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Exponential stability for a Timoshenko-type system with history

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ABSTRACT

In this paper, we consider hyperbolic Timoshenko-type vibrating systems that are coupled to a heat equation modeling an expectedly dissipative effect through heat conduction. We use the semigroup method to prove the exponential stability result with assumptions on past history relaxation function g exponentially decaying for the equal wave-speed case.

1. Introduction

In this paper, we will consider the system

$$\begin{aligned} &\rho_1\varphi_{tt} - k(\varphi_x + \psi)_x = 0, \\ &\rho_2\psi_{tt} - b\psi_{xx} + \int_0^\infty g(s)\psi_{xx}(t-s)\,ds + k(\varphi_x + \psi) + \delta\theta_{tx} = 0, \\ &\rho_3\theta_{tt} - \beta\theta_{txx} - \beta\theta_{xx} + \delta\psi_{xt} = 0 \end{aligned} \tag{1.1}$$

with positive constants ρ_1, ρ_2, ρ_3, k, b, β, δ together with initial conditions

$$\varphi(\cdot,0) = \varphi_0, \qquad \varphi_t(\cdot,0) = \varphi_1, \qquad \psi(\cdot,0) = \varphi_0, \qquad \psi_t(\cdot,0) = \varphi_1, \qquad \theta(\cdot,0) = \theta_0 \tag{1.2}$$

and boundary conditions

$$\varphi_x(0,t) = \varphi_x(1,t) = \psi(0,t) = \psi(1,t) = \theta(0,t) = \theta(1,t) = 0, \tag{1.3}$$

where the functions φ, ψ and θ depend on $(x,t) \in [0,1] \times [0,\infty)$ and model the transverse displacement of a beam with reference configuration $(0,1) \subset R$, the rotation angle of a filament and the temperature difference respectively.

In 1921, Timoshenko [23] gave, as a model for a thick beam, the following system of coupled hyperbolic equations

$$\begin{aligned} &\rho u_{tt} = \big(k(u_x - \varphi)\big)_x \quad \text{in } (0,L) \times (0,+\infty), \\ &I_\rho\varphi_{tt} = (EI\varphi_x)_x + k(u_x - \varphi) \quad \text{in } (0,L) \times (0,+\infty), \end{aligned} \tag{1.4}$$

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where t denotes the time variable and x is the space along the beam of length L, in its equilibrium configuration, u is the transverse displacement of the beam and φ is the rotation angle of the filament of the beam. The coefficients ρ, I_ρ, E, I and k are respectively the density, the polar moment of the inertia of a cross section, Young's modulus of elasticity, the moment of inertia of a cross section, and the shear modulus.

This system (1.4) has been studied by many authors and results concerning existence and asymptotic behavior have been established. Kim and Renardy [9] considered (3.1.4) together with two linear boundary controls of the form

$$k\big(\varphi(L,t)-u_x(L,t)\big)=\alpha u_t(L,t), \quad \forall t \geqslant 0,$$
$$EI\varphi_x(L,t)=-\beta\varphi_t(L,t), \quad \forall t \geqslant 0,$$

and established an exponential decay result. They also provided numerical estimates to the eigenvalues of the operator associated with system (3.1.4). Yan [24] generalized the result of [9] by considering nonlinear boundary conditions of the form

$$k\big(\varphi(L,t)-u_x(L,t)\big)=f_1\big(u_t(L,t)\big), \quad \forall t \geqslant 0,$$
$$-EI\varphi_x(L,t)=f_2\big(\varphi_t(L,t)\big), \quad \forall t \geqslant 0,$$

where f_1, f_2 are functions with polynomial growth near the origin. Raposo et al. [18] studied (3.1.4) with homogeneous Dirichlet boundary conditions and two linear frictional dampings and proved that the energy decays exponentially. This result is similar to the one by Taylor [21] but, as he mentioned, the originality in his work lies on the semigroup theory method, which was developed by Liu and Zheng [11]. Soufyane and Wehbe [20] showed that it is possible to stabilize uniformly (3.1.4) by using a unique locally distributed feedback. They considered

$$\rho u_{tt}=\big(k(u_x-\varphi)\big)_x, \quad (x,t)\in(0,L)\times(0,+\infty),$$
$$I_\rho\varphi_{tt}=(EI\varphi_x)_x+k(u_x-\varphi)-b(x)\varphi(t), \quad (x,t)\in(0,L)\times(0,+\infty),$$

where $b(x)$ is a positive and continuous function satisfying

$$b(x)\geqslant b_0>0, \quad \forall x\in[a_0,a_1].$$

They proved that the uniform stability holds if and only if the wave speeds are equal; otherwise only the asymptotic stability has been proved. This result improves an earlier one by Shi and Feng [19], where an exponential decay of the solution energy of (3.1.4), together with two locally distributed feedbacks, had been proved. Ammar-Khodja et al. [2] considered a linear Timoshenko-type system with memory of the form

$$\rho_1\varphi_{tt}-k(\varphi_x+\psi)_x=0,$$
$$\rho_2\psi_{tt}-b\psi_{xx}+\int_0^t g(t-s)\psi_{xx}\,ds+k(\varphi_x+\psi)=0$$

in $(0,L)\times(0,\infty)$, together with homogeneous boundary conditions. They used the multiplier techniques and proved that the system is uniformly stable if and only if the wave speeds are equal ($\frac{k}{\rho_1}=\frac{b}{\rho_2}$) and g decays uniformly. Precisely, they proved an exponential decay if g decays in an exponential rate and polynomial decay if g decays in a polynomial rate. They also required some extra technical conditions on both g' and g'' to obtain their result. Guesmia and Messaoudi [6] obtained the same uniform decay results without imposing those extra technical conditions on g' and g''. Recently, Messaoudi and Mustafa [13] improved the results of [2,6] by allowing more general relaxation functions. They established a more general decay result, from which the exponential and the polynomial decay results are only special cases, later, Fernández Sare and Muñoz Rivera [4] considered a similar Timoshenko-type system with a past history of the form

$$\rho_1\varphi_{tt}-k(\varphi_x+\psi)_x=0,$$
$$\rho_2\psi_{tt}-b\psi_{xx}+\int_0^\infty g(s)\psi_{xx}(t-s)\,ds+k(\varphi_x+\psi)=0,$$

where ρ_1, ρ_2, k, b are positive constants and g is a positive twice differentiable function satisfying, for some constants $k_0,k_1,k_2>0$,

$$g(t)>0, \qquad -k_0g(t)\leqslant g'(t)\leqslant -k_1g(t), \qquad \big|g''\big|\leqslant k_2(t), \quad \forall t\geqslant 0,$$
$$\hat{b}=b-\int_0^\infty g(s)\,ds>0,$$

and showed that the dissipation given by the history term is strong enough to stabilize the system exponentially if and only if the wave speeds are equal. They also proved that the solution decays polynomially for the case of different wave speeds.

For Timoshenko systems in thermoelasticity, Muñoz Rivera and Racke [16] considered

$$\begin{aligned}
&\rho_1\varphi_{tt} - \sigma(\varphi_x, \psi) = 0, \quad (x,t) \in (0,L) \times (0,+\infty),\\
&\rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \gamma\theta_x = 0, \quad (x,t) \in (0,L) \times (0,+\infty),\\
&\rho_3\theta_t - k\theta_{xx} + \gamma\psi_{tx} = 0, \quad (x,t) \in (0,L) \times (0,+\infty),
\end{aligned}$$

under appropriate conditions of σ, ρ_i, b, k, γ, they proved several exponential decay results for the linearized system and a nonexponential stability result for the case of different wave speeds. Messaoudi et al. [14] studied

$$\begin{aligned}
&\rho_1\varphi_{tt} - \sigma(\varphi_x, \psi) + \mu\varphi_t = 0, \quad (x,t) \in (0,L) \times (0,+\infty),\\
&\rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \beta\theta_x = 0, \quad (x,t) \in (0,L) \times (0,+\infty),\\
&\rho_3\theta_t + \gamma q_x + \delta\psi_{tx} = 0, \quad (x,t) \in (0,L) \times (0,+\infty),\\
&\tau_0 q_t + q + k\theta_x = 0, \quad (x,t) \in (0,L) \times (0,+\infty),
\end{aligned}$$

where $\varphi = \varphi(x,t)$ is the displacement vector, $\psi = \psi(x,t)$ is the rotation angle of the filament, $\theta(x,t)$ is the temperature difference, $q = q(x,t)$ is the heat flux vector, ρ_1, ρ_2, ρ_3, b, k, γ, δ, μ, τ_0 are positive constants. The nonlinear function σ is assumed to be sufficiently smooth and satisfy

$$\sigma_{\varphi_x}(0,0) = \sigma_\psi(0,0) = k$$

and

$$\sigma_{\varphi_x\varphi_x}(0,0) = \sigma_{\varphi_x\psi}(0,0) = \sigma_{\psi\psi} = 0.$$

Several exponential decay results for both linear and nonlinear cases have been established. Also Messaoudi and Said-Houari [15] considered a Timoshenko-type system of thermoelasticity type III and proved an exponential decay similar to the one in [14,16]. Recently, Fernández Sare and Racke [5] considered hyperbolic Timoshenko-type vibrating systems that are coupled to a heat equation modeling an expectedly dissipative effect through heat conduction

$$\begin{aligned}
&\rho_1\varphi_{tt} - k(\varphi_x + \psi)_x = 0,\\
&\rho_2\psi_{tt} - b\psi_{xx} + \int_0^\infty g(s)\psi_{xx}(t-s)\,ds + k(\varphi_x + \psi) + \delta\theta_x = 0,\\
&\rho_3\theta_t - q_x + \delta\psi_{xt} = 0,\\
&\tau q_t + \beta q + \theta_x = 0
\end{aligned} \tag{1.5}$$

with an exponential decaying kernel g, for $\tau = 0$, they proved the exponential stable if and only if the wave speeds are equal.

We would like to mention other works in [1,10,12,15,3,7,8,22,23] for related models.

In the present work, we consider system (1.1), that is, we use the semigroup method to prove the exponential stability result for equal wave speed case.

2. Preliminaries

In order to state our main result we make the following hypotheses:

$g : R^+ \to R^+$ is a differentiable function such that

$$g(0) > 0, \qquad \exists k_0 g(t) \leqslant g'(t) \leqslant -k_1 g(t), \qquad \left|g''(t)\right| \leqslant k_2 g(t), \tag{2.1}$$

$$b - \int_0^\infty g(s)\,ds = \bar{b} > 0. \tag{2.2}$$

We introduce

$$\eta(x,t,s) = \psi(x,t) - \psi(x,t-s), \quad t,s \geqslant 0, \tag{2.3}$$

then we have reformulated system (1.1) to

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x + \psi)_x = 0, \\ \rho_2\psi_{tt} - \bar{b}\psi_{xx} - \int_0^\infty g(s)\eta^t_{xx}(t-s)\,ds + k(\varphi_x + \psi) + \delta\theta_{tx} = 0, \\ \rho_3\theta_{tt} - \beta\theta_{txx} - \beta\theta_{xx} + \delta\psi_{xt} = 0, \\ \eta_t + \eta_s - \psi_t = 0 \end{cases} \tag{2.4}$$

with initial conditions

$$\varphi(\cdot,0) = \varphi_0, \qquad \varphi_t(\cdot,0) = \varphi_1, \qquad \psi(\cdot,0) = \varphi_0, \qquad \psi_t(\cdot,0) = \varphi_1, \qquad \theta(\cdot,0) = \theta_0, \qquad \eta(x,0,s) = \eta_0 \tag{2.5}$$

and boundary conditions

$$\varphi_x(0,t) = \varphi_x(1,t) = \psi(0,t) = \psi(1,t) = \theta(0,t) = \theta(1,t) = 0. \tag{2.6}$$

We shall use the semigroup method to demonstrate the exponential stability, for this purpose we rewrite the system (2.4) as evolution equation for

$$U = (\varphi, \varphi_t, \psi, \psi_t, \theta, \theta_t, \eta)^T \equiv (u_1, u_2, u_3, u_4, u_5, u_6, u_7),$$
$$U_t = \mathcal{A}U, \qquad U(0) = U_0,$$

where $U_0 = (\varphi_0, \varphi_1, \psi_0, \psi_1, \theta_0, \theta_1, \eta_0)$, and $\mathcal{A}$ is the differential operator

$$\mathcal{A} = \begin{pmatrix} 0 & Id & 0 & 0 & 0 & 0 & 0 \\ \frac{k}{\rho_1}\partial_x^2 & 0 & \frac{k}{\rho_1}\partial x & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & Id & 0 & 0 & 0 \\ -\frac{k}{\rho_2}\partial_x & 0 & \frac{\bar{b}}{\rho_2}\partial_x^2 - \frac{k}{\rho_2}Id & 0 & 0 & -\frac{\delta}{\rho_2}\partial_x & -\frac{1}{\rho_2}\int_0^\infty g(s)\partial_x^2(s)\,ds \\ 0 & 0 & 0 & 0 & 0 & Id & 0 \\ 0 & 0 & 0 & -\frac{\delta}{\rho_3} & \frac{\beta}{\rho_3}\partial_x^2 & \frac{\beta}{\rho_3}\partial_x^2 & 0 \\ 0 & 0 & 0 & Id & 0 & 0 & -\partial_s \end{pmatrix}.$$

Let

$$\mathcal{H} = H^1_*(0,1) \times L^2_*(0,1) \times H^1_0(0,1) \times L^2(0,1) \times H^1_0(0,1) \times L^2(0,1) \times L^2_g\big(\mathbf{R}, H^1_0\big)$$

be the Hilbert space with

$$L^2_*(0,1) := \left\{ v \in L^2(0,1) \;\Big|\; \int_0^1 v(x)\,dx = 0 \right\},$$

$$H^1_*(0,1) := \left\{ v \in H^1(0,1) \;\Big|\; \int_0^1 v(x)\,dx = 0 \right\}$$

and norm given by

$$\|U\|^2_{\mathcal{H}} = \rho_1\|u^2\|^2_{L^2} + \rho_2\|u^4\|^2_{L^2} + k\|u^1_x + u^3\|^2_{L^2} + \rho_3\|u^6\|^2_{L^2} + \rho_3\|u^5_x\|^2_{L^2} + \rho_1\|u^7\|^2_{L^2_g(\mathbf{R}^+, H^1_0)}$$

is a Hilbert space, where $L^2_g(\mathbf{R}^+, H^1_0)$ denotes the Hilbert space of H^1_0-valued functions on $\mathbf{R}^+$, endowed with the inner product

$$\langle \varphi, \psi \rangle_{L^2_g(\mathbf{R}^+, H^1_0)} = \int_0^1 \int_0^\infty g(s)\varphi_x(x,s)\psi_x(x,s)\,ds\,dx.$$

The domain of the operator $\mathcal{A}$ is defined by

$$D(\mathcal{A}) = \bigg\{ U \in \mathcal{H} \,\bigg|\, u^1 \in H^2(0,1),\ u^1_x \in H^1_0(0,1),\ u^2 \in H^1_*(0,1),$$
$$u^4 \in H^1_0(0,1),\ u^5_x \in H^2(0,1) \cap H^1_0(0,1),\ u^7_s \in L^2_g(\mathbf{R}^+, H^1_0),$$
$$\bar{b}u^3 + \int_0^\infty g(s) u^7(x,s)\,ds \in H^2(0,1) \cap H^1_0(0,1),\ u^7(x,0) = 0 \bigg\}.$$

It is not difficult to prove that the operator $\mathcal{A}$ is the infinitesimal generator of a C_0 contraction semigroup [17].

We shall use the following well-known result from the semigroup theory [11].

Lemma 2.1. *A semigroup of contractions $\{e^{t\mathcal{A}}\}_{t\geqslant 0}$ in a Hilbert space with norm $\|\cdot\|$ is exponentially stable if and only if*

(i) *the resolvent set $\rho(\mathcal{A})$ of $\mathcal{A}$ contains the imaginary axis and*

(ii)
$$\lim_{\lambda\to\pm\infty} \sup \left\| (i\lambda Id - \mathcal{A})^{-1} \right\| < \infty$$

hold.

3. Exponential stability for $\frac{k}{\rho_1} = \frac{b}{\rho_2}$

In this section we will show that the system is exponentially stable infinity provided the condition

$$\frac{k}{\rho_1} = \frac{b}{\rho_2} \tag{3.1}$$

holds. Once more we use Lemma 2.1, and we have to check if the two conditions hold:

$$i\mathbf{R} \subset \rho(\mathcal{A}) \tag{3.2}$$

and

$$\exists C > 0,\ \forall \lambda \in \mathbf{R}:\quad \left\| (i\lambda Id - \mathcal{A})^{-1} \right\|_{\mathcal{H}} \leqslant C. \tag{3.3}$$

First we will show (3.2) using contradiction arguments. In fact, suppose that (3.2) is not true. Then there exist $\omega \in \mathbf{R}$, a sequence $(\beta_n)_n \in \mathbf{R}$ with $\beta_n \to \omega$, $|\beta| < |\omega|$ and a sequence of functions

$$U_n = \left(u^1_n, u^2_n, u^3_n, u^4_n, u^5_n, u^6_n, u^7_n\right) \in D(\mathcal{A}) \quad \text{with } \|U_n\|_{\mathcal{H}} = 1 \tag{3.4}$$

such that, as $n \to \infty$,

$$i\beta_n U_n - \mathcal{A}U_n \to 0 \quad \text{in } \mathcal{H}, \tag{3.5}$$

that is,

$$i\beta_n u^1_n - u^2_n \to 0 \quad \text{in } H^1_*(0,1), \tag{3.6}$$
$$i\beta_n \rho_1 u^2_n - k\left(u^1_{n,x} + u^3_n\right)_x \to 0 \quad \text{in } L^2_*(0,1), \tag{3.7}$$
$$i\beta_n u^3_n - u^4_n \to 0 \quad \text{in } H^1_0(0,1), \tag{3.8}$$
$$i\beta_n \rho_2 u^4_n - \bar{b}u^3_{n,xx} - \int_0^\infty g(s) u^7_{n,xx}\,ds + k\left(u^1_{n,x} + u^3_n\right) + \delta u^6_{n,x} \to 0 \quad \text{in } L^2(0,1), \tag{3.9}$$
$$i\beta_n u^5_n - u^6_n \to 0 \quad \text{in } H^1_0(0,1), \tag{3.10}$$
$$i\beta_n \rho_3 u^6_n - \beta u^5_{n,xx} - \beta u^6_{n,xx} + \delta u^4_{n,x} \to 0 \quad \text{in } L^2(0,1), \tag{3.11}$$
$$i\beta_n u^7_n + u^7_{n,s} - u^4_n \to 0 \quad \text{in } L^2_g\left(\mathbf{R}^+, H^1_0\right). \tag{3.12}$$

Taking the inner product of (3.5) with U_n in $\mathcal{H}$ and then taking its real part yields

$$-\mathrm{Re}\langle \mathcal{A}U_n, U_n\rangle_{\mathcal{H}} = -\frac{1}{2}\int_0^1\int_0^\infty g'(s)\left|u_{n,x}^7\right|^2 ds\,dx + \beta\int_0^1 \left|u_{n,x}^6\right|^2 dx \to 0.$$

Using the hypotheses on g, we find that

$$u_n^7 \to 0 \quad \text{in } L_g^2\big(\mathbf{R}^+, H_0^1\big), \tag{3.13}$$

$$u_n^6 \to 0 \quad \text{in } H_0^1(0,1) \hookrightarrow L^2(0,1), \tag{3.14}$$

inserting (3.14) into (3.10), we obtain

$$u_n^5 \to 0 \quad \text{in } H_0^1(0,1) \hookrightarrow L^2(0,1). \tag{3.15}$$

Then using (3.4), we find that

$$\rho_1\left\|u_n^2\right\|_{L^2}^2 + \rho_2\left\|u_n^4\right\|_{L^2}^2 + \beta\left\|u_{n,x}^3\right\|_{L^2}^2 + k\left\|u_{n,x}^1 + u_n^3\right\|_{L^2}^2 \to 1. \tag{3.16}$$

On the other hand, denote $(f,g)_I$ to represent the inner product of f, g in the space I, taking the inner product of (3.6) with $\rho_1 u_n^2$ in L_*^2 and (3.7) with $\rho_1 u_n^1$ in L_*^2, respectively, yields

$$i\rho_1\beta_n\big(u_n^1, u_n^2\big)_{L^2} - \rho_1\left\|u_n^2\right\|_{L^2}^2 \to 0$$

and

$$i\rho_1\beta_n\big(u_n^2, u_n^1\big)_{L^2} - k\big(u_{n,x}^1 + u_n^3, u_{n,x}^1\big)_{L^2} \to 0.$$

Adding and taking the real part, we get

$$k\,\mathrm{Re}\big(u_{n,x}^1 + u_n^3, u_{n,x}^1\big)_{L^2} - \rho_1\left\|u_n^2\right\|_{L^2}^2 \to 0. \tag{3.17}$$

Analogously, taking the inner product of (3.8) with $\rho_2 u_n^4$ in $L^2(0,1)$ and (3.9) with u_n^3 in $L^2(0,1)$, respectively, yields

$$i\rho_2\beta_n\big(u_n^3, u_n^4\big)_{L^2} - \rho_2\left\|u_n^4\right\|_{L^2}^2 \to 0 \tag{3.18}$$

and

$$i\rho_2\beta_n\big(u_n^4, u_n^3\big)_{L^2} + \bar{b}\left\|u_{n,x}^3\right\|_{L^2}^2 + \int_0^\infty g(s)\big(u_{n,x}^7, u_{n,x}^3\big)_{L^2} \to 0. \tag{3.19}$$

Note that from (3.13), (3.14), we have

$$\int_0^\infty g(s)\big(u_{n,x}^7, u_{n,x}^3\big)_{L^2}\,ds + \delta\big(u_{n,x}^6, u_n^3\big)_{L^2} \to 0,$$

this used in (3.19) results in

$$i\rho_2\beta_n\big(u_n^4, u_n^3\big)_{L^2} + \bar{b}\left\|u_{n,x}^3\right\|_{L^2}^2 + k\big(u_{n,x}^1 + u_n^3, u_n^3\big)_{L^2} \to 0. \tag{3.20}$$

Adding (3.18) and (3.20) and taking real part, we get

$$-\rho_2\left\|u_n^4\right\|_{L^2}^2 + \bar{b}\left\|u_{n,x}^3\right\|_{L^2}^2 + k\,\mathrm{Re}\big(u_{n,x}^1 + u_n^3, u_n^3\big)_{L^2} \to 0, \tag{3.21}$$

and adding (3.17) with (3.21), we have

$$\bar{b}\left\|u_{n,x}^3\right\|_{L^2}^2 + k\left\|u_{n,x}^1 + u_n^3\right\|_{L^2}^2 - \rho_1\left\|u_n^2\right\|_{L^2}^2 - \rho_2\left\|u_n^2\right\|_{L^2}^4 \to 0. \tag{3.22}$$

Consequently, from (3.16) and (3.22), we deduce that

$$\bar{b}\left\|u_{n,x}^3\right\|_{L^2}^2 + k\left\|u_{n,x}^1 + u_n^3\right\|_{L^2}^2 \to \frac{1}{2}, \tag{3.23}$$

$$\rho_1\left\|u_n^2\right\|_{L^2}^2 + \rho_2\left\|u_n^2\right\|_{L^2}^4 \to \frac{1}{2}. \tag{3.24}$$

Also, it is clear that $s \mapsto \frac{1}{\beta_n^2}u_n^4 \in L_g^2(\mathbf{R}^+, H_0^1)$. Then multiplying (3.12) with $\frac{1}{\beta_n^2}u_n^4$ in $L_g^2(\mathbf{R}^+, H_0^1)$ gives

$$i\left(u_n^7, \frac{u_n^4}{\beta_n}\right)_{L_g^2} + \frac{1}{\beta_n^2}\left(u_{n,s}^7, u_n^4\right)_{L_g^2} - \frac{1}{\beta_n^2}\left(u_n^4, u_n^4\right)_{L_g^2} \to 0. \tag{3.25}$$

Using (3.8) we have that $\frac{u_n^4}{\beta_n}$ is bounded in $H_0^1(0,1)$, and using (3.13) we get that the first term of (3.26) converges to zero. This yields

$$b_0\left\|\frac{u_n^4}{\beta_n}\right\|_{H_0^1}^2 - \frac{1}{\beta_n^2}\int_0^\infty g(s)\left(u_{n,s}^7, u_n^4\right)_{H_0^1} ds \to 0, \tag{3.26}$$

where $b_0 = \int_0^\infty g(s)\,ds$. We now prove that the second term in (3.26) converges to zero. In fact, using again that $\frac{u_n^4}{\beta_n}$ is bounded in $H_0^1(0,1)$, (2.1) and (3.13), we have

$$\begin{aligned}\left|-\frac{1}{\beta_n^2}\int_0^\infty g(s)\left(u_{n,s}^7, u_n^4\right)_{H_0^1} ds\right| &= \frac{1}{|\beta_n|}\left|-\int_0^\infty g'(s)\left(u_{n,s}^7, \frac{u_n^4}{\beta_n}\right)_{H_0^1} ds\right| \\ &\leqslant \frac{k_0}{|\beta_n|}\left\|\frac{u_n^4}{\beta_n}\right\|_{H_0^1}\int_0^\infty g(s)\left\|u_n^7(s)\right\|_{H_0^1} ds \\ &\leqslant \frac{k_0\sqrt{b_0}}{|\beta_n|}\left\|\frac{u_n^4}{\beta_n}\right\|_{H_0^1}\left\|u_n^7\right\|_{L_g^2} \to 0. \end{aligned} \tag{3.27}$$

Therefore, we can deduce from (3.26) that

$$\frac{u_n^4}{\beta_n} \to 0 \quad \text{in } H_0^1(0,1),$$

it follows from (3.8) that

$$u_n^3 \to 0 \quad \text{in } H_0^1(0,1), \tag{3.28}$$

and using (3.28) in (3.23), we get

$$k\left\|u_{n,x}^1 + u_n^3\right\|_{L^2}^2 \to \frac{1}{2}. \tag{3.29}$$

We want to show that this is a contradiction if the basic condition (3.1) holds.

Multiplying (3.9) by $(u_{n,x}^1 + u_n^3)$ in $L^2(0,1)$, we have

$$\begin{aligned} &i\beta_n\rho_2\left(u_n^4, u_{n,x}^1 + u_n^3\right)_{L^2} - \left(\bar{b}u_{n,xx}^3 + \int_0^\infty g(s)u_{n,x}^7(s)\,ds, \left(u_{n,x}^1 + u_n^3\right)_x\right)_{L^2} \\ &\quad + k\left\|u_{n,x}^1 + u_n^3\right\|_{L^2}^2 + \delta\left(u_{n,x}^6, u_{n,x}^1 + u_n^3\right)_{L^2} \to 0. \end{aligned} \tag{3.30}$$

Note that by (3.14), we have that the last term of (3.30) converges to zero. Then we get

$$i\beta_n\rho_2\left(u_n^4, u_{n,x}^1 + u_n^3\right)_{L^2} - \left(\bar{b}u_{n,xx}^3 + \int_0^\infty g(s)u_{n,x}^7(s)\,ds, \left(u_{n,x}^1 + u_n^3\right)_x\right)_{L^2} + k\left\|u_{n,x}^1 + u_n^3\right\|_{L^2}^2 \to 0. \tag{3.31}$$

Also, multiplying (3.7) by $\frac{1}{k}(\bar{b}u_{n,x}^3 + \int_0^\infty g(s)u_{n,x}^7(s)\,ds)$ in $L^2(0,1)$ results in

$$-i\frac{\rho_1}{k}\beta_n\left(\bar{b}u_{n,x}^3 + \int_0^\infty g(s)u_{n,x}^7(s)\,ds, u_n^2\right)_{L^2} - \left(\bar{b}u_{n,x}^3 + \int_0^\infty g(s)u_{n,x}^7(s)\,ds, \left(u_{n,x}^1 + u_n^3\right)_x\right)_{L^2} \to 0. \tag{3.32}$$

Then, adding (3.31) and (3.32), we obtain

$$i\beta_n\rho_2\left(u_n^4, u_{n,x}^1 + u_n^3\right)_{L^2} - i\frac{\rho_1}{k}\beta_n\left(\bar{b}u_{n,x}^3 + \int_0^\infty g(s)u_{n,x}^7(s)\,ds, u_n^2\right)_{L^2} + k\left\|u_{n,x}^1 + u_n^3\right\|_{L^2}^2 \to 0. \tag{3.33}$$

On the other hand, multiplying (3.6) by $\rho_2 u_n^4$, (3.8) by $i\rho_2\beta_n u_n^3$ and $-\rho_2 u_{n,x}^2$ in $L^2(0,1)$, respectively, yields

$$-i\beta_n\rho_2\big(u_n^4, u_{n,x}^1\big)_{L^2} - \rho_2\big(u_n^4, u_{n,x}^2\big)_{L^2} \to 0, \tag{3.34}$$

$$\beta_n^2\rho_2\big\|u_n^3\big\|_{L^2}^2 + i\beta_n\rho_2\big(u_n^4, u_n^3\big)_{L^2} \to 0, \tag{3.35}$$

$$-i\beta_n\rho_2\big(u_n^3, u_{n,x}^2\big)_{L^2} + \rho_2\big(u_n^4, u_{n,x}^2\big)_{L^2} \to 0. \tag{3.36}$$

Since $u_n^3 \to 0$ in $H_0^1(0,1) \hookrightarrow L^2(0,1)$, we obtain from (3.35) that

$$i\beta_n\rho_2\big(u_n^4, u_n^3\big)_{L^2} \to 0. \tag{3.37}$$

Adding (3.33), (3.34) and (3.37), we deduce that

$$-i\beta_n\rho_2\big(u_n^3, u_{n,x}^2\big)_{L^2} - i\frac{\rho_1}{k}\beta_n\left(\bar{b}u_{n,x}^3 + \int_0^\infty g(s)u_{n,x}^7(s)\,ds, u_n^2\right)_{L^2} + k\big\|u_{n,x}^1 + u_n^3\big\|_{L^2}^2 \to 0. \tag{3.38}$$

Now, from (3.12), we have

$$i\beta_n u_{n,x}^7 + u_{n,xs}^7 - u_{n,x}^4 \to 0 \quad \text{in } L_g^2\big(\mathbf{R}^+, L^2\big),$$

then, multiplying by $\frac{\rho_1}{k}u_n^2$ in $L_g^2(\mathbf{R}^+, L^2)$ results in

$$i\beta_n\frac{\rho_1}{k}\big(u_{n,x}^7, u_n^2\big)_{L_g^2(\mathbf{R}^+,L^2)} + \frac{\rho_1}{k}\big(u_{n,xs}^7, u_n^2\big)_{L_g^2(\mathbf{R}^+,L^2)} - i\beta_n\frac{\rho_1 b_0}{k}\big(u_{n,x}^4, u_n^2\big)_{L^2} \to 0. \tag{3.39}$$

Using similar argument used in (3.27), we can conclude

$$\frac{\rho_1}{k}\big(u_{n,sx}^7, u_n^2\big)_{L_g^2(\mathbf{R}^+,L^2)} \to 0,$$

then it follows from (3.39) that

$$\frac{\rho_1 b_0}{k}\big(u_{n,x}^4, u_n^2\big)_{L^2} \to 0. \tag{3.40}$$

Multiplying (3.8) by $-\frac{\rho_1 b_0}{k}u_{n,x}^2$ in $L^2(0,1)$ yields

$$i\beta_n\frac{\rho_1 b_0}{k}\big(u_n^3, u_{n,x}^2\big)_{L^2} + \frac{\rho_1 b_0}{k}\big(u_{n,x}^4, u_n^2\big)_{L^2} \to 0, \tag{3.41}$$

then, adding (3.40) and (3.41), we get

$$i\beta_n\frac{\rho_1 b_0}{k}\big(u_n^3, u_{n,x}^2\big)_{L^2} \to 0. \tag{3.42}$$

Finally, adding (3.38) and (3.42), we obtain

$$-i\beta_n\rho_2\big(u_n^3, u_{n,x}^2\big)_{L^2} + i\beta_n\frac{\rho_1\bar{b}}{k}\big(u_n^3, u_{n,x}^2\big)_{L^2} + i\frac{\rho_1 b_0}{k}\big(u_n^3, u_{n,x}^2\big)_{L^2} + k\big\|u_{n,x}^1 + u_n^3\big\|_{L^2}^2 \to 0,$$

and using that $\bar{b} = b - b_0$, we obtain

$$i\beta_n b\left(\frac{\rho_1}{k} - \frac{\rho_2}{b}\right)\big(u_n^3, u_{n,x}^2\big)_{L^2} + k\big\|u_{n,x}^1 + u_n^3\big\|_{L^2}^2 \to 0,$$

using (3.1), then

$$k\big\|u_{n,x}^1 + u_n^3\big\|_{L^2}^2 \to 0,$$

which contradicts (3.29). Thus (3.2) is proved.

To complete the result about exponential stability, we now prove (3.3). Note again the resolvent equation $(i\lambda Id - \mathcal{A})U = F \in \mathcal{H}$ is given by

$$i\lambda u^1 - u^2 = f^1, \tag{3.43}$$

$$i\lambda \rho_1 u^2 - k\big(u^1_x + u^3\big)_x = \rho_1 f^2, \tag{3.44}$$

$$i\lambda u^3 - u^4 = f^3, \tag{3.45}$$

$$i\lambda \rho_2 u^4 - \bar{b} u^3_{xx} - \int_0^\infty g(s) u^7_{xx}(s)\,ds + k\big(u^1_x + u^3\big) + \delta u^6_x = \rho_2 f^4, \tag{3.46}$$

$$i\lambda u^5 - u^6 = f^5, \tag{3.47}$$

$$i\lambda \rho_3 u^6 - \beta u^6_{xx} - \beta u^5_{xx} + \delta u^4_x = \rho_3 f^6, \tag{3.48}$$

$$i\lambda u^7 + u^7_s - u^4 = f^7, \tag{3.49}$$

where $b_0 = \int_0^\infty g(s)\,ds$, $\bar{b} = b - b_0 > 0$. To prove (3.3) we will use a series of the lemmas.

Lemma 3.1. *Let us suppose that the conditions* (2.1) *and* (2.2) *on g hold. Then there exists a positive constant C, being independent of F such that*

$$\beta \int_0^1 \big|u^6_x\big|^2\,dx + \int_0^1 \int_0^\infty g(s)\big|u^7_x\big|\,ds\,dx \leqslant C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}.$$

Proof. Multiplying (3.44) by u^2 (in $L^2(0,1)$), we get

$$i\lambda\rho_2 \int_0^1 \big|u^2\big|^2\,dx + k\int_0^1 \big(u^1_x + u^3\big)\overline{u^2_x}\,dx = \rho_1 \int_0^1 f^2\overline{u^2}\,dx$$

and, using Eq. (3.43),

$$i\lambda\rho_2 \int_0^1 \big|u^2\big|^2\,dx - i\lambda k\int_0^1 \big(u^1_x + u^3\big)\overline{u^1_x}\,dx = \rho_1 \int_0^1 f^2\overline{u^2}\,dx + k\int_0^1 \big(u^1_x + u^3\big)\overline{f^1_x}\,dx. \tag{3.50}$$

On the other hand, multiplying Eq. (3.46) by u^4 and integrating over $[0,1]$, we get

$$i\lambda\rho_2 \int_0^1 \big|u^4\big|^2\,dx + \bar{b}\int_0^1 u^3_x\overline{u^4_x}\,dx + \int_0^1\int_0^\infty g(s)u^7_x\overline{u^4_x}\,ds\,dx + k\int_0^1 \big(u^1_x + u^3\big)\overline{u^4}\,dx + \delta\int_0^1 u^6_x\overline{u^4}\,dx = \rho_2\int_0^1 f^4\overline{u^4}\,dx. \tag{3.51}$$

Substituting u^4 given by (3.49), (3.45) into (3.51), we get

$$\begin{aligned} &i\lambda\rho_2 \int_0^1 \big|u^4\big|^2\,dx - i\lambda\bar{b}\int_0^1 \big|u^3_x\big|^2\,dx - i\lambda\int_0^1\int_0^\infty g(s)\big|u^7_x\big|^2\,ds\,dx \\ &\quad - i\lambda k\int_0^1 \big(u^1_x + u^3\big)\overline{u^3}\,dx + \int_0^1\int_0^\infty g(s)u^7_x\overline{u^7_{xs}}\,ds\,dx + \delta\int_0^1 u^6_x\overline{u^4}\,dx \\ &= \rho_2\int_0^1 f^4\overline{u^4}\,dx + \bar{b}\int_0^1 u^3_x\overline{f^3_x}\,dx + k\int_0^1\big(u^1_x + u^3\big)\overline{f^3}\,dx + \int_0^1\int_0^\infty g(s)u^7_x\overline{f^7_x}\,ds\,dx. \end{aligned} \tag{3.52}$$

Also, multiplying Eq. (3.48) by $\overline{u^6}$, we obtain

$$i\lambda\rho_3 \int_0^1 \big|u^6\big|^2\,dx + \beta\int_0^1 \big|u^6_x\big|^2\,dx - \beta\int_0^1 u^5_{xx}u^6\,dx = \rho_3\int_0^1 f^6\overline{u^6}\,dx. \tag{3.53}$$

Inserting (3.47) into (3.53), and adding (3.51), (3.52), using (2.1) and taking the real part our conclusion follows. □

Lemma 3.2. *With the same hypotheses as in Lemma* 3.1, *there exists* $C > 0$ *such that*

$$\rho_2 \int_0^1 |u^4|^2\,dx \leqslant C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + C\|U\|_{\mathcal{H}}^{1/2}\|F\|_{\mathcal{H}}^{1/2}\left(\|u_x^3\|_{L^2} + \|u_x^1 + u^3\|_{L^2}\right).$$

Proof. Multiplying (3.46) by $\int_0^\infty g(s)\overline{u^7}\,ds$ in $L^2(0,1)$, we get

$$i\lambda\rho_2 \int_0^1\int_0^\infty g(s)\overline{u_x^7}u_x^3\,ds\,dx + \int_0^1\left|\int_0^\infty g(s)u_x^7\,ds\right|^2 dx + k\int_0^1\int_0^\infty g(s)\left(u_x^1 + u^3\right)\overline{u^7}\,ds\,dx - \delta\int_0^1\int_0^\infty g(s)\overline{u_x^7}u^6\,ds\,dx$$
$$= \rho_2 \int_0^1\int_0^\infty g(s)\overline{u^7} f^4\,ds\,dx. \tag{3.54}$$

From Lemma 3.1, we obtain

$$\int_0^1\left|\int_0^\infty g(s)u_x^7\,ds\right|^2 dx \leqslant \int_0^\infty g(s)\,ds\int_0^1\int_0^\infty g(s)|u_x^7|^2\,ds\,dx \leqslant C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}$$

and

$$\mathrm{Re}\left\{\delta\int_0^1\int_0^\infty g(s)\overline{u_x^7}u^6\,ds\,dx\right\} \leqslant C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}.$$

Substituting $i\lambda u^7$ given by (3.49) into (3.54), using

$$\mathrm{Re}\left\{\delta\int_0^1\int_0^\infty g(s)\overline{u_s^7}u^4\,ds\,dx\right\} \leqslant \frac{\rho_2}{2}\int_0^1 |u^4|^2\,dx + C\int_0^1\int_0^\infty |g'(s)||u_x^7|^2\,ds\,dx$$

and using (2.1), our conclusion now immediately follows from Lemma 3.1. □

Lemma 3.3. *With the same hypotheses as in Lemma* 3.1, *for any* $\varepsilon_1 > 0$ *there exists* $C_{\varepsilon_1} > 0$, *at most depending on* ε_1, *such that*

$$\bar{b}\int_0^1 |u_x^3|^2\,dx \leqslant C_{\varepsilon_1} \leqslant C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + C\|U\|_{\mathcal{H}}^{1/2}\|F\|_{\mathcal{H}}^{1/2}\|u_x^1 + u^3\|_{L^2} + \varepsilon_1\rho_1\|u^2\|_{L^2}^2.$$

Proof. Multiplying of (3.46) by u^3 yields

$$i\lambda\rho_2\int_0^1 u^4\overline{u^3}\,dx + \bar{b}\int_0^1 |u_x^3|^2\,dx + \int_0^1\int_0^\infty g(s)u_x^7\overline{u_x^3}\,ds\,dx$$
$$+ k\int_0^1 \left(u_x^1 + u^3\right)\overline{u^3}\,dx + \delta\int_0^1 u_x^6\overline{u^3}\,dx = \rho_2\int_0^1 f^4\overline{u^3}\,dx. \tag{3.55}$$

Substituting $i\lambda u^3$ given by (3.45) into (3.55), we get

$$\bar{b}\int_0^1 |u_x^3|^2\,dx + k\int_0^1 \left(u_x^1 + u^3\right)\overline{u^3}\,dx$$
$$= \rho_2\int_0^1 |u^4|^2\,dx - \int_0^1\int_0^\infty g(s)u_x^7\overline{u_x^3}\,ds\,dx + \delta\int_0^1 u^6\overline{u_x^3}\,dx + \rho_2\int_0^1 f^4\overline{u^3}\,dx + \rho_2\int_0^1 u^4\overline{f^3}\,dx. \tag{3.56}$$

On the other hand, multiplying (3.44) by $\int_0^x \overline{u^3(y)}\,dy$ we get

$$i\lambda\rho_1 \int_0^1 u^2\left(\int_0^x \overline{u^3(y)}\,dy\right)dx - k\int_0^1 \left(u_x^1+u^3\right)_x\left(\int_0^x \overline{u^3(y)}\,dy\right)dx = \rho_1\int_0^1 f^2\left(\int_0^x \overline{u^3(y)}\,dy\right)dx. \tag{3.57}$$

Using (3.45), we have

$$k\int_0^1 \left(u_x^1+u^3\right)\overline{u^3}\,dx = \rho_1\int_0^1 u^2\left(\int_0^x \overline{u^4(y)}\,dy\right)dx$$
$$+\rho_1\int_0^1 u^2\left(\int_0^x \overline{f^3(y)}\,dy\right)dx + \rho_1\int_0^1 f^2\left(\int_0^x \overline{u^3(y)}\,dy\right)dx. \tag{3.58}$$

Finally, using (3.58) in (3.56) and using that

$$\operatorname{Re}\left\{\rho_1\int_0^1 f^2\overline{\int_0^x u^4(y)\,dy}\,dx\right\} \leqslant \varepsilon_1\rho_1\left\|u^2\right\|_{L^2}^2 + C_{\varepsilon_1}\rho_2\left\|u^4\right\|_{L^2}^2,$$

taking the real part and using Lemmas 3.1 and 3.2, our conclusion follows. □

Our next step is to estimate the term $\|u_x^1+u^3\|_{L^2}^2$. Here we shall use condition (3.1).

Lemma 3.4. *With the same hypotheses as in Lemma* 3.1, *together with condition* (3.1), *for any* $\varepsilon_2>0$ *there exists* $C_{\varepsilon_2}>0$, *at most depending on* ε_2, *such that*

$$k\int_0^1 \left|u_x^1+u^3\right|^2 dx \leqslant C_{\varepsilon_2}\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + (\varepsilon_1+\varepsilon_2)\rho_1\left\|u^2\right\|_{L^2}^2.$$

Proof. Multiplying (3.46) by $u_x^1+u^3$, we have

$$i\lambda\rho_2\int_0^1 u^4\overline{\left(u_x^1+u^3\right)}\,dx + k\int_0^1 \left|u_x^1+u^3\right|dx + \delta\int_0^1 u_x^6\overline{\left(u_x^1+u^3\right)}\,dx$$
$$+\int_0^1\left(\bar{b}u_x^3+\int_0^\infty g(s)u_x^7\,ds\right)\overline{\left(u_x^1+u^3\right)_x}\,dx = \rho_2\int_0^1 f^4\overline{\left(u_x^1+u^3\right)}\,dx. \tag{3.59}$$

Substituting $(u_x^1+u^3)_x$ given by (3.44) into (3.59), we get

$$\underbrace{i\lambda\rho_2\int_0^1 u^4\overline{u_x^1}\,dx}_{I_1} + \underbrace{i\lambda\rho_2\int_0^1 u^4\overline{u^3}\,dx}_{I_2} - i\lambda\frac{\bar{b}\rho_1}{k}\int_0^1 u_x^3\overline{u^2}\,dx + k\int_0^1\left|u_x^1+u^3\right|^2 dx$$
$$+\delta\int_0^1 u_x^6\overline{\left(u_x^1+u^3\right)}\,dx - \underbrace{i\lambda\frac{\rho_1}{k}\int_0^1\int_0^\infty g(s)u_x^7\overline{u^2}\,ds\,dx}_{I_3} - \frac{\rho_1}{k}\int_0^1\int_0^\infty g(s)u_x^7\overline{f^2}\,ds\,dx - \frac{\bar{b}\rho_1}{k}\int_0^1 u_x^3\overline{f^2}\,dx$$
$$=\rho_2\int_0^1 f^4\overline{\left(u_x^1+u^3\right)}\,dx. \tag{3.60}$$

Substitute u^1 given by (3.43) and u^4 given by (3.45) into (I_1), then

$$I_1 = i\lambda\rho_2\int_0^1 u^3\overline{u_x^2}\,dx - \rho_2\int_0^1 u^4\overline{f_x^1}\,dx + \rho_2\int_0^1 f^3\overline{u_x^2}\,dx. \tag{3.61}$$

Using (3.45) we get

$$I_2 = -\rho_2\int_0^1 |u^4|^2\,dx - \rho_2\int_0^1 u^4\overline{f^3}\,dx. \tag{3.62}$$

Finally, a substitution of u^7 given by (3.49) yields

$$I_3 = \frac{\rho_1}{k}\int_0^1\int_0^\infty g(s)u_{xs}^7\overline{u^2}\,ds\,dx - \frac{\rho_1 b_0}{k}\int_0^1 u_x^4\overline{u^2}\,dx - \frac{\rho_1}{k}\int_0^1\int_0^\infty g(s)f_x^7\overline{u^2}\,ds\,dx.$$

From (3.45) we can rewrite I_3 as

$$I_3 = -\frac{\rho_1}{k}\int_0^1\int_0^\infty g'(s)u_x^7\overline{u^2}\,ds\,dx - i\lambda\frac{\rho_1 b_0}{k}\int_0^1 u_x^3\overline{u^2}\,dx + \frac{\rho_1 b_0}{k}\int_0^1 f_x^3\overline{u^2}\,dx - \frac{\rho_1}{k}\int_0^1\int_0^\infty g(s)f_x^7\overline{u^2}\,ds\,dx. \tag{3.63}$$

Using (3.61)–(3.63) in (3.60), we get

$$\begin{aligned}
i\lambda b\left(\frac{\rho_1}{k} - \frac{\rho_2}{b}\right)\int_0^1 u^3\overline{u_x^2}\,dx + k\int_0^1 |u_x^1 + u^3|^2\,dx &= \rho_2\int_0^1 |u^4|^2\,dx - \delta\int_0^1 u_x^6\overline{(u_x^1 + u^3)}\,dx \\
&\quad + \frac{\rho_1}{k}\int_0^1\int_0^\infty g'(s)u_x^7\overline{u^2}\,ds\,dx + \frac{\rho_1\bar{b}}{k}\int_0^1 u_x^3\overline{f^2}\,dx \\
&\quad + \frac{\rho_1}{k}\int_0^1\int_0^\infty g(s)u_x^7\overline{f^2}\,ds\,dx + \rho_2\int_0^1 f^4\overline{(u_x^1 + u^3)}\,dx \\
&\quad + \rho_2\int_0^1 u^4\overline{f^3}\,dx + \rho_2\int_0^1 u^3\overline{f_x^1}\,dx + \left(\rho_2 - \frac{\rho_1 b_0}{k}\right)\int_0^1 f_x^3\overline{u^2}\,dx \\
&\quad + \frac{\rho_1}{k}\int_0^1\int_0^\infty g(s)f_x^7\overline{u^2}\,ds\,dx.
\end{aligned}$$

Now, using (3.1) and the previous lemmas, our claim follows. □

Lemma 3.5. *There exists $C > 0$ such that*

$$\rho_1\int_0^1 |u^2|^2\,dx \leqslant C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + 4k\|u_x^1 + u^3\|_{L^2}^2.$$

Proof. Multiplying Eq. (3.44) by u^1, we get

$$\underbrace{i\lambda\rho_1\int_0^1 u^2\overline{u^1}\,dx}_{I_4} + k\int_0^1 (u_x^1 + u^3)\overline{u_x^1}\,dx = \rho_1\int_0^1 f^2\overline{u^1}\,dx.$$

Substituting u^1 given by (3.43) into I_4 and taking real parts, we get

$$\rho_1 \int_0^1 |u^2|^2 \, dx \leqslant C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + 2k\|u_x^1 + u^3\|_{L^2}^2 + C\|u_x^3\|_{L^2}^2.$$

Using Lemma 3.3, for ε_1 sufficiently small, our conclusion follows. □

Lemma 3.6. *With the same hypotheses as in Lemma* 3.1, *for any $\varepsilon_3 > 0$ there exists $C_{\varepsilon_3} > 0$, at most depending on ε_3, such that*

$$\frac{\beta}{4} \int_0^1 |u_x^5|^2 \, dx \leqslant C_{\varepsilon_3}\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + \frac{2\delta^2}{\beta}\|u_x^4\|^2 \, dx. \tag{3.64}$$

Proof. Multiplying Eq. (3.48) by u^5, we have

$$i\lambda\rho_3 \int_0^1 u^6\overline{u^5} \, dx - \beta \int_0^1 u_{xx}^6\overline{u^5} \, dx - \beta \int_0^1 u_{xx}^5\overline{u^5} \, dx + \delta \int_0^1 u_x^4\overline{u^5} \, dx = \rho_3 \int_0^1 f^6\overline{u^5} \, dx. \tag{3.65}$$

Substituting u^5 given by (3.43) into (3.64), taking real parts, and using Lemmas 3.1–3.5 and Young's inequality, our conclusion follows. □

Now we are in the position to prove the main result of this section.

Theorem 3.1. *Let us assume hypotheses* (2.1) *and* (2.2) *on g and suppose that condition* (3.1) *holds. Then the heat conduction Timoshenko system is exponentially stable.*

Proof. It remains to show (3.3). Let $U = (u^1, u^2, u^3, u^4, u^5, u^6, u^7)^T$, $F = (f^1, f^2, f^3, f^4, f^5, f^6, f^7)^T$ satisfy (3.43)–(3.49), then, from Lemma 3.1, we get

$$\rho_3\|u^6\|_{L^2}^2 + \|u^7\|_{L_g^2}^2 \leqslant C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}. \tag{3.66}$$

From Lemma 3.2, for $\varepsilon_2 > 0$, there exists $C_1 > 0$ such that

$$\rho_2\|u^4\|_{L^2}^2 \leqslant C_1\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \frac{\bar{b}}{2}\|u_x^3\|_{L^2}^2 + \frac{\varepsilon_2}{2}k\|u_x^1 + u^3\|_{L^2}^2. \tag{3.67}$$

Also, from Lemma 3.3, we obtain

$$\bar{b}\|u_x^3\|_{L^2}^2 \leqslant C_{\varepsilon_1}\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \varepsilon_1\rho_1\|u^2\|_{L^2}^2 + \varepsilon_2 k\|u_x^1 + u^3\|_{L^2}^2. \tag{3.68}$$

On the other hand, from Lemma 3.5, we have

$$k\|u_x^1 + u^3\|_{L^2}^2 \leqslant C_3\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + (\varepsilon_1 + \varepsilon_2)\rho_1\|u^2\|_{L^2}^2. \tag{3.69}$$

From Lemma 3.5, we obtain

$$2(\varepsilon_1 + \varepsilon_2)\rho_1\|u^2\|_{L^2}^2 \leqslant 2(\varepsilon_1 + \varepsilon_2)\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + 8(\varepsilon_1 + \varepsilon_2)k\|u_x^1 + u^3\|_{L^2}^2. \tag{3.70}$$

Adding (3.66) and (3.67), we get

$$(1 - 8\varepsilon_1 + \varepsilon_2)k\|u_x^1 + u^3\|_{L^2}^2 + (\varepsilon_1 + \varepsilon_2)\rho_1\|u^2\|_{L^2}^2 \leqslant C\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}. \tag{3.71}$$

Finally, from Lemma 3.6, we get

$$\frac{\varepsilon_3\beta}{4} \int_0^1 |u_x^5|^2 \, dx \leqslant C_{\varepsilon_3}\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + \frac{2\delta^2\varepsilon_3}{\beta}\|u_x^4\|^2 \, dx. \tag{3.72}$$

From (3.66), (3.67), (3.68), (3.71) and (3.72), we obtain for ε_1, ε_2, ε_3 sufficiently small, that there exists $C > 0$ independent of λ, F, U such that

$$\|U\|_{\mathcal{H}}^2 \leqslant C\|F\|_{\mathcal{H}}^2,$$

this completes the proof. □

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