Global convergence of a reaction–diffusion predator–prey model with stage structure and nonlocal delays

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Abstract

In this paper, a Lotka–Volterra type reaction–diffusion predator–prey model with stage structure for the prey and nonlocal delays due to gestation of the predator is investigated. In the case of a general domain, sufficient conditions are obtained for the global convergence of positive solutions of the proposed problem by using the energy function method. Numerical simulations are carried out to illustrate the main results. © 2007 Elsevier Ltd. All rights reserved.

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1. Introduction

Stage structure is a natural phenomenon and represents, for example, the division of a population into immature and mature individuals. As is common, the dynamics—eating habits, susceptibility to predators etc.—are often quite different in these two sub-populations. Hence, it is of ecological importance to investigate the effects of such a subdivision on the interaction of species.

Population models with stage structure are of current research interest in mathematical biology. They can exhibit phenomena similar to those of partial differential equations and many important physiological parameters can be incorporated [1]. Moreover, they are often much simpler than the corresponding models governed by partial differential equations. In [2], Chen proposed a stage-structured single-species population model without time delay. Let $N_i(t)$ and $N_m(t)$ denote the immature and mature population densities at time $t$, respectively. Then the following stage-structured single-species population model was discussed in [2]:

$$\begin{align*}
\dot{N}_i(t) &= B(t) - D_i(t) - W(t), \\
\dot{N}_m(t) &= \alpha W(t) - D_m(t).
\end{align*}$$

(1.1)
In (1.1), $B(t)$ is the birth rate of the immature population at time $t$; $D_i(t)$ and $D_m(t)$ are the death rates of the immature and mature at time $t$, respectively; $W(t)$ represents the transformation rate of the immature into the mature; $\alpha$ is the probability of the successful transformation of the immature into the mature. If it is assumed in model (1.1) that the birth rate obeys the Malthus rule, i.e., $B(t) = aN_i(t)$, the death rates of the immature and mature populations are logistic, and the transformation rate of the immature into mature is proportional to the immature population, i.e.,

$$D_i(t) = r_iN_i(t) + b_iN_i^2(t), \quad D_m(t) = r_mN_m(t) + b_mN_m^2(t),$$

and $W(t) = bN_i(t)$. Then we recover the model proposed by Chen for a single species with stage structure:

$$\dot{N}_i(t) = aN_i(t) - r_iN_i(t) - b_iN_i^2(t) - bN_i(t),$$

$$\dot{N}_m(t) = bN_i(t) - r_mN_m(t) - b_mN_m^2(t),$$

where $b = 1/\tau$ is the transformation rate of the immature into the mature in unit time and $\tau$ is the maturity. Following the work by Chen [2], many authors studied different kinds of stage-structured models and a significant body of work has been carried out (see, for example, [3–11]).

We note that the spatial content of the environment has been ignored in the aforementioned models. These models above have been traditionally formulated in relation to the time evolution of uniform population distributions in the habitat and are as such governed by ordinary differential equations. However, as argued in [12], in many ecological systems, the species under consideration may disperse spatially as well as evolving in time. This spatial dispersal or diffusion arises from the tendency of certain species to migrate towards regions of lower population density, mainly due to resource limitation: in regions of high population density, food will become scarce, and individuals will tend to migrate to regions of lower population density. In recent years, the effect of spatial dispersion of population in a bounded habitat has been taken into consideration, and in this situation the governing equations for the population densities are described by a system of reaction–diffusion equations.

An ecologically interesting and mathematically challenging problem is to determine under what condition the time-dependent solution converges to a positive steady-state solution, and what role is played by the effect of diffusion and time delays (see, for example, Pao [13–16]). It is argued that in more realistic ecological models, any delays should be spatially inhomogeneous, that is, the delay affects both the temporal and spatial variables. This is due to the fact that any given individual may not necessarily have been at the same spatial location at previous times. Such delays are called nonlocal. Recently, great attention has been paid to the study of ecological models with nonlocal delays (see, for example, Boshaba and Ruan [17], Britton [18], Gourley [19], Gourley and Britton [20,21], Gourley and Ruan [22,23], Yamada [24,25]).

Motivated by the work on a single-species model with stage structure by Chen [2], the work on competition model with nonlocal delays by Gourley and Ruan [22] and the work on predator–prey model with nonlocal delay by Yamada [25], in the present paper we discuss the following stage-structured reaction–diffusion predator–prey model with nonlocal delays

$$\frac{\partial u_1}{\partial t} = D_1\Delta u_1 + au_2(t, x) - r_1u_1(t, x) - a_{11}u_1^2(t, x) - bu_1(t, x) - a_{13}u_1(t, x)u_3(t, x),$$

$$\frac{\partial u_2}{\partial t} = D_2\Delta u_2 + bu_1(t, x) - r_2u_2(t, x) - a_{22}u_2^2(t, x) - a_{23}u_2(t, x)u_3(t, x),$$

$$\frac{\partial u_3}{\partial t} = D_3\Delta u_3 + u_3(t, x) \left( -r_3 + a_{31} \int_\Omega \int_{t-s}^{t} K_1(x, y, t-s)u_1(s, y)dsdy + a_{32} \int_\Omega \int_{t-s}^{t} K_2(x, y, t-s)u_2(s, y)dsdy - a_{33}u_3(t, x) \right)$$

for $t > 0$, $x \in \Omega$, with homogeneous Neumann boundary conditions

$$\frac{\partial u_1}{\partial v} = \frac{\partial u_2}{\partial v} = \frac{\partial u_3}{\partial v} = 0, \quad t > 0, \quad x \in \partial \Omega,$$

and initial conditions

$$u_i(\theta, x) = \phi_i(\theta, x), \quad (i = 1, 2, 3), \quad (\theta, x) \in (-\infty, 0] \times \tilde{\Omega},$$

(1.5)
In system (1.3), $u_1(t, x)$ and $u_2(t, x)$ represent the densities of the immature and mature prey populations at time $t$ and location $x$, respectively; $u_3(t, x)$ denotes the density of the predator population at time $t$ and location $x$. $\Omega$ is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial \Omega$, $\partial / \partial n$ denotes the outward normal derivative on $\partial \Omega$. The boundary condition in (1.4) implies that the populations do not move across the boundary $\partial \Omega$. The parameters $a, b, a_{11}, a_{13}, a_{22}, a_{32}, a_{33}, r_1, r_2, r_3, D_1, D_2$ and $D_3$ are positive constants. $\phi_1(\theta, x), \phi_2(\theta, x)$ and $\phi_3(\theta, x)$ are nonnegative and Hölder continuous and satisfy $\partial \phi_1 / \partial t = \partial \phi_2 / \partial t = \partial \phi_3 / \partial t = 0$ in $(-\infty, 0) \times \partial \Omega$.

The model is derived under the following assumptions.

(A1) The prey population: the birth rate is proportional to the existing mature population with a proportionality $a > 0$; the death rate of the immature population and the transformation rate from the immature individuals to mature individuals are proportional to the existing immature population with proportionality constants $r_1 > 0$ and $b > 0$, respectively. The death rate of the mature population is proportional to the existing mature population with a proportionality $r_2 > 0$; $a_{11}$ and $a_{22}$ are the intra-specific competition rates of the immature and mature populations, respectively.

(A2) The predator population: the growth of the species is of Lotka–Volterra nature. $a_{13}$ and $a_{23}$ are the capturing rates of the predator on the immature and mature prey, respectively; $r_3 > 0$ is the death rate of the predator; $a_{33} > 0$ is the intra-specific competition rate; $a_{31} / a_{13}$ and $a_{32} / a_{23}$ are the conversion rates of the predator by feeding on the immature and mature prey, respectively; the term \[ \int_{\Omega} \int_{-\infty}^{t} K_i(x, y, t-s)u_i(s, y)ds \ dy \]

represents a time delay due to the gestation of the predator, that is, mature adult predators can only contribute to the reproduction of predator biomass.

In system (1.3), we assume that the kernels $K_i(x, y, t)$ ($i = 1, 2$) depend on both the spatial and the temporal variables. The delay in this type of model formulation is called a spatio-temporal delay or nonlocal delay. The idea of this formulation is to account for the drift of individuals to their present position (at time $t$) from all previous times (see, for example Gourley and Britton [20,21], Gourley and Ruan [22], Yamada [25], Gourley and So [26] and the references cited therein). Here, we assume this drift cannot be viewed as being sufficiently small so as to be purely a local (in time) phenomenon (as in the previous paper).

In the present paper, we shall further assume that

\[ K_i(x, y, t) = G_i(x, y, t)k_i(t), \quad x, y \in \Omega, k_i(t) \geq 0, \]

\[ \int_{\Omega} G_i(x, y, t)dx = \int_{\Omega} G_i(x, y, t)dy = 1, \quad t \geq 0, \]

\[ \int_{0}^{\infty} k_i(t)dt = 1, \quad tk_i(t) \in L^1((0, \infty); R), \quad (1.6) \]

where $G_i(x, y, t)$ ($i = 1, 2$) are nonnegative functions which are continuous in $(x, y) \in \tilde{\Omega} \times \tilde{\Omega}$ for each $t \in [0, \infty)$ and measurable in $t \in [0, \infty)$ for each pair $(x, y) \in \tilde{\Omega} \times \tilde{\Omega}$.

Throughout this paper, we need the following notations.

For $1 \leq p \leq \infty$, let $L^p(\Omega)$ denote the Banach space of Lesbegue measurable functions $u$ on $\Omega$ satisfying

\[ \| u \|_p = \left\{ \left( \int_{\Omega} |u(x)|^p dx \right)^{1/p} \right\} \text{ if } 1 \leq p < \infty, \]

\[ \text{ess sup}_{x \in \Omega} |u(x)| < \infty \text{ if } p = \infty. \]

In particular, if $p = 2$, $L^2(\Omega)$ becomes a Hilbert space with the usual inner product $\langle \cdot, \cdot \rangle$ and $\| \cdot \|_2^2 = \langle \cdot, \cdot \rangle$. Let $\| \cdot \|_2$ denote the norm in $L^2((0, T); L^2(\Omega; R))$, i.e.,

\[ \| u \|_2 = \left( \int_{0}^{T} \| u(s) \|_2^2 ds \right)^{1/2}. \]
Further, for \( m \in \mathbb{N}, 1 \leq p \leq +\infty \), the Sobolev space \( W^{m,p}(\Omega) \) is defined by
\[
W^{m,p}(\Omega) = \{ f \in L^p(\Omega) : \forall |\alpha| \leq m, \partial_x^\alpha f \in L^p(\Omega) \},
\]
where \( \alpha = (\alpha_1, \ldots, \alpha_n), |\alpha| = \alpha_1 + \cdots + \alpha_n \), and the derivatives \( \partial_x^\alpha f = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} f \) are taken in a weak sense.

When endowed with the norm
\[
\| f \|_{m,p,\Omega} = \sum_{|\alpha| \leq m} \| \partial_x^\alpha f \|_p,
\]
\( W^{m,p}(\Omega) \) is a Banach space (see, for example, Adams [27]).

This paper is organized as follows. In the next section, we shall discuss the global attractivity of each of the nonnegative steady-state solutions of system (1.3) with boundary conditions (1.4) and initial conditions (1.5) by using the energy function method. Numerical simulations are presented in Section 3 to illustrate our main results. In Section 4, a discussion is given to conclude this work.

2. Convergence

In this section, we study the global attractivity of each of the nonnegative steady-state solutions of problem (1.3)–(1.5) by using the energy function method.

It is clear that system (1.3) always has a trivial steady-state solution \( E_0(0, 0, 0) \). It is easy to show that system (1.3) admits a semi-trivial steady-state solution \( E_1(u_1^0, u_2^0, 0) \) if the following holds:

\[ (H1) \quad r_2(r_1 + b) < ab, \]

where \( (u_1^0, u_2^0) \) is the unique solution of the following system of algebra equations
\[
au_2 - (r_1 + b)u_1 - a_11u_1^2 = 0,
\]
\[
bu_1 - r_2u_2 - a_22u_2^2 = 0.
\]

It is readily seen that \((0, 0, 0)\) and \((M_1, M_2, M_3)\) are a pair of coupled lower–upper solutions of problem (1.3)–(1.5), where

\[
M_1 = \max \left\{ \frac{a}{a_11} \sup_{\theta \leq 0} \| \phi_1(\theta, \cdot) \|_{C(\hat{\Omega} ; R)} \right\},
M_2 = \max \left\{ \frac{a}{a_11} \sup_{\theta \leq 0} \| \phi_2(\theta, \cdot) \|_{C(\hat{\Omega} ; R)} \right\},
M_3 = \max \left\{ \frac{a_{31}M_1 + a_{32}M_2}{a_{33}}, \sup_{\theta \leq 0} \| \phi_3(\theta, \cdot) \|_{C(\hat{\Omega} ; R)} \right\}.
\]

Hence, the global existence of solutions \((u_1(t, x), u_2(t, x), u_3(t, x))\) to problem (1.3)–(1.5) can be derived based on the theory of upper–lower solution pairs (see, for example, Redlinger [28]). It follows that \( 0 \leq u_i(t, x) \leq M_i(i = 1, 2, 3) \) for \((t, x) \in (-\infty, \infty) \times \hat{\Omega} \). By the strong maximum principle, if \( \phi_i(0, x) \neq 0(i = 1, 2, 3) \), we have \( u_i(t, x) > 0(i = 1, 2, 3) \) for all \( t > 0, x \in \hat{\Omega} \).

We are now in a position to state and prove our main result on the global attractivity of the positive uniform steady-state solution to system (1.3).

**Theorem 2.1.** Let \((u_1(t, x), u_2(t, x), u_3(t, x))\) be a solution of system (1.3) with boundary conditions (1.4) and initial conditions (1.5), \( \phi_i(0, x) \neq 0(i = 1, 2, 3) \). Suppose that system (1.3) has a unique positive constant steady-state solution \( E^*(u_1^*, u_2^*, u_3^*) \). If one of the following conditions holds:

\[ (H2) \quad a_{13} = a_{31} = 0, a_{22}a_{33} > a_{23}a_{32}; \]
\[ (H3) \quad a_{23} = a_{32} = 0, a_{11}a_{33} > a_{13}a_{31}. \]
\[
\text{Then}\quad \lim_{t\to +\infty} (u_1(t, x), u_2(t, x), u_3(t, x)) = (u^*_1, u^*_2, u^*_3) \quad \text{uniformly for } x \in \bar{\Omega}.
\]

**Proof.** System (1.3) can be rewritten as

\[
\frac{\partial u_1}{\partial t} = D_1 \Delta u_1 + \frac{a}{u_1^*}[(-u_2(t, x))(u_1(t, x) - u_1^*) + u_1(t, x)(u_2(t, x) - u_2^*)] - u_1(t, x)[a_{11}(u_1(t, x) - u_1^*) + a_{13}(u_3(t, x) - u_3^*)],
\]

\[
\frac{\partial u_2}{\partial t} = D_2 \Delta u_2 + \frac{b}{u_2^*}[(-u_1(t, x))(u_2(t, x) - u_2^*) + u_2(t, x)(u_1(t, x) - u_1^*)] - u_2(t, x)[a_{22}(u_2(t, x) - u_2^*) + a_{23}(u_3(t, x) - u_3^*)],
\]

\[
\frac{\partial u_3}{\partial t} = D_3 \Delta u_3 + u_3(t, x) \left\{ a_{31} \int_{\bar{\Omega}} \int_{\Omega} K_1(x, y, t - s)(u_1(s, y) - u_1^* )dsdy + a_{32} \int_{\bar{\Omega}} \int_{\Omega} K_2(x, y, t - s)(u_2(s, y) - u_2^* )dsdy - a_{33}(u_3(t, x) - u_3^*) \right\}.
\]

Define

\[
V_1(t) = \sum_{i=1}^{3} \alpha_i \int_{\Omega} \left( u_i(t, x) - u_i^* - u_i^* \ln \frac{u_i(t, x)}{u_i^*} \right) dx,
\]

where \( \alpha_3 = 1, \alpha_1 \) and \( \alpha_2 \) are positive constants to be determined.

Calculating the derivative of \( V_1(t) \) along the positive solution of system (1.3), we derive that

\[
\frac{d}{dt} V_1(t) = \sum_{i=1}^{2} \alpha_i \int_{\Omega} \frac{\partial u_i}{\partial t} \left( 1 - \frac{u_i^*}{u_i} \right) dx + \int_{\Omega} \frac{\partial u_3}{\partial t} \left( 1 - \frac{u_3^*}{u_3} \right) dx
\]

\[
= -\alpha_1 D_1 u_1^* \int_{\Omega} \left| \nabla u_1 \right|^2 dx + \frac{\alpha_1 a}{u_1^*} \left\{ - \int_{\Omega} \frac{u_2(t, x)}{u_1(t, x)} (u_1(t, x) - u_1^*)^2 dx + \int_{\Omega} (u_1(t, x) - u_1^*)(u_2(t, x) - u_2^*) dx \right\} - \alpha_{11} \int_{\Omega} (u_1(t, x) - u_1^*)^2 dx
\]

\[-\alpha_{13} \int_{\Omega} (u_1(t, x) - u_1^*)(u_3(t, x) - u_3^*) dx
\]

\[-\alpha_2 D_2 u_2^* \int_{\Omega} \left| \nabla u_2 \right|^2 dx + \frac{\alpha_2 b}{u_2^*} \left\{ - \int_{\Omega} \frac{u_1(t, x)}{u_2(t, x)} (u_2(t, x) - u_2^*)^2 dx + \int_{\Omega} (u_1(t, x) - u_1^*)(u_2(t, x) - u_2^*) dx \right\} - \alpha_2 \int_{\Omega} (u_2(t, x) - u_2^*)^2 dx
\]

\[+ a_{23} \int_{\Omega} (u_2(t, x) - u_2^*)(u_3(t, x) - u_3^*) dx - D_3 u_3^* \int_{\Omega} \left| \nabla u_3 \right|^2 dx - a_{33} \int_{\Omega} (u_3(t, x) - u_3^*)^2 dx
\]

\[+ a_{31} \int_{\Omega} \int_{\Omega} K_1(x, y, t - s)(u_1(s, y) - u_1^*)(u_3(t, x) - u_3^*) dsdy dx + a_{32} \int_{\Omega} \int_{\Omega} K_2(x, y, t - s)(u_2(s, y) - u_2^*)(u_3(t, x) - u_3^*) dsdy dx.
\]

Taking \( \alpha_1 a / u_1^* = \alpha_2 b / u_2^* \), then it follows from (2.4) that

\[
\frac{d}{dt} V_1(t) = -2 \sum_{i=1}^{2} \alpha_i D_i u_i^* \int_{\Omega} \left| \nabla u_i \right|^2 dx - D_3 u_3^* \int_{\Omega} \left| \nabla u_3 \right|^2 dx
\]
\[-\frac{\alpha_1 a}{u_1^*} \int_\Omega \left[ \frac{u_2(t,x)}{u_1(t,x)} (u_1(t,x) - u_1^*) - \frac{u_1(t,x)}{u_2(t,x)} (u_2(t,x) - u_2^*) \right]^2 \, dx \]

\[-\alpha_1 a_{11} \int_\Omega (u_1(t,x) - u_1^*)^2 \, dx - \alpha_1 a_{13} \int_\Omega (u_1(t,x) - u_1^*) (u_3(t,x) - u_3^*) \, dx \]

\[-\alpha_2 a_{22} \int_\Omega (u_2(t,x) - u_2^*)^2 \, dx \]

\[-\alpha_2 a_{23} \int_\Omega (u_2(t,x) - u_2^*) (u_3(t,x) - u_3^*) \, dx - \alpha_3 \int_\Omega (u_3(t,x) - u_3^*)^2 \, dx \]

\[+ \alpha_1 \int_\Omega \int_\Omega \int_{-\infty}^t K_1(x,y,t-s) (u_1(s,y) - u_1^*) (u_3(t,x) - u_3^*) \, ds \, dy \, dx \]

\[+ \alpha_3 \int_\Omega \int_\Omega \int_{-\infty}^t K_2(x,y,t-s) (u_2(s,y) - u_2^*) (u_3(t,x) - u_3^*) \, ds \, dy \, dx. \]  

(2.5)

Using the inequality $ab \leq \frac{1}{2} \lambda a^2 + \frac{1}{2\lambda} b^2$, we derive from (2.5) that

\[ \frac{d}{dt} V_1(t) \leq - \sum_{i=1}^2 \alpha_i D_i u_i^* \int_\Omega \frac{\left| \nabla u_i \right|^2}{u_i^2} \, dx - D_3 u_3^* \int_\Omega \frac{\left| \nabla u_3 \right|^2}{u_3^2} \, dx - \alpha_1 a_{11} \int_\Omega (u_1(t,x) - u_1^*)^2 \, dx \]

\[+ \alpha_1 a_{13} \int_\Omega \left\{ \frac{1}{2} \lambda_1 (u_1(t,x) - u_1^*)^2 + \frac{1}{2\lambda_1} (u_3(t,x) - u_3^*)^2 \right\} \, dx - \alpha_2 a_{22} \int_\Omega (u_2(t,x) - u_2^*)^2 \, dx \]

\[+ \alpha_2 a_{23} \int_\Omega \left\{ \frac{1}{2} \lambda_2 (u_2(t,x) - u_2^*)^2 + \frac{1}{2\lambda_2} (u_3(t,x) - u_3^*)^2 \right\} \, dx \]

\[+ \alpha_1 \int_\Omega \int_\Omega \int_{-\infty}^t K_1(x,y,t-s) \left\{ \frac{1}{2} \lambda_1 (u_1(s,y) - u_1^*)^2 + \frac{1}{2\lambda_1} (u_3(t,x) - u_3^*)^2 \right\} \, ds \, dy \, dx \]

\[+ \alpha_3 \int_\Omega \int_\Omega \int_{-\infty}^t K_2(x,y,t-s) \left\{ \frac{1}{2} \lambda_2 (u_2(s,y) - u_2^*)^2 + \frac{1}{2\lambda_2} (u_3(t,x) - u_3^*)^2 \right\} \, ds \, dy \, dx \]

\[- a_3 \int_\Omega (u_3(t,x) - u_3^*)^2 \, dx. \]  

(2.6)

Noting the property of $K(x,y,t)$ in (1.6) we get from (2.6) that

\[ \frac{d}{dt} V_1(t) \leq - \sum_{i=1}^2 \alpha_i D_i u_i^* \int_\Omega \frac{\left| \nabla u_i \right|^2}{u_i^2} \, dx - D_3 u_3^* \int_\Omega \frac{\left| \nabla u_3 \right|^2}{u_3^2} \, dx \]

\[+ \left( \alpha_1 - \frac{1}{2} \lambda_1 a_{13} \right) \int_\Omega (u_1(t,x) - u_1^*)^2 \, dx - \alpha_2 \left( a_{22} - \frac{1}{2} \lambda_2 a_{23} \right) \int_\Omega (u_2(t,x) - u_2^*)^2 \, dx \]
that for $i = 1, 2,$
\[
\int_{\Omega} \int_{t-r}^{t} K_i(x, y, r)(u_i(t, y) - u_i^*)^2 dy dx = \int_{\Omega} \int_{0}^{\infty} K_i(x, y, r)(u_i(t, y) - u_i^*)^2 dr dy dx
\]
\[
= \int_{\Omega} (u_i(t, y) - u_i^*)^2 dy,
\]
we derive from (2.9) that
\[
\frac{d}{dt} V(t) \leq - \sum_{i=1}^{2} \alpha_i D_i u_i^* \int_{\Omega} \frac{|\nabla u_i|^2}{u_i} dx - D_3 u_3^* \int_{\Omega} \frac{|\nabla u_3|^2}{u_3} dx
\]
\[
- \alpha_1 \left( a_{11} - \frac{1}{2} \lambda_1 a_{13} \right) \int_{\Omega} (u_1(t, x) - u_1^*)^2 dx - \alpha_2 \left( a_{22} - \frac{1}{2} \lambda_2 a_{23} \right) \int_{\Omega} (u_2(t, x) - u_2^*)^2 dx
\]
\[
- \left( a_{33} - \frac{1}{2} \lambda_1 a_{13} - \frac{1}{2} \lambda_1 a_{31} - \frac{1}{2} \lambda_1 a_{32} \right) \int_{\Omega} (u_3(t, x) - u_3^*)^2 dx
\]
\[
- \frac{1}{2} \lambda_1 a_{13} \int_{\Omega} \int_{0}^{\infty} K_1(x, y, r)(u_1(t, y) - u_1^*)^2 dr dy dx
\]
\[
+ \frac{1}{2} \lambda_2 a_{23} \int_{\Omega} \int_{0}^{\infty} K_2(x, y, r)(u_2(t, y) - u_2^*)^2 dr dy dx.
\]

For any $T > 0$, integrating (2.10) over $[0, T]$, we derive that
\[
\sum_{i=1}^{2} \alpha_i D_i u_i^* \left\| \frac{\nabla u_i}{u_i} \right\|^2_2 + D_3 u_3^* \left\| \frac{\nabla u_3}{u_3} \right\|^2_2 + \alpha_1 a_{11} \| u_1 - u_1^* \|_2^2 + \alpha_2 a_{22} \| u_2 - u_2^* \|_2^2 + a_{33} \| u_3 - u_3^* \|_2^2
\]
\[
\leq V(0) \left[ \frac{1}{2} \lambda_1 (\alpha_1 a_{13} + a_{31}) \| u_1 - u_1^* \|_2^2 + \frac{1}{2} \lambda_2 (\alpha_2 a_{23} + a_{32}) \| u_2 - u_2^* \|_2^2 \right] 
\]
If \( a_{23} = 0, \ a_{32} = 0 \), we first choose
\[
\lambda_1 = \frac{\alpha_1a_{13} + a_{31}}{2a_{33}}.
\]
Then it follows from (2.11) that
\[
\sum_{i=1}^{2} \alpha_i D_i u_i^* \| \frac{\nabla u_i}{u_i} \|_2^2 + D_3 u_3^* \| \frac{\nabla u_3}{u_3} \|_2^2 + \alpha_1a_{11} \| u_1 - u_1^* \|_2^2 + \alpha_2a_{22} \| u_2 - u_2^* \|_2^2 \\
\leq V(0) + \frac{(\alpha_1a_{13} + a_{31})^2}{4a_{33}} \| u_1 - u_1^* \|_2^2.
\]
(2.12)

Noting that \( a_{11}a_{33} > a_{13}a_{31} \), we can choose \( \alpha_1 > 0 \) such that
\[
\alpha_1a_{11} > \frac{(\alpha_1a_{13} + a_{31})^2}{4a_{33}}.
\]
We therefore derive that
\[
\| \frac{\nabla u_1}{u_1} \|_2 \leq C_1, \quad \| \frac{\nabla u_2}{u_2} \|_2 \leq C_2,
\]
(2.13)
and
\[
\| u_1 - u_1^* \|_2 \leq C_3, \quad \| u_2 - u_2^* \|_2 \leq C_4,
\]
(2.14)
for some constants \( C_i \) (\( i = 1, 2, 3, 4 \)) independent of \( T \).

Noting that \( u_1(t, x), u_2(t, x) \) are bounded above, it follows from (2.13) that
\[
\| \nabla u_1 \|_2 \leq C_5, \quad \| \nabla u_2 \|_2 \leq C_6
\]
(2.15)
for some positive constants \( C_5 \) and \( C_6 \) independent of \( T \). It therefore follows from (2.13) and (2.15) that \( u_1(t, x) - u_1^*, \ u_2(t, x) - u_2^* \in L^2((0, \infty); W^{1,2}(\Omega; \mathbb{R})) \). Hence, we have
\[
\lim_{t \to +\infty} \| u_1(t) - u_1^* \|_{W^{1,2}} = 0, \quad \lim_{t \to +\infty} \| u_2(t) - u_2^* \|_{W^{1,2}} = 0.
\]
It follows from the Sobolev compact embedding theorem (see, for example, [27]) that
\[
\lim_{t \to +\infty} \| u_1(t) - u_1^* \|_{C(\bar{\Omega}; \mathbb{R})} = 0, \quad \lim_{t \to +\infty} \| u_2(t) - u_2^* \|_{C(\bar{\Omega}; \mathbb{R})} = 0.
\]
In a similar way, by choosing
\[
\lambda_1 = \frac{2\alpha_1a_{11}}{\alpha_1a_{13} + a_{31}},
\]
one can show that
\[
\| \nabla u_3 \|_2 \leq C_7, \quad \| u_3 - u_3^* \|_2 \leq C_8
\]
for some positive constants \( C_7 \) and \( C_8 \) independent of \( T > 0 \). We therefore get
\[
\lim_{t \to +\infty} \| u_3(t) - u_3^* \|_{C(\bar{\Omega}; \mathbb{R})} = 0.
\]
If \( a_{13} = a_{31} = 0, \ a_{22}a_{33} > a_{23}a_{32} \), using several similar arguments, one can also derive
\[
\lim_{t \to +\infty} \| u_i(t) - u_i^* \|_{C(\bar{\Omega}; \mathbb{R})} = 0 \quad (i = 1, 2, 3),
\]
This completes the proof. \( \Box \)
Theorem 2.2. Let \((u_1(t, x), u_2(t, x), u_3(t, x))\) be a solution of system (1.3) with boundary conditions (1.4) and initial conditions (1.5), \(\phi_i(0, x) \neq 0 (i = 1, 2, 3)\). Let (H1), (H2) or (H1), (H3) hold. Assume further that

(H4) \(a_{31}u_1^0 + a_{32}u_2^0 < r_3\).

Then

\[
\lim_{t \to +\infty} (u_1(t, x), u_2(t, x), u_3(t, x)) = (u_1^0, u_2^0, 0) \quad \text{uniformly for } x \in \Omega.
\]

Proof. System (1.3) can be rewritten as

\[
\frac{\partial u_1}{\partial t} = D_1 \Delta u_1 + \frac{a}{u_1^0}[-u_2(t, x)(u_1(t, x) - u_1^0) + u_1(t, x)(u_2(t, x) - u_2^0)]
- a_{11}u_1(t, x)(u_1(t, x) - u_1^0) - a_{13}u_1(t, x)u_3(t, x),
\]

\[
\frac{\partial u_2}{\partial t} = D_2 \Delta u_2 + \frac{b}{u_2^0}[-u_1(t, x)(u_2(t, x) - u_2^0) + u_2(t, x)(u_1(t, x) - u_1^0)]
- a_{22}u_2(t, x)(u_2(t, x) - u_2^0) - a_{23}u_2(t, x)u_3(t, x),
\]

\[
\frac{\partial u_3}{\partial t} = D_3 \Delta u_3 + u_3(t, x) \left[ -r_3 + a_{31}u_1^0 + a_{32}u_2^0 - a_{33}u_3(t, x) 
+ a_{31} \int_{\Omega} \int_{-\infty}^{t} K_1(x, y, t - s)(u_1(s, y) - u_1^0)dyds
+ a_{32} \int_{\Omega} \int_{-\infty}^{t} K_2(x, y, t - s)(u_2(s, y) - u_2^0)dyds \right].
\]

Define

\[
V_1(t) = \sum_{i=1}^{2} \alpha_i \int_{\Omega} \left( u_i - u_i^0 - u_i^0 \ln \frac{u_i}{u_i^0} \right) dx + \int_{\Omega} u_3(t, x) dx,
\]

where \(\alpha_1\) and \(\alpha_2\) are positive constants to be determined.

Calculating the derivative of \(V_1(t)\) along a positive solution of problem (1.3)–(1.5), it follows that

\[
\frac{d}{dt} V_1(t) = \sum_{i=1}^{2} \alpha_i \int_{\Omega} \frac{\partial u_i}{\partial t} \left( 1 - \frac{u_i^0}{u_i} \right) dx + \int_{\Omega} \frac{\partial u_3}{\partial t} dx
= -\alpha_1 D_1 u_1^0 \int_{\Omega} \frac{|\nabla u_1|^2}{u_1^2} dx + \alpha_1 a \left[ -\int_{\Omega} \frac{u_2(t, x)}{u_1(t, x)} (u_1(t, x) - u_1^0)^2 dx 
+ \int_{\Omega} (u_1(t, x) - u_1^0)(u_2(t, x) - u_2^0) dx \right]
- \alpha_1 a_{11} \int_{\Omega} (u_1(t, x) - u_1^0)^2 dx - \alpha_1 a_{13} \int_{\Omega} u_3(t, x)(u_1(t, x) - u_1^0) dx
- \alpha_2 D_2 u_2^0 \int_{\Omega} \frac{|\nabla u_2|^2}{u_2^2} dx + \alpha_2 b \left[ -\int_{\Omega} \frac{u_1(t, x)}{u_2(t, x)} (u_2(t, x) - u_2^0)^2 dx 
+ \int_{\Omega} (u_1(t, x) - u_1^0)(u_2(t, x) - u_2^0) dx \right]
- \alpha_2 a_{22} \int_{\Omega} (u_2(t, x) - u_2^0)^2 dx - \alpha_2 a_{23} \int_{\Omega} u_3(t, x)(u_2(t, x) - u_2^0) dx
+ (-r_3 + a_{31}u_1^0 + a_{32}u_2^0) \int_{\Omega} u_3(t, x) dx - a_{33} \int_{\Omega} u_3(t, x) dx
\]
Using the inequality
\[ \int_0^t \int_\Omega \int_{-\infty}^t K_1(x, y, t - s)u_3(t, x)(u_1(s, y) - u_1^0)dx\,dy\,ds \]
\[ + \alpha_3 \int_0^t \int_\Omega \int_{-\infty}^t K_2(x, y, t - s)u_3(t, x)(u_2(s, y) - u_2^0)dx\,dy\,ds. \]  
(2.18)

Taking \( \alpha_1\alpha_1^0 = \alpha_2\beta/\beta_2^0 \), and noting that \( \alpha_3\alpha_1^0 + \alpha_2\beta_2^0 < \gamma_3 \), we derive from (2.18) that
\[
\frac{d}{dt} V_1(t) \leq -\alpha_1 D_1 u_1^0 \int_\Omega \frac{|\nabla u_1|^2}{u_1^2} dx - \alpha_2 D_2 u_2^0 \int_\Omega \frac{|\nabla u_2|^2}{u_2^2} dx 
- \alpha_1 a_{11} \int_\Omega (u_1(t, x) - u_1^0)^2 dx
- \alpha_1 a_{13} \int_\Omega u_3(t, x)(u_1(t, x) - u_1^0)dx
- \alpha_2 a_{22} \int_\Omega (u_2(t, x) - u_2^0)^2 dx
- \alpha_2 a_{23} \int_\Omega u_3(t, x)(u_2(t, x) - u_2^0)dx
+ \alpha_1 \int_\Omega \int_\Omega \int_{-\infty}^t K_1(x, y, t - s)u_3(t, x)(u_1(s, y) - u_1^0)dx\,dy\,ds
+ \alpha_2 \int_\Omega \int_\Omega \int_{-\infty}^t K_2(x, y, t - s)u_3(t, x)(u_2(s, y) - u_2^0)dx\,dy\,ds.
\]  
(2.19)

Using the inequality \( ab \leq \frac{1}{2} \lambda a^2 + \frac{1}{2\lambda} b^2 \), it follows from (2.19) that
\[
\frac{d}{dt} V_1(t) \leq -\alpha_1 D_1 u_1^0 \int_\Omega \frac{|\nabla u_1|^2}{u_1^2} dx - \alpha_2 D_2 u_2^0 \int_\Omega \frac{|\nabla u_2|^2}{u_2^2} dx
- \alpha_1 \left( a_{11} \frac{1}{2} \lambda_{11} \right) \int_\Omega (u_1(t, x) - u_1^0)^2 dx
- \alpha_2 \left( a_{22} \frac{1}{2} \lambda_{22} \right) \int_\Omega (u_2(t, x) - u_2^0)^2 dx
- \left\{ a_{33} - \frac{1}{2\lambda_1} (a_{11} + a_{31}) - \frac{1}{2\lambda_2} (a_{22} + a_{32}) \right\} \int_\Omega u_3^2(t, x)dx
+ \frac{1}{2} \lambda a_{31} \int_\Omega \int_\Omega \int_0^\infty K_1(x, y, t - r)(u_1(t, y) - u_1^0)^2 dr\,dy\,dx
+ \frac{1}{2} \lambda a_{32} \int_\Omega \int_\Omega \int_0^\infty K_2(x, y, t - r)(u_2(t, y) - u_2^0)^2 dr\,dy\,dx.
\]  
(2.20)

Define
\[
V(t) = V_1(t) + \frac{1}{2} \lambda_1 a_{31} \int_\Omega \int_\Omega \int_0^\infty \int_{t-r}^t K_1(x, y, t - r)(u_1(t, y) - u_1^0)^2 dl\,dr\,dy\,dx
+ \frac{1}{2} \lambda_2 a_{32} \int_\Omega \int_\Omega \int_0^\infty \int_{t-r}^t K_2(x, y, t - r)(u_2(t, y) - u_2^0)^2 dl\,dr\,dy\,dx.
\]  
(2.21)

It follows from (2.20) and (2.21) that
\[
\frac{d}{dt} V(t) \leq -\alpha_1 D_1 u_1^0 \int_\Omega \frac{|\nabla u_1|^2}{u_1^2} dx - \alpha_2 D_2 u_2^0 \int_\Omega \frac{|\nabla u_2|^2}{u_2^2} dx
- \left\{ a_{11} - \frac{1}{2} \lambda_1 (a_{11} + a_{31}) \right\} \int_\Omega (u_1(t, x) - u_1^0)^2 dx
- \left\{ a_{22} - \frac{1}{2} \lambda_2 (a_{22} + a_{32}) \right\} \int_\Omega (u_2(t, x) - u_2^0)^2 dx
- \left\{ a_{33} - \frac{1}{2\lambda_1} (a_{11} + a_{31}) - \frac{1}{2\lambda_2} (a_{22} + a_{32}) \right\} \int_\Omega u_3^2(t, x)dx.
\]  
(2.22)
For any $T > 0$, integrating (2.22) over $[0, T]$ we derive
\[ \alpha_1 D_1 u_1^0 \left\| \nabla u_1 \right\|_2^2 + \alpha_2 D_2 u_2^0 \left\| \nabla u_2 \right\|_2^2 + \alpha_1 a_{11} \| u_1 - u_1^0 \|_2^2 + \alpha_2 a_{22} \| u_2 - u_2^0 \|_2^2 + a_{33} \| u_3 \|_2^2 \]
\[ \leq V(0) + \frac{1}{2} \lambda_1 (\alpha_1 a_{13} + a_{31}) \| u_1 - u_1^0 \|_2^2 + \frac{1}{2} \lambda_2 (\alpha_2 a_{23} + a_{32}) \| u_2 - u_2^0 \|_2^2 \]
\[ + \left[ \frac{1}{2 \lambda_1} (\alpha_1 a_{13} + a_{31}) + \frac{1}{2 \lambda_2} (\alpha_2 a_{23} + a_{32}) \right] \| u_3 \|_2^2. \]

Using similar arguments to those in the proof of Theorem 2.1 we can show that
\[ \lim_{t \to +\infty} \| u_i(t) - u_i^0 \|_{C(\Omega; R)} = 0 \quad (i = 1, 2), \quad \lim_{t \to +\infty} \| u_3(t) \|_{C(\Omega; R)} = 0. \]

This completes the proof. \( \square \)

**Theorem 2.3.** Let \((u_1(t, x), u_2(t, x), u_3(t, x))\) be a solution of system (1.3) with boundary conditions (1.4) and initial conditions (1.5), \( \phi_i(0, x) \neq 0 (i = 1, 2, 3) \). Assume the following holds:

(H5) \( ab \leq r_2(b + r_1) \).

Then
\[ \lim_{t \to +\infty} (u_1(t, x), u_2(t, x), u_3(t, x)) = (0, 0, 0) \text{ uniformly for } x \in \bar{\Omega}. \]

**Proof.** Define
\[ V_1(t) = \frac{b}{r_1 + b} \int_\Omega u_1(t, x)dx + \int_\Omega u_2(t, x)dx + \alpha \int_\Omega u_3(t, x)dx, \quad (2.23) \]

where \( \alpha \) is a positive constant to be determined.

Calculating the derivative of \( V_1(t) \) along the positive solution of system (1.3), it follows that
\[ \frac{d}{dt} V_1(t) = \frac{b}{r_1 + b} \int_\Omega \frac{\partial u_1}{\partial t} dx + \int_\Omega \frac{\partial u_2}{\partial t} dx + \alpha \int_\Omega \frac{\partial u_3}{\partial t} dx \]
\[ = -\frac{b a_{11}}{r_1 + b} \int_\Omega u_1^2(t, x)dx - \frac{b a_{12}}{r_1 + b} \int_\Omega u_1(t, x)u_3(t, x)dx + \left( \frac{ab}{r_1 + b} - r_2 \right) \int_\Omega u_2(t, x)dx \]
\[ - a_{22} \int_\Omega u_2^2(t, x)dx - a_{23} \int_\Omega u_2(t, x)u_3(t, x)dx - \alpha \int_\Omega u_3(t, x)dx - \alpha a_{33} \int_\Omega u_3^2(t, x)dx \]
\[ + \alpha a_{31} \int_\Omega \int_\Omega \int_{-\infty}^t K_1(x, y, t - s)u_3(t, x)u_1(s, y)dsdydx \]
\[ + \alpha a_{32} \int_\Omega \int_\Omega \int_{-\infty}^t K_2(x, y, t - s)u_3(t, x)u_2(s, y)dsdydx. \]

Noting that \( ab \leq r_2(r_1 + b) \), we derive that
\[ \frac{d}{dr} V_1(t) \leq -\frac{b a_{11}}{r_1 + b} \int_\Omega u_1^2(t, x)dx - a_{22} \int_\Omega u_2^2(t, x)dx - \alpha a_{33} \int_\Omega u_3^2(t, x)dx \]
\[ + \alpha a_{31} \int_\Omega \int_\Omega \int_0^t K_1(x, y, r)u_3(t, x)u_1(t - r, y)drdydx \]
\[ + \alpha a_{32} \int_\Omega \int_\Omega \int_0^t K_2(x, y, r)u_3(t, x)u_2(t - r, y)drdydx. \quad (2.24) \]

Using the inequality \( ab \leq \frac{1}{2} \lambda_1 a^2 + \frac{1}{2 \lambda_2} b^2 \), it follows from (2.24) that
\[ \frac{d}{dr} V_1(t) \leq -\frac{b a_{11}}{r_1 + b} \int_\Omega u_1^2(t, x)dx - a_{22} \int_\Omega u_2^2(t, x)dx - \alpha \left( a_{33} - \frac{1}{2 \lambda_1} a_{31} - \frac{1}{2 \lambda_2} a_{32} \right) \int_\Omega u_3^2(t, x)dx \]
\[ V(t) = V_1(t) + \frac{1}{2} \lambda_1 \alpha a_{31} \int_{\Omega} \int_{\Omega} \int_{0}^{\infty} K_1(x, y, r)u_1^2(t - r, y)drdydx \]
\[ + \frac{1}{2} \lambda_2 \alpha a_{32} \int_{\Omega} \int_{\Omega} \int_{0}^{\infty} K_2(x, y, r)u_2^2(t - r, y)drdydx. \]  

(2.25)

Define
\[ V(t) = V_1(t) + \frac{1}{2} \lambda_1 \alpha a_{31} \int_{\Omega} \int_{\Omega} \int_{0}^{\infty} \int_{t-r}^{t} K_1(x, y, r)u_1^2(l, y)dl dr dy dx \]
\[ + \frac{1}{2} \lambda_2 \alpha a_{32} \int_{\Omega} \int_{\Omega} \int_{0}^{\infty} \int_{t-r}^{t} K_2(x, y, r)u_2^2(l, y)dl dr dy dx. \]  

(2.26)

We derive from (2.25) and (2.26) that
\[ \frac{d}{dt} V(t) \leq - \left( \frac{ba_{11}}{r_1 + b} - \frac{1}{2} \lambda_1 \alpha a_{31} \right) \int_{\Omega} u_1^2(t, x)dx - \left( a_{22} - \frac{1}{2} \lambda_2 \alpha a_{32} \right) \int_{\Omega} u_2^2(t, x)dx \]
\[ - \alpha \left( a_{33} - \frac{1}{2} \lambda_1 a_{31} - \frac{1}{2} \lambda_2 a_{32} \right) \int_{\Omega} u_3^2(t, x)dx. \]  

(2.27)

Taking
\[ \lambda_1 = \frac{ba_{11}}{\alpha a_{31} (r_1 + b)}, \quad \lambda_2 = \frac{a_{22}}{\alpha a_{32}}, \]

it follows from (2.27) that
\[ \frac{d}{dt} V(t) \leq - \frac{ba_{11}}{2(r_1 + b)} \int_{\Omega} u_1^2(t, x)dx - \frac{1}{2} a_{22} \int_{\Omega} u_2^2(t, x)dx \]
\[ - \alpha \left( a_{33} - \alpha \left( \frac{a_{31} (r_1 + b)}{2ba_{11}} + \frac{a_{32}}{2a_{22}} \right) \right) \int_{\Omega} u_3^2(t, x)dx. \]  

(2.28)

Choosing
\[ \alpha = \frac{a_{33}}{2 \left( \frac{a_{31} (r_1 + b)}{2ba_{11}} + \frac{a_{32}}{2a_{22}} \right)}, \]

then \( V(t) \) is non-increasing in \( [0, \infty) \). For any \( T > 0 \), integrating (2.28) over \( [0, T] \), we derive that
\[ \frac{ba_{11}}{2(r_1 + b)} ||u_1||_2^2 + \frac{1}{2} a_{22} ||u_2||_2^2 + \frac{1}{2} \alpha a_{33} ||u_3||_2^2 \leq V(0). \]

We therefore have
\[ ||u_1||_2 \leq C_1, \quad ||u_2||_2 \leq C_2, \quad ||u_3||_2 \leq C_3 \]  

(2.29)

for some constants \( C_i \) (\( i = 1, 2, 3 \)) independent of \( T \).

Using Green’s identity, it follows that
\[ D_1 \int_{0}^{T} \int_{\Omega} |\nabla u_1(s, x)|^2 dx ds = -D_1 \int_{0}^{T} \int_{\Omega} u_1(s, x) \Delta u_1(s, x) dx ds \]
\[ = -\int_{0}^{T} \int_{\Omega} u_1(s, x) \frac{\partial}{\partial s} u_1(s, x) dx ds + a \int_{0}^{T} \int_{\Omega} u_1(s, x) u_2(s, x) dx ds \]
\[ - (r_1 + b) \int_{0}^{T} \int_{\Omega} u_1^2(s, x) dx ds - a_{11} \int_{0}^{T} \int_{\Omega} u_2^2(s, x) dx ds \]
\[ - a_{13} \int_{0}^{T} \int_{\Omega} u_1^2(s, x) u_3(s, x) dx ds. \]
We note that
\[
\int_0^T \int_\Omega u_1(s, x) \frac{\partial}{\partial s} u_1(s, x) dx ds = \int_0^T \int_\Omega \frac{1}{2} \frac{\partial}{\partial s} u_1^2(s, x) dx ds
\]
\[
= \frac{1}{2} \int_\Omega u_1^2(T, x) dx - \frac{1}{2} \int_\Omega u_1^2(0, x) dx \leq M_1 \text{mes } \Omega,
\]
\[
\int_0^T \int_\Omega u_1(s, x) u_2(s, x) dx ds \leq \frac{1}{2} \int_0^T \int_\Omega u_1^2(s, x) dx ds + \frac{1}{2} \int_0^T \int_\Omega u_2^2(s, x) dx ds,
\]
\[
\int_0^T \int_\Omega u_1^3(s, x) dx ds \leq M_1 \int_0^T \int_\Omega u_1^2(s, x) dx ds,
\]
\[
\int_0^T \int_\Omega u_1^2(s, x) u_3(s, x) dx ds \leq M_3 \int_0^T \int_\Omega u_1^2(s, x) dx ds.
\]

We therefore derive
\[
\|\nabla u_1\|_2 \leq C_4
\]
for some constant $C_4 > 0$ independent of $T$. In a similar way, one can prove
\[
\|\nabla u_2\|_2 \leq C_5, \quad \|\nabla u_3\|_2 \leq C_6
\]
for some constants $C_5$ and $C_6$ independent of $T$. It therefore follows as above that
\[
\lim_{t \to +\infty} \|u_i(t)\|_{C(\overline{D}; R)} = 0 \quad (i = 1, 2, 3).
\]

This completes the proof. 

\section{Numerical simulations}

In this section, we give some examples to illustrate our main results on the convergence of problem (1.3) with boundary conditions (1.4) and initial conditions (1.5).

In the following we always take $\Omega = [0, \pi]$, $K_i(x, y, t) = G_i(x, y, t)k_i(t)$, where $k_i(t) = \frac{1}{\tau} e^{-t/\tau}$, and
\[
G_i(x, y, t) = \frac{1}{\pi} + \frac{2}{\pi} \sum_{n=1}^{\infty} e^{-D_i n^2 t} \cos nx \cos ny.
\]

Here $G_i(x, y, t)$ is the solution of
\[
\frac{\partial G_i}{\partial t} = D_i \frac{\partial^2 G_i}{\partial x^2}, \quad \frac{\partial G_i}{\partial x} = 0 \quad \text{at } x = 0, \pi, \quad G_i(x, y, 0) = \delta(x - y).
\]

It is readily seen that $G_i$ also satisfies
\[
\frac{\partial G_i}{\partial t} = D_i \frac{\partial^2 G_i}{\partial y^2}, \quad \frac{\partial G_i}{\partial y} = 0 \quad \text{at } y = 0, \pi, \quad G_i(x, y, 0) = \delta(x - y).
\]

(See, for example, Gourley and So \cite{26} for a description of where this kind of kernel arises.)

Define
\[
Q_1(t, x) = \int_{-\infty}^{t} \int_0^\pi G_1(x, y, t - s) \frac{1}{\tau} e^{-(t-s)/\tau} u_1(s, y) dy ds,
\]
\[
Q_2(t, x) = \int_{-\infty}^{t} \int_0^\pi G_2(x, y, t - s) \frac{1}{\tau} e^{-(t-s)/\tau} u_2(s, y) dy ds.
\]

Differentiating (3.4) with respect to $t$, it follows that
\[
\frac{\partial Q_1}{\partial t} = D_1 \frac{\partial^2 Q_1}{\partial x^2} + \frac{1}{\tau} (u_1(t, x) - Q_1(t, x)).
\]
Fig. 1. The temporal solution found by numerical integration of system (3.5) with $a = 3$, $b = 2$, $r_1 = r_2 = r_3 = 0.1$, $a_{11} = 4$, $a_{22} = 3$, $a_{33} = 3$, $a_{13} = 2$, $a_{31} = 2$, $a_{23} = a_{32} = 0$, $k_i(t) = (1/\tau) e^{t/\tau}$, $\tau = 1$, $\Omega = [0, \pi]$ and $(\phi_1, \phi_2, \phi_3, \phi_4) \equiv (0.3593 + 0.0005 \sin 3x, 0.4730 + 0.0005 \sin 3x, 0.3593 + 0.0005 \sin 3x, 0.2062 + 0.0005 \sin 3x, 0.3593 + 0.0005 \sin 3x)$. 

\[ \frac{\partial Q_2}{\partial t} = D_2 \frac{\partial^2 Q_2}{\partial x^2} + \frac{1}{\tau} (u_2(t, x) - Q_2(t, x)). \]

Clearly, we have $\frac{\partial Q_i}{\partial x} = 0$ at $x = 0, \pi$. Therefore, in the case of the one dimensional domain $[0, \pi]$, system (1.3) can be replaced by

\[
\begin{align*}
\frac{\partial u_1}{\partial t} & = D_1 \frac{\partial^2 u_1}{\partial x^2} + a u_2(t, x) - r_1 u_1(t, x) - a_{11} u_1^2(t, x) - b u_1(t, x) - a_{13} u_1(t, x) u_3(t, x), \\
\frac{\partial u_2}{\partial t} & = D_2 \frac{\partial^2 u_2}{\partial x^2} + b u_1(t, x) - r_2 u_2(t, x) - a_{22} u_2^2(t, x) - a_{23} u_2(t, x) u_3(t, x), \\
\frac{\partial u_3}{\partial t} & = D_3 \frac{\partial^2 u_3}{\partial x^2} + u_3(t, x) (-r_3 + a_{31} Q_1(t, x) + a_{32} Q_2(t, x) - a_{33} u_3(t, x)), \\
\frac{\partial Q_1}{\partial t} & = D_1 \frac{\partial^2 Q_1}{\partial x^2} + \frac{1}{\tau} (u_1(t, x) - Q_1(t, x)), \\
\frac{\partial Q_2}{\partial t} & = D_2 \frac{\partial^2 Q_2}{\partial x^2} + \frac{1}{\tau} (u_2(t, x) - Q_2(t, x)),
\end{align*}
\]  

\hspace{1cm} \text{(3.5)}

with homogeneous Neumann boundary conditions for each component. It is easily seen that, in this case, systems (1.3) and (3.5) are equivalent if

\[
Q_i(0, x) = \int_{-\infty}^{t} \int_0^\pi G_i(x, y, -s) \frac{1}{\tau} e^{s/\tau} u_i(s, y) dy ds \quad (i = 1, 2). 
\]  

\hspace{1cm} \text{(3.6)}

Referring to [26], solving the fourth and the fifth equations of (3.5) independently of the others, we have
with boundary conditions we see that using the standard MATLAB algorithm (see Theorem 2.1 has a unique positive and initial conditions Fig. 1 with boundary conditions Theorem 2.1 admits a unique positive uniform equilibrium will converge we see that the positive solution Fig. 2. The temporal solution found by numerical integration of system \( \tilde{\mathbf{u}}(t, x) = \mathbf{Q}_i(t, x) + \int_{-\infty}^{t} \int_{0}^{\pi} G_i(x, y, t - s) \frac{1}{\tau} e^{-(t-s)/\tau} u_i(s, y) dy ds, \)

where \( \tilde{\mathbf{Q}}_i(t, x) \) satisfies

\[
\frac{\partial \tilde{\mathbf{Q}}_i}{\partial t} = D_i \frac{\partial^2 \tilde{\mathbf{Q}}_i}{\partial x^2} - \frac{1}{\tau} \tilde{\mathbf{Q}}_i, \quad \frac{\partial \tilde{\mathbf{Q}}_i}{\partial x} = 0 \quad \text{at} \quad x = 0, \pi, i = 1, 2.
\]

It is easy to show that \( \tilde{\mathbf{Q}}(t, x) \rightarrow 0 \) as \( t \rightarrow +\infty \). Hence, we are assured that the use of initial data not satisfying (3.6) has only a transient effect on the solution dynamics.

**Example 1.** In system (1.3), let \( a = 3, b = 2, r_1 = r_2 = r_3 = 0.1, a_{11} = 4, a_{22} = 3, a_{33} = 3, a_{13} = a_{31} = 0, a_{23} = 3, a_{32} = 2, k_i(t) = (1/\tau)e^{t/\tau} \), \( \tau = 1, \Omega = [0, \pi] \) and \( \phi_1, \phi_2, \phi_3 \equiv (0.3172 + 0.005 \sin 3x, 0.3562 + 0.005 \sin 3x, 0.2041 + 0.005 \sin 3x, 0.3562 + 0.005 \sin 3x) \). At \( t = 005 \sin 3x, 33 \equiv 31, 32 \equiv 31, 32 \equiv (0.3172 + 0.005 \sin 3x, 0.3562 + 0.005 \sin 3x, 0.2041 + 0.005 \sin 3x, 0.3562 + 0.005 \sin 3x) \).

\[
\tilde{\mathbf{Q}}_i(t, x) = \tilde{\mathbf{Q}}_i(t, x) + \int_{-\infty}^{t} \int_{0}^{\pi} G_i(x, y, t - s) \frac{1}{\tau} e^{-(t-s)/\tau} u_i(s, y) dy ds, \]

where \( G_i(t, x, y) \) satisfies

\[
\frac{\partial \tilde{\mathbf{Q}}_i}{\partial t} = D_i \frac{\partial^2 \tilde{\mathbf{Q}}_i}{\partial x^2} - \frac{1}{\tau} \tilde{\mathbf{Q}}_i, \quad \frac{\partial \tilde{\mathbf{Q}}_i}{\partial x} = 0 \quad \text{at} \quad x = 0, \pi, i = 1, 2.
\]

It is easy to show that \( \tilde{\mathbf{Q}}(t, x) \rightarrow 0 \) as \( t \rightarrow +\infty \). Hence, we are assured that the use of initial data not satisfying (3.6) has only a transient effect on the solution dynamics.

**Example 2.** In system (1.3), let \( a = 3, b = 2, r_1 = r_2 = r_3 = 0.1, a_{11} = 4, a_{22} = 3, a_{33} = 3, a_{13} = a_{31} = 0, a_{23} = 3, a_{32} = 2, k_i(t) = (1/\tau)e^{t/\tau} \). It is easy to show that system (1.3) admits a unique positive uniform equilibrium \( E^*(0.3172, 0.3562, 0.2041) \). We note that \( a_{22}a_{33} - a_{23}a_{32} = 3 > 0 \). By Theorem 2.1 we see that the positive solution \( (u_1(t, x), u_2(t, x), u_3(t, x)) \) of system (1.3) with boundary conditions (1.4) and initial conditions (1.5) will converge to the unique positive uniform equilibrium \( E^* \) as \( t \rightarrow +\infty \). Numerical simulation illustrates this observation (see Fig. 2).
Example 3. In system (1.3), let $a = 3, b = 2, r_1 = r_2 = 0.1, r_3 = 1, a_{11} = 4, a_{22} = 3, a_{33} = 4, a_{13} = 2, a_{23} = a_{32} = 0, \tau = 1, k_i(t) = (1/\tau)e^{it}$. It is easy to show that system (1.3) admits a unique semi-trivial uniform equilibrium $E_1(0.4059, 0.5038, 0)$, and $a_{11}a_{33} - a_{13}a_{31} = 12 > 0$. By Theorem 2.2 we see that the positive solution $(u_1(t, x), u_2(t, x), u_3(t, x))$ of system (1.3) with boundary conditions (1.4) and initial conditions (1.5) will converge uniformly to the unique semi-trivial uniform equilibrium $E_1$ as $t \to +\infty$. Numerical simulation illustrates this observation (see Fig. 3).

Example 4. In system (1.3), let $a = 1, b = 2, r_1 = r_2 = 0.1, r_3 = 1, a_{11} = 2, a_{22} = 3, a_{33} = 3, a_{13} = 3, a_{23} = 3, a_{32} = 2, \tau = 1, k_i(t) = (1/\tau)e^{it}$. It is trivial to show that $ab - r_2(r_1 + b) = -0.1 < 0$. By Theorem 2.3 we see that the positive solution $(u_1(t, x), u_2(t, x), u_3(t, x))$ of system (1.3) with boundary conditions (1.4) and initial conditions (1.5) will uniformly converge to the trivial uniform equilibrium $E_0(0, 0, 0)$ as $t \to +\infty$. Numerical simulation of (3.5) illustrates this fact (see Fig. 4).

Example 5. We should mention here that we haven’t given a theoretical result on the global attractivity of the positive equilibrium of system (1.3) when the predator feeds on both the immature and the mature prey. In system (1.3), let $a = 3, b = 2, r_1 = r_2 = r_3 = 0.1, a_{11} = 4, a_{22} = 3, a_{33} = 3, a_{13} = a_{31} = a_{23} = a_{32} = 2, k_i(t) = (1/\tau)e^{it}, \tau = 1$. In this case, system (1.3) has a unique positive uniform equilibrium $E^*(0.2383, 0.2936, 0.3213)$. We note that $a_{11}a_{33} - a_{13}a_{31} = 8, a_{22}a_{33} - a_{23}a_{32} = 5$. As shown in Fig. 5, numerical simulation of (3.5) shows that the positive equilibrium $E^*$ is global attractive. We therefore expect when the predator feeds on both the immature and the mature prey, the corresponding system may have similar global dynamics to those described in Theorems 2.1–2.3.

4. Discussion

In this paper, we have incorporated stage structure for prey and spatio-temporal delays into a two-species Lotka–Volterra type predator–prey model. When the predator species feeds only on either immature prey or mature
prey, using the energy function method (see, for example, Yamada [25]), we derived sufficient conditions for the global convergence of the positive solutions to system (1.3) with homogeneous Neumann boundary conditions (1.4) and initial conditions (1.5). By Theorem 2.1, we see that if system (1.3) admits a unique positive constant steady state \( E^* (u_1^*, u_2^*, u_3^*) \), then \( E^* \) is globally attractive if \( a_{11} a_{33} > a_{13} a_{31} \) or \( a_{22} a_{33} > a_{23} a_{32} \). Ecologically, this means that if the intra-specific competitions of the immature prey (mature prey) and the predator dominate the inter-specific interaction between the immature prey (mature prey) and the predator, then both the prey and the predator populations will be permanent. By Theorem 2.2, we know that if (H1), (H2), (H4) or (H1), (H3), (H4) hold, then the semi-trivial uniform equilibrium of problem (1.3)–(1.5) will be globally attractive. Ecologically, this means that both the immature and mature prey species will be permanent; however, the predator population will go to extinction. By Theorem 2.3, the positive solutions of problem (1.3)–(1.5) will approach the trivial equilibrium \( E_0 (0, 0, 0) \) if (H5) holds (in this case, the semi-trivial uniform equilibrium does not exist). Ecologically, both the prey and predator population will go to extinction if the death rates of the mature and immature prey population are large enough and the birth rate and the transformation rate from immature individuals to mature individuals are sufficiently low.

If the prey species in model (1.3) has no stage structure, system (1.3) becomes the following classical Lotka–Volterra predator–prey model with nonlocal delay as previously studied by Yamada [25]:

\[
\begin{align*}
\frac{\partial u_2}{\partial t} &= D_2 \Delta u_2 + u_2(t, x)(a - r_2 - a_{22} u_2(t, x) - a_{23} u_3(t, x)), \\
\frac{\partial u_3}{\partial t} &= D_3 \Delta u_3 + u_3(t, x) \left( -r_3 + a_{32} \int_0^1 \int_{-\infty}^t K_2(x, y, t - s) u_2(x, y) \, ds \, dy - a_{33} u_3(t, x) \right)
\end{align*}
\]

for \( t > 0, x \in \Omega \), with homogeneous Neumann boundary conditions.

It is easy to show that the trivial uniform equilibrium \( E_0 (0, 0, 0) \) of (4.1) is locally unstable. But for problem (1.3)–(1.5) when \( ab \leq r_2 (b + r_1) \), the trivial uniform equilibrium \( E_0 (0, 0, 0) \) is globally attractive. Therefore, we see that the incorporation of stage structure into prey species can have a negative effect on the persistence of the prey population.
population. We note that Theorem 2.2 extends the result of Yamada [25] to include stage structure. However, in order to do this we have had to strengthen the required hypothesis ((H2)–(H3)). Therefore, our result in Theorem 2.2 may have room for improvement.

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