),
\]
which ends the proof. \( \Box \)

**Proposition 2.7.** Let \( V \subset X \) be an irreducible algebraic set of dimension \( k \), where \( 0 \leq k < \dim X \). Then for any \( z^0 \in X \setminus V \) there exists a polynomial \( P \in \mathcal{P}(X) \) such that \( \deg P \leq \deg V \), \( P|_V = 0 \) and \( P(z^0) \neq 0 \).
\textbf{Proof.} Let \( n = \dim X \). Observe that we may assume \( z^0 = 0 \). Indeed, define \( F : X \to X \) by \( F(x) = x - z^0 \). It is clear that \( F \) is a proper polynomial mapping. Hence from \([25, \text{Proposition V.7.1.2}]\), \( F(V) \) is an irreducible algebraic subset and \( \dim F(V) = \dim V \). Moreover, \( G'_{n-k}(X) = \{ F(L) \mid L \in G'_{n-k}(X) \} \) and \( \#(L \cap V) = \#[F(L) \cap F(V)] \) for \( L \in G'_{n-k}(X) \), so from Corollary 2.2 we have

\[
\deg V = \max \left\{ \#(L \cap V) \mid L \in G'_{n-k}(X), \#(L \cap V) < +\infty \right\} = \max \left\{ \#(L \cap F(V)) \mid L \in G'_{n-k}(X), \#(L \cap F(V)) < +\infty \right\} = \deg F(V).
\]

Since \( F(z^0) = 0 \) and \( \deg P \circ F = \deg P \) for every \( P \in \mathcal{P}(X) \), if a polynomial \( P \in \mathcal{P}(X) \) satisfies the assertion for \( F(V) \) and \( z^0 = 0 \), then \( P \circ F \in \mathcal{P}(X) \) satisfies the assertion for \( V \) and \( z^0 = 0 \). Hence from now on \( z^0 = 0 \).

We apply induction with respect to \( m = \dim X - k \). If \( m = 1 \), then from \([25, \text{Corollary VII.11.3.2}]\) the algebraic set \( V \) is a principal variety. So the assertion follows from \([25, \text{Proposition VII.11.5.7}]\). Assume the assertion is true for some \( m \geq 1 \). Suppose that \( m + 1 = \dim X - k \). Let \( \varphi : V \times \mathbb{C} \to X \) be the polynomial mapping defined by \( \varphi(x, t) = xt \) and let \( \tilde{V} = \varphi(V \times \mathbb{C}) \). From \([25, \text{VII.8.3}]\) and \([25, \text{Proposition VII.8.3.2}]\), \( \tilde{V} \) is an algebraic set. Moreover, \( k + 1 = \dim X - m < \dim X \), so from \([26, \text{Corollary 3.15}]\) and \([25, \text{II.1.4}]\) we obtain \( \dim \tilde{V} \leq \dim(V \times \mathbb{C}) = k + 1 < \dim X \). Hence the set \( X \setminus \tilde{V} \) is open and non-empty, so from the definition of \( \tilde{V} \) and from \([25, \text{B.6.11}]\) the set \( U = \{ Y \in G_1(X) : Y \cap \tilde{V} = \emptyset \} \) is open and non-empty. By Sadullaev’s Theorem \([25, \text{VII.7.1}]\) there exists \( Y \in U \) which is a Sadullaev space for \( V \). Let \( X_Y \subseteq X \) be a linear complement of \( Y \). Then \( \dim X_Y = \dim X - 1 \) and \( X = X_Y \oplus Y \). Consider the mapping \( \pi : X_Y \oplus Y \ni x + y \mapsto x \in X_Y \). Since \( Y \) is a Sadullaev space for \( V \), there exists \( C > 0 \) such that

\[
V \subset \{ x + y \in X_Y \oplus Y : |y| < C(1 + |x|) \}.
\]

(2.3)

So for every compact set \( Z \subset X_Y \), the set \( \pi^{-1}(Z) \cap V \) is bounded and obviously closed, hence compact. Therefore the restriction \( \pi|_V : V \to X_Y \) is a proper mapping. So \( \pi(V) \subset X_Y \) is an irreducible algebraic set of dimension \( k \). Moreover, from the definition of \( U \) and the assumption that \( 0 \notin V \) we have \( \pi(0) = 0 \notin \pi(V) \). From the inductive assumption there exists a polynomial \( Q \in \mathcal{P}(X_Y) \) such that \( \deg Q \leq \deg \pi(V) \), \( Q|_{\pi(V)} = 0 \) and \( Q(0) \neq 0 \). Define \( P \in \mathcal{P}(X) \) by \( P(x + y) = Q(x) \) for \( x \in X_Y \), \( y \in Y \). Then \( \deg P = \deg Q \leq \deg \pi(V) \), \( P|_{\pi(V)} = 0 \) and \( P(0) \neq 0 \). From the definition of \( P \) it follows that \( P|_V = 0 \).

To end the proof it is enough to show that \( \deg \pi(V) \leq \deg V \). Since \( \dim X_Y - \dim \pi(V) = m \), there exists \( L \in G'_n(X_Y) \) such that \( \deg \pi(V) = \#(L \cap \pi(V)) \).

Then \( L \oplus Y \in G'_{m+1}(X) \) and from (2.3) we have \( \deg \pi(V) \leq \#((L \oplus Y) \cap V) < +\infty \), so from Corollary 2.2, \( \deg \pi(V) \leq \deg V \). \( \square \)

From Proposition 2.7 we obtain

\textbf{Lemma 2.8.} \textit{Let \( V \subset X \) be an algebraic set. Then there exist \( h_1, \ldots, h_r \in \mathcal{P}(X) \) such that \( \deg h_i \leq \delta(V) \), \( i = 1, \ldots, r \), and \( V = \{ z \in X : h_1(z) = \cdots = h_r(z) = 0 \} \).}

\textbf{Proof.} Let \( V = V_1 \cup \cdots \cup V_s \) be the decomposition into irreducible components. By Proposition 2.7 and \([25, \text{A.9.2}]\) for each \( j = 1, \ldots, s \) there exist \( g_{1,j}, \ldots, g_{l,j} \in \mathcal{P}(X) \) with \( \deg g_{i,j} \leq \delta(V) \). \( \square \)
deg $V_j$ such that $V_j = \{z \in X: g_{i,j}(z) = \cdots = g_l(z) = 0\}$. Let $h_1, \ldots, h_r$ be all polynomials of the form $g_{i,1} \cdot g_{i,2} \cdots g_{i,s}$. Then $\deg h_i \leq \delta(V)$ and $V = \{z \in X: h_1(z) = \cdots = h_r(z) = 0\}$. □

3. Proof of Theorem 1.6

Proof. Let $F = (f_1, \ldots, f_m) : \mathbb{C}^n \to \mathbb{C}^m$. Since $\#(F^{-1}(0) \cap V) < +\infty$, we have $\mathcal{L}_\infty(F|V) > -\infty$. So, by Proposition 2.4, there exist $C_1, R_1 > 0$ such that

$$
|F(z)| \geq C_1|z|^\mathcal{L}_\infty(F|V) \quad \text{for } z \in V, |z| \geq R_1. \tag{3.1}
$$

We can assume that

$$
|F(z)| \geq C_1|w|^\mathcal{L}_\infty(F|V) \quad \text{for } z \in V, |z| \geq R_1, |z - w| \leq 1. \tag{3.2}
$$

Indeed, for $|z - w| \leq 1$ we have $|z| - |w| \leq |z - w| \leq 1$, so $|w| - 1 \leq |z| \leq |w| + 1$ and $|w| \geq R_1 - 1$. Assume that $|w| \geq R_1 - 1 > 2$. Then $\frac{1}{2}|w| > 1$. So, $\frac{1}{2}|w| \leq |z| \leq 2|w|$. If $\mathcal{L}_\infty(F|V) > 0$, then

$$
\left(\frac{1}{2}|w|\right)^{\mathcal{L}_\infty(F|V)} \leq |z|^\mathcal{L}_\infty(F|V) \leq (2|w|)^{\mathcal{L}_\infty(F|V)}.
$$

From this inequality and (3.1) we obtain

$$
|F(z)| \geq \frac{C_1}{2^{\mathcal{L}_\infty(F|V)}}|w|^\mathcal{L}_\infty(F|V).
$$

It is obvious that $\frac{C_1}{2^{\mathcal{L}_\infty(F|V)}} \leq C_1$. Thus, diminishing $C_1$ if necessary, we get (3.2). If $\mathcal{L}_\infty(F|V) < 0$, then

$$
\left(\frac{1}{2}|w|\right)^{\mathcal{L}_\infty(F|V)} \geq |z|^\mathcal{L}_\infty(F|V) \geq (2|w|)^{\mathcal{L}_\infty(F|V)}.
$$

From this and (3.1) we obtain

$$
|F(z)| \geq 2^{\mathcal{L}_\infty(F|V)}|w|^\mathcal{L}_\infty(F|V).
$$

Obviously $2^{\mathcal{L}_\infty(F|V)}C_1 \leq C_1$. Thus, diminishing $C_1$ if necessary, we again obtain (3.2).

From the Mean Value Theorem, for every $z, w \in \mathbb{C}^n$ and for any $i$ there is a point $t_i$ on the segment with end points $z$ and $w$ such that

$$
|f_i(z) - f_i(w)| \leq \|
abla f_i(t_i)\| |z - w|. \tag{3.3}
$$

Let

$$
M(w) = \sup\{|\nabla f_i(z)|: |z| \leq |w| + 1, i = 1, \ldots, m\}.
$$

Let $d = \deg F$. Since $\deg f_i \leq d$, we have $\deg \nabla f_i \leq d - 1$. So, there exist some constants $C_2 > 0$ and $R_2 \geq R_1 + 1$ such that

$$
0 \leq M(w) \leq C_2|w|^{d-1} \quad \text{for } |w| \geq R_2. \tag{3.4}
$$

From (3.3) and (3.4), for $|w| \geq R_2, |z - w| \leq 1$ we have

$$
|F(z) - F(w)| \leq M(w)|z - w| \leq C_2|w|^{d-1}|z - w|. \tag{3.5}
$$
Let
\[ W = \left\{ w \in \mathbb{C}^n : \varrho(w, V) \leq \min\left\{ 1, C_1 \frac{|w|}{2C_2} \right\} \right\}. \]

Then
\[ |F(w)| \geq \frac{C_1}{2} |w|^{L_\infty(F|V)} \text{ for } w \in W, \ |w| \geq R_2. \quad (3.6) \]

Indeed, for \( z \in V \) and \( w \in W \) such that \( |z| \geq R_1, \ |w| \geq R_2 \) and \( \varrho(w, V) = |z - w| \), from (3.5) we have
\[
|F(z)| - |F(w)| \leq |F(z) - F(w)| \leq C_2 |w|^{d-1} |z - w| \\
\leq C_2 |w|^{d-1} \frac{C_1}{2C_2} |w|^{L_\infty(F|V)} = \frac{C_1}{2} |w|^{L_\infty(F|V)}. 
\]

From these inequalities and from (3.2) we obtain
\[
|F(w)| \geq |F(z)| - \frac{C_1}{2} |w|^{L_\infty(F|V)} \geq C_1 |w|^{L_\infty(F|V)} - \frac{C_1}{2} |w|^{L_\infty(F|V)} = \frac{C_1}{2} |w|^{L_\infty(F|V)}. 
\]

This, together with the definition of \( W \), gives (3.6).

According to Lemma 2.8 there exists a polynomial mapping
\[ H = (h_1, \ldots, h_r) : \mathbb{C}^n \to \mathbb{C}^r, \]
with \( r \geq n, \) of degree \( k \leq \delta(V) \) such that \( \text{deg } h_i > 0 \) for \( i = 1, \ldots, r \) and
\[ H^{-1}(0) = V. \]

Let
\[ q = k^n (d + 2 - L_\infty(F|V)) + L_\infty(F|V). \]

Observe that
\[ (\delta(V))^n (d + 2 - L_\infty(F|V)) + L_\infty(F|V) \geq q > d. \quad (3.7) \]

Indeed, from Proposition 2.3 we have
\[ L_\infty(F|V) \leq d, \quad (3.8) \]
hence \( d + 2 - L_\infty(F|V) > 0 \). So, from the inequality \( k \leq \delta(V) \) we obtain the left inequality of (3.7). Moreover, \( \delta(V) \geq 1, \) so
\[ q \geq d + 2 - L_\infty(F|V) + L_\infty(F|V) > d. \]

This gives the right inequality of (3.7).

Enlarging \( R_2 \) if necessary, we can assume that
\[ 0 \leq \frac{\varrho(w, V)}{1 + |w|^2} < 1 \text{ for } w \in \mathbb{C}^n, \ |w| \geq R_2. \quad (3.9) \]

Observe that there exists a constant \( C_3 > 0 \) such that
\[ |H(w)| \geq C_3 \left( \frac{\varrho(w, V)}{1 + |w|^2} \right)^k \text{ for } w \in \mathbb{C}^n, \ |w| \geq R_2. \quad (3.10) \]
Indeed, reordering coordinates of \( H \), we can assume that \( k = \deg h_1 \geq \cdots \geq \deg h_r > 0 \). Then \( kn \geq B(\deg h_1, \ldots, \deg h_r; n) \). So, by Theorem 1.5 and (3.9), we get (3.10).

Let

\[
U = \mathbb{C}^n \setminus W.
\]

We will show that there exist constants \( C_4 > 0 \) and \( R_3 \geq R_2 \) such that

\[
|H(z)||z|^q \geq C_4|z|^\mathcal{L}_\infty(F|V)
\]

for \( z \in U \), \(|z| \geq R_3\). \hspace{1cm} (3.11)

Let us consider two cases:

(i) \( \mathcal{L}_\infty(F|V) > d - 1 \),

(ii) \( \mathcal{L}_\infty(F|V) \leq d - 1 \).

If (i) holds, then from the definition of \( W \), there exists a constant \( R_3 \geq R_2 \) such that \( g(z, V) \geq 1 \) for \( z \in U \), \(|z| \geq R_3\). From (3.8) and (3.10) for \( z \in U \), \(|z| \geq R_3\), we have

\[
|H(z)||z|^q \geq C_3 \left( \frac{g(z, V)}{1 + |z|^2} \right)^{kn} |z|^{kn(d + 2 - \mathcal{L}_\infty(F|V)) + \mathcal{L}_\infty(F|V)}
\]

\[
\geq C_3 \frac{kn}{2k^n} |z|^{-2k^n} |z|^{kn(d + 2 - \mathcal{L}_\infty(F|V)) + \mathcal{L}_\infty(F|V)}
\]

\[
\geq C_3 \frac{kn}{2k^n} |z|^{-2k^n} |z|^{2k^n + \mathcal{L}_\infty(F|V)} = C_3 \frac{kn}{2k^n} |z|^{\mathcal{L}_\infty(F|V)}.
\]

So, for \( C_4 = C_3/2k^n \) we get (3.11) in the case considered. If (ii) holds, there exist constants \( C_5 > 0 \) and \( R_3 \geq R_2 \) such that \( g(z, V) \geq C_5|z|^{\mathcal{L}_\infty(F|V) - d + 1} \) for \( z \in U \), \(|z| \geq R_3\). Then from (3.10) for \( z \in U \), \(|z| \geq R_3\) we have

\[
|H(z)||z|^q \geq C_3 \left( \frac{g(z, V)}{1 + |z|^2} \right)^{kn} |z|^{kn(d + 2 - \mathcal{L}_\infty(F|V)) + \mathcal{L}_\infty(F|V)}
\]

\[
\geq C_3 \frac{kn}{2} |z|^{kn(\mathcal{L}_\infty(F|V) - d + 1)} |z|^{-2k^n} |z|^{kn(d + 2 - \mathcal{L}_\infty(F|V)) + \mathcal{L}_\infty(F|V)}
\]

\[
= C_3 \left( \frac{kn}{2} \right)^{kn} |z|^{\mathcal{L}_\infty(F|V)} \geq C_3 \left( \frac{kn}{2} \right)^{kn} |z|^{\mathcal{L}_\infty(F|V)}.
\]

So, for \( C_4 = C_3(C_5/2)^{kn} \) we get (3.11) in case (ii). Thus (3.11) is proved.

Let \( p = -[-q] \) (so \( p \) is the smallest integer greater than or equal to \( q \)). From (3.7) we see that \( p \in \mathbb{N} \). Let \( H : \mathbb{C}^n \to \mathbb{C}^{nr} \) be the polynomial mapping defined by

\[
\tilde{H}(z) = (h_i(z)x_j^p : i = 1, \ldots, r, \ j = 1, \ldots, n), \quad z \in \mathbb{C}^n.
\]

Let

\[
\Phi = (F, \tilde{H}) : \mathbb{C}^n \to \mathbb{C}^m \times \mathbb{C}^{nr}.
\]

Then from (3.7) we get

\[
\deg \Phi \leq -[-(\delta(V))^p (d + 2 - \mathcal{L}_\infty(F|V)) - \mathcal{L}_\infty(F|V)] + \delta(V).
\]

(3.12)
We will show that
\[ \mathcal{L}_\infty(\Phi) = \mathcal{L}_\infty(F|V). \] (3.13)
Indeed, since for \( z \in W \) we have \( |z| \geq R_2 \), from (3.6) we have
\[ |\Phi(z)| \geq |F(z)| \geq \frac{C_1}{2} |z| \mathcal{L}_\infty(F|V) \]
and for \( z \in U, |z| \geq R_3 \), from (3.11) we have
\[ |\Phi(z)| \geq |\bar{H}(z)| \geq C_4 |z| \mathcal{L}_\infty(F|V), \]
it follows that for \( z \in \mathbb{C}^n, |z| \geq R_3 \),
\[ |\Phi(z)| \geq C |z| \mathcal{L}_\infty(F|V), \quad \text{where } C = \min\{ \frac{C_1}{2}, C_4 \}. \]
By Proposition 2.4 there exists a curve \( \varphi : [r, +\infty) \to V \) meromorphic at infinity, with \( \deg \varphi > 0 \), such that
\[ |\Phi(\varphi(t))| = |F(\varphi(t))| \leq C' |\varphi(t)| \mathcal{L}_\infty(F|V) \quad \text{for } t \in [r, +\infty), \]
where \( C' > 0 \) is a constant. Hence we get (3.13).

From Proposition 2.6 there exists a linear mapping \( T \in \Delta_{1,mnr,m} \) such that \( \mathcal{L}_\infty(T \circ \Phi) = \mathcal{L}_\infty(\Phi) \). Then \( T \circ \Phi|_V = F|_V \). Moreover, from (3.12) we obtain
\[ \deg(T \circ \Phi) \leq \deg \Phi \leq -\left[ -\left( \delta(V) \right)^n (d + 2 - \mathcal{L}_\infty(F|V)) - \mathcal{L}_\infty(F|V) \right] + \delta(V). \]
Hence, for the mapping \( G = T \circ \Phi \), from (3.13) we obtain the assertion of Theorem 1.6. \( \square \)

**Remark 3.1.** Let \( H = (h_1, \ldots, h_r) : \mathbb{C}^n \to \mathbb{C}^r, r \geq n \), be a polynomial mapping of degree \( k \leq \delta(V) \) such that \( \deg h_i > 0 \) for \( i = 1, \ldots, r \) and \( H^{-1}(0) = V \). The proof of Theorem 1.6 yields the following estimate:
\[ \deg G \leq -\left[ -\left( B(\deg h_1, \ldots, \deg h_r; n) \right)^n (\deg F + 2 - \mathcal{L}_\infty(F|V)) - \mathcal{L}_\infty(F|V) \right] + B(\deg h_1, \ldots, \deg h_r; n). \]

**4. Proof of Theorem 1.7**

In [6] the authors gave the following examples (see [6, Remark 11.4], [6, Remark 11.5]):

**Example 4.1.** Let \( h : \mathbb{C}^2 \to \mathbb{C} \) be the polynomial defined by
\[ h(x, y) = y^p + (x + y^q)^p, \quad \text{where } p \geq 2, q \geq 1. \] (4.1)
Then
\[ \mathcal{L}_\infty(\nabla h) = \frac{p}{q} - 1. \] (4.2)

**Example 4.2.** Let \( h : \mathbb{C}^2 \to \mathbb{C} \) be the polynomial defined by
\[ h(x, y) = y + y^{1+q}x^{p-q}, \quad \text{where } p > q > 0. \] (4.3)
Then
\[ L_\infty(\nabla h) = -\frac{p}{q}. \quad (4.4) \]

Example 4.3. Let \( H : \mathbb{C}^2 \to \mathbb{C}^2 \) be defined by
\[ H(x, y) = (x, xy - 1), \quad (x, y) \in \mathbb{C}^2. \]
Then
\[ L_\infty(H) = -1. \]

From the above examples not only do we see that for every \( \beta \in \mathbb{Q} \) there exists a polynomial mapping \( H : \mathbb{C}^2 \to \mathbb{C}^2 \) such that \( L_\infty(H) = \beta \), but we are able to estimate the degree of \( H \). Indeed, we have

**Proposition 4.4.** Let \( \beta = k/l \), where \( k \in \mathbb{Z}, \ l \in \mathbb{N} \). Then there exists a polynomial mapping \( H : \mathbb{C}^2 \to \mathbb{C}^2 \) such that

(a) \( L_\infty(H) = \beta \),
(b) \( \deg H \leq (|k| + l) \cdot l. \)

**Proof.** If \( \beta > -1 \), we can write
\[ \beta = \frac{k}{l} = \frac{k + l}{l} - 1, \quad \text{where } k + l \geq 2, \ l \geq 1. \]
Taking \( h \) defined by (4.1), where \( p = k + l, \ q = l \) and setting \( H = \nabla h \), from (4.2) we get \( L_\infty(H) = \beta \) and \( \deg H \leq (k + l) \cdot l \leq (|k| + l) \cdot l. \) If \( \beta < -1 \), taking \( h \) defined by (4.3), where \( p = -k, \ q = l \) and setting \( H = \nabla h \), from (4.4) we get \( L_\infty(H) = \beta \) and \( \deg H \leq (|k| + l) \cdot l. \) In case \( \beta = -1 \), taking \( H = (z_1, z_1z_2 - 1) : \mathbb{C}^2 \to \mathbb{C}^2 \), we get the assertion from Example 4.3. \( \square \)

We now give a version of the Kollár inequality (see Theorem 1.2).

**Proposition 4.5.** Let \( V \subset \mathbb{C}^n \) be an algebraic set with \( 0 < \dim V < n - 2 \). Then there exist \( L \in G_2'(\mathbb{C}^n) \) and \( f \in \mathbb{C}[z_1, \ldots, z_n] \) such that \( V \cap L = \emptyset, \ f|_V = 0, \ f|_L = 1 \) and \( \deg f \leq (\delta(V))^n. \)

**Proof.** By Lemma 2.8 there exist polynomials \( h_1, \ldots, h_r \in \mathbb{C}[z_1, \ldots, z_n] \) such that \( 0 < \deg h_i \leq \delta(V), \ i = 1, \ldots, r, \) and \( V = \{ z \in \mathbb{C}^n : h_1(z) = \cdots = h_r(z) = 0 \} \). Since \( \dim V < n - 2 \), by [25, Corollary VII.11.8] there exists an open and dense set \( G \subset G_2'(\mathbb{C}^n) \) with algebraic complement such that \( V \cap L = \emptyset \) for any \( L \in G \).

Let \( h = h_1 \cdots h_r \) and write \( h = f_0 + \cdots + f_k \), where \( f_i \) is a homogeneous polynomial of degree \( i \) or zero, \( f_k \neq 0 \).

Observe that there exists \( L \in G \) such that \( f_k|_L \neq 0 \). Indeed, assume that, on the contrary, \( L \subset f_k^{-1}(0) \) for every \( L \in G \). Since \( G \) is a dense subset of \( G_2'(\mathbb{C}^n) \), by [25, B.6.11] we obtain
\[ L \subset f_k^{-1}(0) \quad \text{for any } L \in G_2'(\mathbb{C}^n). \quad (4.5) \]
Since \( f_k \neq 0 \) there exists \( z_0 \in \mathbb{C}^n \setminus \{0\} \) such that \( f_k(z_0) \neq 0 \). Observe that, as \( n \geq 2 \), there exists \( z_1 \in \mathbb{C}^n \) such that \( z_0, z_1 \) are linearly independent over \( \mathbb{C} \). Set \( L = z_0 \cdot \mathbb{C} + z_1 \cdot \mathbb{C} \). Then \( L \in G_2'(\mathbb{C}^n) \) and \( f_k|_L \neq 0 \), because \( z_0 \in L \) and \( f_k(z_0) \neq 0 \). This contradicts (4.5).
Using a translation if necessary, we can assume that $0 \in L$. Then $L \in G_2(\mathbb{C}^n)$. Moreover,
\[
\deg h = \deg f_k = \deg f_k|_L = \deg h|_L.
\]
Consequently,
\[
\deg h_i = \deg h_i|_L \quad \text{for } i = 1, \ldots, r.
\]
Since $V \cap L = \emptyset$, the polynomials $h_1|_L, \ldots, h_r|_L$ have no zeros in common. By Theorem 1.2 there exist $g_1, \ldots, g_r \in \mathcal{P}(L)$ such that
\[
\deg(h_i|_L \cdot g_i) \leq B(\deg h_1|_L, \ldots, \deg h_r|_L; n) \quad \text{for } i = 1, \ldots, r
\]
and
\[
g_1 \cdot h_1|_L + \cdots + g_r \cdot h_r|_L = 1.
\]
Let $Y \subset \mathbb{C}^n$ be a linear complement of $L$ and define $\tilde{g}_i : \mathbb{C}^n \to \mathbb{C}^2$ by
\[
\tilde{g}_i(x + y) = g_i(x) \quad \text{for } x \in L, \ y \in Y, \ i = 1, \ldots, r.
\]
Then
\[
\deg(\tilde{g}_i \cdot h_i) = \deg \tilde{g}_i + \deg h_i = \deg g_i + \deg h_i|_L
\]
\[
= \deg(g_i \cdot h_i|_L) \leq B(\deg h_1|_L, \ldots, \deg h_r|_L; n)
\]
\[
= B(\deg h_1, \ldots, \deg h_r; n) \leq (\delta(V))^n \quad \text{for } i = 1, \ldots, r.
\]
So, $f = \tilde{g}_1 h_1 + \cdots + \tilde{g}_r h_r$ is as desired. □

**Proof of Theorem 1.7.** Let $F = (f_1, \ldots, f_m) : \mathbb{C}^n \to \mathbb{C}^m$, $d = \deg F$. According to Proposition 4.5 there exist $L \in G_2(\mathbb{C}^n)$ and $f \in \mathbb{C}[z_1, \ldots, z_n]$ such that
\[
V \cap L = \emptyset, \quad f|_V = 0, \quad f|_L = 1 \quad \text{and} \quad \deg f \leq (\delta(V))^n.
\]
Let $g = 1 - f$. Then
\[
g|_V = 1, \quad g|_L = 0 \quad \text{and} \quad \deg g \leq (\delta(V))^n.
\]
According to Proposition 4.4 there exists a polynomial mapping $H : \mathbb{C}^n \to \mathbb{C}^2$ such that
\[
\mathcal{L}_\infty(H|_L) = \beta \quad \text{and} \quad \deg H \leq (|k| + l) \cdot l.
\]
Using, if necessary, a translation we can assume that $0 \in L$, and so $L \in G_2(\mathbb{C}^n)$. Let $Y \subset \mathbb{C}^n$ be a linear complement of the space $L$ and let $\tilde{H} = (\tilde{h}_1, \tilde{h}_2) : \mathbb{C}^n \to \mathbb{C}^2$ be the polynomial mapping defined by
\[
\tilde{H}(x + y) = H(x) \quad \text{for } x \in L, \ y \in Y.
\]
Then
\[
\mathcal{L}_\infty(\tilde{H}|_L) = \mathcal{L}_\infty(H) = \beta \quad \text{and} \quad \deg \tilde{H} = \deg H \leq (|k| + l) \cdot l.
\]  \hspace{1cm} (4.6)
Let $\Phi : \mathbb{C}^n \to \mathbb{C}^{m+2}$ be the polynomial mapping defined by
\[
\Phi(z) = (g(z)f_1(z), \ldots, g(z)f_m(z), f(z)\tilde{h}_1(z), f(z)\tilde{h}_2(z)) \quad \text{for } z \in \mathbb{C}^n.
\]
Then
\[ \Phi|_{V} = (F, 0) \quad \text{and} \quad \Phi|_{L} = (0, H). \] (4.7)

From (4.7) as well as the definition of \( \Phi \) and from (4.6), we get
\[ L_{\infty}(\Phi|_{V}) = L_{\infty}(F|_{V}) \quad \text{and} \quad L_{\infty}(\Phi|_{L}) = \beta. \]

Since \( \beta < L_{\infty}(F|_{V}) \) and \( L_{\infty}(\Phi|_{V \cup L}) = \min\{L_{\infty}(\Phi|_{V}), L_{\infty}(\Phi|_{L})\} \), we have
\[ L_{\infty}(\Phi|_{V \cup L}) = \beta. \] (4.8)

Observe that
\[ \deg \Phi \leq D, \] (4.9)

where
\[ D = (\delta(V))^n + \max\{d, (|k| + l) \cdot l\}. \]

Indeed,
\[ \deg \Phi = \max\{\deg gF, \deg fH\} = \max\{\deg g + \deg F, \deg f + \deg H\} \leq \max\{(\delta(V))^n + d, (\delta(V))^n + (|k| + l) \cdot l\} \]
\[ = (\delta(V))^n + \max\{d, (|k| + l) \cdot l\} = D. \]

From (4.8) we obtain \#(\Phi^{-1}(0) \cap (V \cup L)) < +\infty. According to Theorem 1.6, there exists a polynomial mapping \( \tilde{G}: \mathbb{C}^n \rightarrow \mathbb{C}^{m+2} \) such that
\[ \Phi|_{V \cup L} = \tilde{G}|_{V \cup L}, \quad L_{\infty}(\tilde{G}) = L_{\infty}(\Phi|_{V \cup L}) = \beta \]

and, by (4.9),
\[ \deg \tilde{G} \leq \deg \Phi \leq -\left[ -\left(\delta(V)\right)^n \left(D + 2 - L_{\infty}(F|_{V})\right) - L_{\infty}(F|_{V}) \right] + \delta(V). \] (4.10)

By Proposition 2.6 there exists a linear mapping \( T \in \Delta(m + 2, m) \) such that \( L_{\infty}(T \circ \tilde{G}) = L_{\infty}(\tilde{G}) = \beta \). Then, by (4.7), we get
\[ T \circ \tilde{G}|_{V} = T \circ \Phi|_{V} = T \circ (F, 0)|_{V} = F|_{V}. \]

Moreover, (4.10) gives
\[ \deg T \circ \tilde{G} \leq \deg \tilde{G} \leq -\left[ -\left(\delta(V)\right)^n \left(D + 2 - L_{\infty}(F|_{V})\right) - L_{\infty}(F|_{V}) \right] + \delta(V). \]

So, for \( G = T \circ \tilde{G} \), we obtain the assertion of Theorem 1.7. \( \square \)

5. Remarks on estimating the Łojasiewicz exponent

In this section we show that under some additional assumptions, from Theorem 1.3 we can obtain Theorem 1.4.

Remark 5.1. Let \( F = (f_1, \ldots, f_m): \mathbb{C}^n \rightarrow \mathbb{C}^m, m \geq n \geq 2, \) be a polynomial mapping, \( \deg f_j > 0, j = 1, \ldots, m. \) Let \( V = \{z \in \mathbb{C}^n: h_1(z) = \cdots = h_s(z) = 0\}, \) where \( h_1, \ldots, h_s \in \mathbb{C}[z_1, \ldots, z_n], \) \( \deg h_i > 0, i = 1, \ldots, s. \) If \( \dim V > 0 \) and \( v = \#(F^{-1}(0) \cap V) < +\infty, \) then there exists a polynomial mapping \( G: \mathbb{C}^n \rightarrow \mathbb{C}^m \) satisfying the following conditions:
(a) $F|_V = G|_V$,
(b) $\deg G \leq \max\{\deg f_1, \ldots, \deg f_m, \deg h_1, \ldots, \deg h_s\}$,
(c) $\mathcal{L}_\infty(G) \geq d_{m+s} - B(d_1, \ldots, d_{m+s}; n) + \nu$, where $d_1 \geq \cdots \geq d_{m+s} > 0$ is the non-decreasing rearrangement of the numbers $\deg f_1, \ldots, \deg f_m, \deg h_1, \ldots, \deg h_s$.

Indeed, let $H = (h_1, \ldots, h_s) : \mathbb{C}^n \to \mathbb{C}^s$. By our assumption the set $(F, H)^{-1}(0) = F^{-1}(0) \cap V$ is finite. Moreover, $\mathcal{L}_\infty(F|V) = \mathcal{L}_\infty((F, H)|V)$. Using Theorem 1.3 we have

$$\mathcal{L}_\infty((F, H)) \geq d_{m+s} - B(d_1, \ldots, d_{m+s}; n) + \nu.$$ 

From the estimate $\mathcal{L}_\infty((F, H)) \leq \mathcal{L}_\infty((F, H)|V) = \mathcal{L}_\infty(F|V)$ we infer, by Proposition 2.6, that there exists $T \in \Delta(m + s, m)$ such that $G = T \circ (F, H)$ satisfies the assertion.

From Remark 5.1 we get the following estimate of the Łojasiewicz exponent of a regular mapping.

**Remark 5.2.** Let $F = (f_1, \ldots, f_m) : \mathbb{C}^n \to \mathbb{C}^m$, $m \geq n \geq 2$, be a polynomial mapping, $\deg f_j > 0$, $j = 1, \ldots, m$. Let $V = \{z \in \mathbb{C}^n : h_1(z) = \cdots = h_s(z) = 0\}$, where $h_1, \ldots, h_s \in \mathbb{C}[z_1, \ldots, z_n]$, $\deg h_i > 0$, $i = 1, \ldots, s$. If $\dim V > 0$ and $\nu = \#(F^{-1}(0) \cap V) < +\infty$, then

$$\mathcal{L}_\infty(F|V) \geq d_{m+s} - B(d_1, \ldots, d_{m+s}; n) + \nu,$$

where $d_1 \geq \cdots \geq d_{m+s} > 0$ is the non-decreasing rearrangement of the numbers $\deg f_1, \ldots, \deg f_m, \deg h_1, \ldots, \deg h_s$.

Indeed, by Remark 5.1 there exists a polynomial mapping $G = (g_1, \ldots, g_m) : \mathbb{C}^n \to \mathbb{C}^m$ such that $G|_V = F|_V$ and

$$\mathcal{L}_\infty(G) \geq d_{m+s} - B(d_1, \ldots, d_{m+s}; n) + \nu.$$ 

Having the inequality $\mathcal{L}_\infty(F|V) \geq \mathcal{L}_\infty(G)$ we get the assertion.

Remark 5.2 yields Theorem 1.4 in the case when $\dim V = n - s > 0$, $\deg V = \deg h_1 \cdots \deg h_s$ and $\deg h_j \geq \deg f_j$ for $i = 1, \ldots, s$, $j = 1, \ldots, m$.

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**References**