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Finite Matrix Groups over Nilpotent Group Rings*

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We study groups of matrices $SGL_n(\mathbb{Z}\Gamma)$ of augmentation one over the integral group ring $\mathbb{Z}\Gamma$ of a nilpotent group Γ. We relate the torsion of $SGL_n(\mathbb{Z}\Gamma)$ to the torsion of Γ. We prove that all abelian p-subgroups of $SGL_n(\mathbb{Z}\Gamma)$ can be stably diagonalized. Also, all finite subgroups of $SGL_n(\mathbb{Z}\Gamma)$ can be embedded into the diagonal $\Gamma^n < SGL_n(\mathbb{Z}\Gamma)$. We apply matrix results to show that if Γ is nilpotent-by-(Π'-finite) then all finite Π-groups of normalized units in $\mathbb{Z}\Gamma$ can be embedded into Γ.

0. INTRODUCTION

The group $\mathcal{U}\mathbb{Z}\Gamma$ of invertible elements of an integral group ring $\mathbb{Z}\Gamma$ is both an important and interesting algebraic object. On the one hand, it is a significant invariant of the group ring. On the other hand, it has strong links with algebraic K-theory and hence finds useful applications outside algebra.

It is customary and also convenient to restrict attention to those units $u = \sum u_g g$ which lie in the kernel of the augmentation homomorphism ϵ: $\mathbb{Z}\Gamma \to \mathbb{Z}$, $\epsilon(u) = \sum u_g$. They form a subgroup $\mathcal{U}_1\mathbb{Z}\Gamma$ of index two in $\mathcal{U}\mathbb{Z}\Gamma$ and in fact $\mathcal{U}\mathbb{Z}\Gamma \approx \mathcal{U}_1\mathbb{Z}\Gamma \times \{+1, -1\}$.

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Obviously, all elements of the group Γ sit inside $\mathcal{U}_1\mathbb{Z}\Gamma$. However, if Γ has elements of finite order then, in most cases, there exist other, "nontrivial" units. The relation between torsion of Γ and the structure of $\mathcal{U}_1\mathbb{Z}\Gamma$ is far from being understood. For example, the old conjecture stating that for groups without torsion holds $\mathcal{U}_1\mathbb{Z}\Gamma = \Gamma$ still seems to be out of reach.

Much more can be said about units which are of finite order. For one thing, they cannot appear in group rings of torsion free groups, as follows from

0.1. THEOREM [7, Thm.VI.2.1]. *If $u \in \mathcal{U}_1\mathbb{Z}\Gamma$ is of order p^n then Γ also contains an element of order p^n.*

Hans Zassenhaus suggested the possibility of a very strong correlation between the torsion of $\mathcal{U}_1\mathbb{Z}\Gamma$ and that of Γ. He conjectured that for a finite group Γ any torsion unit u must be of the form $\gamma g \gamma^{-1}$ for some $g \in \Gamma$ and some unit γ from the rational group algebra $\mathbb{Q}\Gamma$. This conjecture has been verified for several classes of finite groups. For nilpotent groups, Weiss was able to prove a stronger version of the Zassenhaus Conjecture:

0.2. THEOREM [11]. *If Γ is a finite nilpotent group then every finite subgroup $U < \mathcal{U}_1\mathbb{Z}\Gamma$ is of the form $\gamma G \gamma^{-1}$ for some subgroup $G < \Gamma$ and some unit $\gamma \in \mathbb{Q}\Gamma$.*

More generally, we may look for torsion elements in the matrix group $GL_n(\mathbb{Z}\Gamma)$. They arise naturally when studying units in $\mathbb{Z}\tilde{\Gamma}$ and $|\tilde{\Gamma} : \Gamma| = n$, via the coset representation. Matrices over group rings also play an important role in studying group automorphisms; see [9, p. 201]. Of course, the torsion coming from the subgroup $GL_n(\mathbb{Z})$ is not "caused" by the group ring. To sort out the *group ring torsion* from the *integral torsion*, one observes that the map induced on matrices by the augmentation of coefficients $\epsilon_*: GL_n(\mathbb{Z}\Gamma) \to GL_n(\mathbb{Z})$ is a split epimorphism. We will study its kernel $SGL_n(\mathbb{Z}\Gamma) = \{X | \epsilon_*(X) = I\}$.

When Γ is a finite p-group then there is a matrix version of Theorem 0.2.

0.3. THEOREM [10]. *Let Γ be a finite p-group. Every finite subgroup $U < SGL_n(\mathbb{Z}\Gamma)$ is conjugate inside $GL_n(\mathbb{Q}\Gamma)$ to a subgroup of the diagonal $\Gamma \times \cdots \times \Gamma < SGL_n(\mathbb{Z}\Gamma)$.*

In this paper, we study finite subgroups of $SGL_n(\mathbb{Z}\Gamma)$ for nilpotent groups Γ. The main result states that any such subgroup has an isomorphic copy inside the diagonal group $\Gamma \times \cdots \times \Gamma < SGL_n(\mathbb{Z}\Gamma)$. The basic tool is a result in K-theory of nilpotent groups (4.1) which implies stable diagonalization of certain periodic matrices. This, together with residual techniques, reduces the problem to finite groups where we apply Weiss' result

(0.2). As a consequence we obtain solutions to Problems 40 and 33 (for p-elements) in [8].

The paper consists of eight sections.

In Section 1, we investigate the torsion of the matrix groups $SGL_n(R\Gamma)$ and of its subgroups. In particular, we prove in (1.8) that for an arbitrary group Γ with a finite normal nilpotent subgroup N, the kernel of the natural homomorphism $GL_n(\mathbb{Z}\Gamma) \to GL_n(\mathbb{Z}[\Gamma/N])$ can have p-torsion only for those primes p which divide the order of N.

In Section 2, we study the torsion in $SGL_n(\mathbb{Z}\Gamma)$ for polycyclic-by-finite groups Γ. We prove in (2.2) that if $SGL_n(\mathbb{Z}\Gamma)$ has p-torsion then the group Γ has p-torsion as well. We conclude in (2.4) that if Γ is nilpotent then all finite subgroups of $SGL_n(\mathbb{Z}\Gamma)$ are also nilpotent.

In Section 3, we consider representations of finite groups over associative algebras. We introduce the notion of stable conjugation of such representations and investigate in (3.4) its relation to the Bass rank map.

In Section 4, we prove a result from K-theory of nilpotent groups which is later applied together with (3.4) to study the torsion in $SGL_n(\mathbb{Z}\Gamma)$ for such groups. It was posed as Problem 40 in [8].

In Section 5, we study cyclic subgroups $C_{p^r} < SGL_n(\mathbb{Z}\Gamma)$. We extend here our result from [2] stating that such subgroups can be stably diagonalized, provided the torsion classes do not fuse in Γ. In (5.1), we prove the same for all nilpotent groups. As a corollary, we obtain a solution of Problem 33 (for p-elements) in [8].

In Section 6, we apply the results from Sections 2 and 5 to investigate finite subgroups of $SGL_n(\mathbb{Z}\Gamma)$ for nilpotent groups Γ. In (6.4), we prove that any such subgroup has an isomorphic copy inside the diagonal subgroup $\Gamma \times \cdots \times \Gamma < SGL_n(\mathbb{Z}\Gamma)$.

It is used in Section 7, to extend the results from Section 5 from cyclic p-groups to arbitrary abelian p-subgroups of $SGL_n(\mathbb{Z}\Gamma)$. We show in (7.2) that they also can be stably diagonalized.

In Section 8, we prove two results about integral units. We show in (8.2) that the integral group ring $\mathbb{Z}\Gamma$ of a finitely generated nilpotent group Γ determines both the torsion subgroup $T(\Gamma)$ and the quotient group $\Gamma/T(\Gamma)$. Finally, we employ the matrices to prove in (8.3) another embedding theorem: if Γ is an extension of a nilpotent group N by any finite Π'-group then all finite Π-subgroups $U < \mathscr{U}_1\mathbb{Z}\Gamma$ can be embedded into N.

1. TORSION IN MATRIX GROUPS OVER GROUP RINGS

Let R be a cummutative domain of characteristic zero. In this section we study the relation between the torsion of a group Γ and the torsion of different subgroups in $SGL_n(R\Gamma)$.

First we introduce a notation: for an ideal I in a ring S, we define

$$I^{\omega} = \bigcap_{i=1}^{\infty} I^{i} \quad \text{and} \quad I^{\omega^{r+1}} = \left(I^{\omega^{r}}\right)^{\omega} \qquad \text{for } r \geq 1.$$

Most results of this section will follow from properties of the augmentation ideal $\Delta_R(\Gamma)$ and the following simple lemma.

1.1. LEMMA. *Suppose that a prime number p is invertible in R. Let S be an R-algebra with an ideal I such that $I^{\omega^r} = (0)$ for some $r \geq 1$. Then*

(i) *The subgroup of units $\mathscr{U}(S) \cap (1 + I)$ has no p-torsion elements.*

(ii) *The kernel of the natural map $GL_n(S) \to GL_n(S/I)$, $n \geq 1$, has no p-torsion.*

Proof. (i) Suppose that for some $x \in I$ we have $(1+x)^p = 1$. Then $1 + \sum_{i=1}^{p} \binom{p}{i} x^i = 1$ and so $-p \cdot x = \sum_{i=2}^{p} \binom{p}{i} x^i$. Consequently, if x belongs to an ideal J then it also belongs to J^2 and hence to J^{ω}. Applying this observation consecutively to the ideals $I, I^{\omega}, I^{\omega^2}, \ldots,$ we obtain at the end that $x \in I^{\omega^r} = (0)$.

(ii) Apply part (i) to the algebra of matrices $M_n(S)$ and its ideal $M_n(I)$. ∎

1.2. PROPOSITION. *Suppose that a group Γ has a normal subgroup G such that for a prime p and an exponent $r \geq 1$ holds $\Delta_{R[1/p]}^{\omega^r}(G) = (0)$. Then the kernel of the natural homomorphism $GL_n(R\Gamma) \to GL_n(R[\Gamma/G])$ has no p-torsion elements.*

Proof. In the commutative diagram

$$\begin{array}{ccc} GL_n(R\Gamma) & \to & GL_n(R[\Gamma/G]) \\ \downarrow & & \downarrow \\ GL_n(R[1/p]\Gamma) & \to & GL_n(R[1/p][\Gamma/G]) \end{array}$$

the vertical arrows are injections and hence we may assume that $R = R[1/p]$.

It is easy to check that for any ideal $J \subseteq RG$ we have $(R\Gamma \cdot J)^{\omega} = R\Gamma \cdot J^{\omega}$ and hence also $(R\Gamma \cdot J)^{\omega^r} = R\Gamma \cdot J^{\omega^r}$. Therefore we can apply Lemma 1.1(ii) with $S = R\Gamma$ and $I = RT \cdot \Delta_R(G)$. ∎

The intersection condition is satisfied for a large class of groups. Let K be the field of fractions of R.

1.3. PROPOSITION. *If a group G has a subnormal series $\langle 1\rangle = G_0 < \cdots < G_k = G$ with all factor groups G_i/G_{i-1} torsion free abelian then $\Delta_K^{\omega^k}(G) = (0)$.*

Proof. We proceed by induction on the length k of the subnormal series. For $k = 0$ our statement is obvious. If $k > 0$ consider the natural map $\pi_*\colon KG \to K[G/G_{k-1}] = KA$. It maps the ideal $\Delta_K^{\omega}(G)$ to $\Delta_K^{\omega}(A)$ which is zero by [3, p. 87]. Therefore, we have $\Delta_K^{\omega}(G) \subseteq \ker(\pi_*) = KG \cdot \Delta_K(G_{k-1})$ and by induction

$$\begin{aligned}\Delta_K^{\omega^k}(G) &= \left(\Delta_K^{\omega}(G)\right)^{\omega^{k-1}} \subseteq \left(KG \cdot \Delta_K(G_{k-1})\right)^{\omega^{k-1}} \\ &= KG \cdot \Delta_K^{\omega^{k-1}}(G_{k-1}) = (0).\end{aligned}$$ ∎

1.4. THEOREM. *If a group Γ has a subnormal series $\langle 1\rangle = G_0 < \cdots < G_k = G \lhd \Gamma$ with the factor groups G_i/G_{i-1}, $1 \le i \le k$, torsion free abelian then the kernel of the natural homomorphism $GL_n(R\Gamma) \to GL_n(R[\Gamma/G])$ is torsion free for all $n \ge 1$.*

Proof. For each prime p, we have $\Delta_{R[1/p]}^{\omega^k}(G) \subseteq \Delta_K^{\omega^k}(G) = (0)$, by Proposition 1.3. From Proposition 1.2, it then follows that our kernel has no p-torsion for all primes p. ∎

In particular, when we put $G = \Gamma$ in Theorem 1.4, we obtain:

1.5. COROLLARY. *If a group Γ has a subnormal series $\langle 1\rangle = \Gamma_0 < \cdots < \Gamma_k = \Gamma$ with all factor groups Γ_i/Γ_{i-1} torsion free abelian then all matrix groups $SGL_n(R\Gamma)$ are torsion free.*

The subgroup $G \lhd \Gamma$ considered in Theorem 1.4 was solvable and torsion free. Now, we consider the kernel of $GL_n(R\Gamma) \to GL_n(R[\Gamma/G])$ in the case when G is a finitely generated nilpotent group, possibly with torsion elements.

1.6. LEMMA. *Let P be a finite p-group. If $\bigcap_{i=1}^{\infty} p^k R = (0)$ then for any prime $q \neq p$ we have $\Delta_{R[1/q]}^{\omega}(P) = 0$.*

Proof. Let us write $S = R[1/q]$. We still have $\bigcap_{i=1}^{\infty} p^i S = (0)$. We know from [4, Lemma 8.1.17] that the image of $\Delta_S(P)$ in $(S/pS)[P]$ is a nilpotent ideal. Therefore, for some exponent $k \ge 1$ holds $\Delta_S^k(P) \subseteq pS[P]$. Then $\Delta_S^{\omega}(P) \subseteq (\bigcap_{i=1}^{\infty} p^i S)[P] = (0)$. ∎

Let Π be a set of prime numbers. As usual, we write Π' for the complementary set of primes. We say that Γ is a Π-group if it has no elements of order $p \in \Pi'$.

1.7. THEOREM. *Let a group Γ have a normal finitely generated nilpotent Π-subgroup N. If $\bigcap_{i=1}^{\infty} p^i R = (0)$ holds for all $p \in \Pi$, then the kernel of the natural homomorphism $\pi_*: GL_n(R\Gamma) \to GL_n(R[\Gamma/N])$ is a Π-group.*

Proof. Suppose that for some $X \in \ker(\pi_*)$ and a prime $q \notin \Pi$ holds $X^q = I$. Let $T(N)$ denote the torsion part of N. We can factor the map π_* as

$$GL_n(R\Gamma) \overset{\alpha_*}{\to} GL_n(R[\Gamma/T(N)]) \overset{\beta_*}{\to} GL_n(R[\Gamma/N]).$$

As $\ker(\beta) \approx N/T(N)$ is a (poly-$\mathbb{Z}$)-group, it follows from Theorem 1.4 that $\ker(\beta_*)$ is torsion free and hence $\alpha_*(X) = I$. Therefore, we can assume that $N = T(N)$.

Now we proceed by induction on the number r of Sylow p-subgroups in N. For $r = 0$, our claim is obviously true. Assume that $r > 0$ and write N as a product of its Sylow p-subgroups: $N = P_1 \times \cdots \times P_r$. Then α_* can be factored as

$$GL_n(R\Gamma) \overset{\gamma_*}{\to} GL_n(R[\Gamma/P_1]) \overset{\delta_*}{\to} GL_n(R[\Gamma/N]).$$

The kernel of $\delta: \Gamma/P_1 \to \Gamma/N$ has $r - 1$ Sylow p-subgroups so, by induction, the kernel of δ_* has no q-torsion and hence $\gamma_*(X) = I$. From Lemma 1.6 and Proposition 1.2, it then follows that $X = I$. ∎

Because the ring $\mathbb{Z}$ of rational integers satisfies $\bigcap_{i=1}^{\infty} p^i \mathbb{Z} = (0)$ for all primes p, we obtain:

1.8. COROLLARY. *Suppose that a group Γ has a finite nilpotent normal subgroup N. The kernel of the natural homomorphism $GL_n(\mathbb{Z}\Gamma) \to GL_n(\mathbb{Z}[\Gamma/N])$ has p-torsion only for the primes p dividing the order of N.*

2. MATRICES OVER POLYCYCLIC-BY-FINITE GROUP RINGS

Now we turn to polycyclic-by-finite groups and investigate the relation between the torsion of $SGL_n(\mathbb{Z}\Gamma)$ and that of Γ. We will use the Bass rank map (see [1]). It is a function defined for any associative R-algebra S by the formula

$$r: \bigcup_{n \in \mathbb{N}} M_n(S) \to S/[S,S], \qquad r(X) = \text{Trace}(X) \bmod [S,S].$$

Here $S/[S,S]$ denotes the R-module S divided by the submodule $[S,S]$ spanned by all elements of the form $ab - ba$, $a, b \in S$. We are going to use repeatedly the following, easy to check, observation: if the algebra S is not commutative then the traces of conjugate matrices do not need to be equal. However, their Bass ranks are equal.

In this section, we take $R = \mathbb{Z}$ and $S = \mathbb{Z}\Gamma$. Then $S/[S, S]$ is a free $\mathbb{Z}$-module $\mathbb{Z}[Cl(\Gamma)]$ spanned by the set $Cl(\Gamma)$ of conjugate classes of Γ.

2.1. LEMMA. *Suppose that a matrix $X \in GL_n(\mathbb{Z}\Gamma)$ satisfies $X^k = I$. If $r(X^i) = n \cdot \{1\}$ holds for $0 < i < k$ then $X = I$.*

Proof. Consider the idempotent matrix

$$E = \frac{1}{k} \sum_{i=0}^{k-1} X^i.$$

Then $r(E) = n \cdot \{1\}$ which, by [1, Cor. 8.10], implies $E = I$. But then $X = XE = E = I$. ∎

2.2. THEOREM. *If a polycyclic-by-finite group Γ has no p-torsion then also the matrix groups $SGL_n(\mathbb{Z}\Gamma)$ have no p-torsion.*

Proof. Suppose that we have a matrix $X \in SGL_n(\mathbb{Z}\Gamma)$ of order p. Let $r(X) = \sum t_{\mathscr{C}}(X) \cdot \mathscr{C}$ be its Bass rank. From [8, Lemma 48.6], we know that if $t_{\mathscr{C}}(X) \neq 0$ then $\mathscr{C}$ is either a conjugacy class of a p-torsion element or $\mathscr{C} = \{1\}$.

If Γ has no p-torsion elements then we must have $r(X) = t_{\{1\}}\mathrm{Tr}(X) \cdot \{1\}$. But then

$$t_{\{1\}}\mathrm{Tr}(X) = \sum_{\mathscr{C}} t_{\mathscr{C}}\mathrm{Tr}(X) = \epsilon(\mathrm{Tr}(X)) = \mathrm{Tr}(\epsilon_*(X)) = \mathrm{Tr}(I) = n.$$

Obviously the same holds for all powers of X. Now Lemma 2.1 concludes the proof. ∎

2.3. COROLLARY. *If a polycyclic-by-finite group Γ has p-torsion only, then every finite subgroup of $SGL_n(\mathbb{Z}\Gamma)$ is a p-group.*

2.4 THEOREM. *If Γ is a nilpotent group then all finite subgroups of $SGL_n(\mathbb{Z}\Gamma)$ are also nilpotent.*

Proof. Let U be a finite subgroup of $SGL_n(\mathbb{Z}\Gamma)$. By restricting to the subgroup generated by the supports of all elements of $\mathbb{Z}\Gamma$ appearing in U, we may assume that Γ is a finitely generated nilpotent group, and hence polycyclic.

For each prime p dividing the order of U let J_p be the kernel of the natural ring homomorphism $\pi_p\colon M_n(\mathbb{Z}\Gamma) \to M_n(\mathbb{Z}[\Gamma/T_p(\Gamma)])$, where $T_p(\Gamma) < \Gamma$ is the subgroup consisting of all elements of p-power order. We prove that $U_p = U \cap (1 + J_p)$ coincides with the set of all elements of U which are of p-power order.

On the one hand, the group $SGL_n(\mathbb{Z}[\Gamma/T_p(\Gamma)])$ has no p-torsion, by Theorem 2.2. Therefore all p-elements of U must be mapped by π_p to the identity matrix; i.e., they belong to U_p. On the other hand, U_p is contained in the kernel of π_p: $GL_n(\mathbb{Z}\Gamma) \to GL_n(\mathbb{Z}[\Gamma/T_p(\Gamma)])$ and hence all its elements must be of p-power order, by Corollary 1.8.

It is easy to see that for any ideal $J \subseteq M_n(\mathbb{Z}\Gamma)$ the intersection $U \cap (1 + J)$ is a normal subgroup of U. Therefore each U_p is a normal Sylow p-subgroup of U. It follows that U is nilpotent. ∎

3. REPRESENTATIONS OF GROUPS OVER ALGEBRAS

Let S be an associative K-algebra with a unit element over a field K of characteristic zero. By a representation of a group G over S, we mean a group homomorphism $G \to GL_n(S)$. For example, we always have the regular representation ρ: $G \to GL_d(K) \subseteq GL_d(S)$, where $d = |G|$.

In the classical representation theory, we take $S = K$. Two representations are considered equivalent if they are conjugate. For more general algebras we usually have $\tilde{K}_0(S) \neq 0$. Taking this into account, we introduce the notion of stable conjugation of group representations over an algebra. Let ϕ, ψ: $G \to GL_n(S)$ be two representations of a finite group G over S.

3.1. DEFINITION. Representations ϕ and ψ are *stably conjugate* if there exists a natural number k and a matrix $Y \in GL_{n+kd}(S)$ such that $(\phi \oplus \rho^k)(g) = Y \cdot (\psi \oplus \rho^k)(g) \cdot Y^{-1}$ for all $g \in G$. We write then $\phi \sim_s \psi$.

3.2. *Remark.* Notice that when the group G is cyclic and the field K contains a primitive dth root of unity then the regular representation ρ can be diagonalized and we obtain the definition of the stable conjugation of matrices which was considered in [2].

Consider the algebra $SG = S \otimes_K KG$. Each representation $G \to GL_n(S)$ determines on S^n an SG-module structure in the usual way. It is obvious from the definition that two representations are stably conjugate iff the corresponding SG-modules are stably isomorphic. To study those modules we use again the rank function r. For any ring with unit R and a finitely generated projective R-module P we have $P \approx ER^n$ for some idempotent matrix $E \in M_n(R)$. The formula $[P] \mapsto r(E)$ defines the well-known natural transformation r_R: $K_0(R) \to HH_0(R) = R/[R, R]$, where $HH_0(-)$ is the 0th Hochschild homology functor.

3.3. LEMMA. *Suppose that G is a finite group and K is its splitting field of characteristic zero. If r_S: $K_0(S) \to S/[S, S]$ is injective then r_{SG}: $K_0(SG) \to SG/[SG, SG]$ is injective.*

Proof. By the Wedderburn Theorem, we have $SG \approx S \otimes KG \approx S \otimes \bigoplus_{i=1}^{h} M_{n_i}(K) \approx \bigoplus_{i=1}^{h} M_{n_i}(S)$. We obtain a commutative diagram of K-vector spaces

$$\begin{array}{ccccc} K_0(SG) & \overset{\approx}{\to} & \bigoplus_{i=1}^{h} K_0\big(M_{n_i}(S)\big) & \overset{\approx}{\to} & \bigoplus_{i=1}^{h} K_0(S) \\ \downarrow r_{SG} & & \downarrow \oplus r_{M_{n_i}(s)} & & \downarrow \oplus r_S \\ HH_0(SG) & \overset{\approx}{\to} & \bigoplus_{i=1}^{h} HH_0\big(M_{n_i}(S)\big) & \overset{\approx}{\to} & \bigoplus_{i=1}^{h} HH_0(S). \end{array}$$

The left-hand square commutes because of the naturality of the rank map. The right-hand square is a direct sum of diagrams arising from

$$\begin{array}{ccc} M_k\big(M_{n_i}(S)\big) & \overset{\approx}{\to} & M_{k\cdot n_i}(S) \\ \downarrow \mathrm{Tr}_{M_{n_i}(S)} & & \downarrow \mathrm{Tr}_S \\ M_{n_i}(S) & \overset{\mathrm{Tr}_S}{\to} & S \end{array}$$

which clearly commute. In the first diagram, the horizontal arrows, as well as the right vertical arrow, are injective. Therefore the left vertical arrow is injective as well. ∎

The rank map is strongly related to stable conjugation. Namely, we have:

3.4. PROPOSITION. *Suppose that the rank map* $r\colon K_0(S) \to S/[S,S]$ *is injective. If* G *is a finite group and* K *is its splitting field then for any representations* $\phi, \psi\colon G \to GL_n(S)$ *the following conditions are equivalent*:

(i) $r \cdot \phi = r \cdot \psi$;

(ii) $\phi \sim_s \psi$.

Proof. (i) ⇒ (ii) Let P and Q be the SG-modules corresponding to ϕ and ψ. Because P and Q are S-projective and G is finite of order invertible in S, the standard Maschke argument shows that they are projective as SG-modules. Hence $[P],[Q] \in K_0(SG)$. From (i) it follows that $r_{SG}[P] = r_{SG}[Q]$. Lemma 3.3 implies that P and Q are stably isomorphic.

(ii) ⇒ (i) Take any element $g \in G$ and consider its class sum $c_g = \Sigma\{h | h$ conjugate to $g\}$. It is a central element of KG. Let t denote the number of elements in the conjugacy class of g. Because the rank function attains the same value on conjugate matrices, we have

$$(r\cdot\phi)(g) = \frac{1}{t}(r\cdot\phi)(c_g) = \frac{1}{t}(r\cdot\psi)(c_g) = (r\cdot\psi)(g).$$ ∎

4. K_0 OF NILPOTENT GROUP ALGEBRAS

We want to apply Proposition 3.4 when S is the group algebra $K\Gamma$ of another group. To do so, we must know whether the rank map $r\colon K_0(K\Gamma) \to S/[S,S] = K[Cl(\Gamma)]$ is injective. In [2], we proved that it is true when K is a field of characteristic zero and Γ is a finitely generated nilpotent group such that the inclusion of its torsion part $T < \Gamma$ does not fuse the conjugacy classes. Here we extend this result to all finitely generated nilpotent groups, thus solving Problem 40 from [8].

4.1. THEOREM. *If Γ is a finitely generated nilpotent group and K is a characteristic zero splitting field for T then the rank map $r\colon K_0(K\Gamma) \to K[Cl(\Gamma)]$ is injective.*

Proof. In [2, Lemma 3], we proved that the inclusion $i: T \subseteq \Gamma$ induces an epimorphism $i_*\colon K_0(KT) \to K_0(K\Gamma)$.

The group Γ acts on T by conjugation. By composing these automorphisms with representations of T over K, we obtain an action of Γ on the set of finitely generated projective KT-modules. It is obvious that irreducible modules form a Γ-invariant subset. Let $\mathrm{Irr}(KT)$ be a finite, Γ-invariant set of irreducible KT-modules which contains exactly one module from each isomorphism class. Then $\mathrm{Irr}(KT)$ forms a basis for the free abelian group $K_0(KT)$. Hence the $\mathbb{Z}$-linear extension of the above Γ-action converts $K_0(KT)$ into a permutation $\mathbb{Z}\Gamma$-module.

If two KT-modules lie in the same Γ-orbit then the $K\Gamma$-modules induced from them are clearly isomorphic. Hence i_* factors to an epimorphism

$$i_*\colon K_0(KT) \otimes_{\mathbb{Z}\Gamma} \mathbb{Z} \to K_0(K\Gamma).$$

Similarly, Γ acts on the set $Cl(T)$ of conjugacy classes in T. This action gives us another permutation $\mathbb{Z}\Gamma$-module $K[Cl(T)]$ together with an obvious homomorphism

$$\hat{i}\colon K[Cl(T)] \otimes_{\mathbb{Z}\Gamma} \mathbb{Z} \to K[Cl(\Gamma)].$$

Moreover, the map $\hat{i}$ is injective because $K[Cl(T)] \otimes_{\mathbb{Z}\Gamma} \mathbb{Z} \approx K[Cl(T)/\Gamma]$ and $\hat{i}$ is just a linear extension of an embedding of bases: $Cl(T)/\Gamma \to Cl(\Gamma)$.

Let $r_T\colon K_0(KT) \to K[Cl(T)]$ be the Bass rank map for the group T. From the definition of r_T it is easy to see that it is a Γ-map. Hence we obtain a commutative diagram

$$\begin{array}{ccc} K_0(KT)\otimes_{\mathbb{Z}\Gamma}\mathbb{Z} & \overset{i_*}{\rightarrow} & K_0(K\Gamma) \\ \downarrow r_T\otimes 1 & & \downarrow r_\Gamma \\ K[Cl(T)]\otimes_{\mathbb{Z}\Gamma}\mathbb{Z} & \overset{\hat{i}}{\rightarrow} & K[Cl(\Gamma)] \end{array}$$

with the bottom horizontal arrow injective and the top horizontal arrow surjective. We need to prove that the right vertical arrow is injective. From the diagram, it follows that it is enough to prove the injectivity of the left vertical arrow.

To this end let us consider the functor $\mathcal{F}$ from the category of $\mathbb{Z}\Gamma$-modules to the category of K-vector spaces, defined by $\mathcal{F}(A)=A\otimes_{\mathbb{Z}\Gamma}K$. We shall consider it as a composition of other functors in two different ways.

On the one hand we can write $\mathcal{F}(A)=(A\otimes_{\mathbb{Z}}K)\otimes_{K\Gamma}K$. From the representation theory of finite groups we know that the map $r_T: K_0(KT)\to K[Cl(T)]$ embeds $K_0(KT)$ as a full lattice in the vector space of class functions $K[Cl(T)]$. Hence the inside functor $-\otimes_{\mathbb{Z}}K$ applied to r_T gives an isomorphism. Therefore the map

$$\mathcal{F}(r_T): K_0(KT)\otimes_{\mathbb{Z}\Gamma}K \overset{\approx}{\rightarrow} K[Cl(T)]\otimes_{\mathbb{Z}\Gamma}K$$

is an isomorphism as well.

On the other hand, the functor $\mathcal{F}$ can be viewed as the composition:

$$\mathcal{F}(A)=(A\otimes_{\mathbb{Z}\Gamma}\mathbb{Z})\otimes_{\mathbb{Z}}K.$$

The outside functor is exact, as K it is a flat $\mathbb{Z}$-module. Therefore

$$\ker\big(r_T\otimes_{\mathbb{Z}\Gamma}\mathrm{id}: K_0(KT)\otimes_{\mathbb{Z}\Gamma}\mathbb{Z}\to K[Cl(T)]\otimes_{\mathbb{Z}\Gamma}\mathbb{Z}\big)\otimes_{\mathbb{Z}}K$$
$$\approx \ker\Big(K_0(KT)\otimes_{\mathbb{Z}\Gamma}K\overset{\approx}{\rightarrow}K[Cl(T)]\otimes_{\mathbb{Z}\Gamma}K\Big)=0.$$

It follows that the finitely generated abelian group

$$\ker\big(r_T\otimes_{\mathbb{Z}\Gamma}\mathrm{id}: K_0(KT)\otimes_{\mathbb{Z}\Gamma}\mathbb{Z}\to K[Cl(T)]\otimes_{\mathbb{Z}\Gamma}\mathbb{Z}\big)$$

is finite. To finish the proof it is enough to notice that $K_0(KT)\otimes_{\mathbb{Z}\Gamma}\mathbb{Z}$ is a free abelian group. In fact: $K_0(KT)\otimes_{\mathbb{Z}\Gamma}\mathbb{Z}\approx\mathbb{Z}[\mathrm{Irr}(KT)]\otimes_{\mathbb{Z}\Gamma}\mathbb{Z}\approx\mathbb{Z}[\mathrm{Irr}(KT)/\Gamma]$. ∎

5. STABLE DIAGONALIZATION OF CYCLIC p-GROUPS

Let Γ be a nilpotent group and let p be a prime number. We have proved in [2, Prop. 17] that for every representation $\phi: C_{p^\alpha}\to SGL_n(\mathbb{Z}\Gamma)$

there exists another representation $\psi\colon C_{p^\alpha} \to SGL_n(\mathbb{Z}\Gamma)$ such that

(i) $\psi(x) = (g_1(x), \ldots, g_n(x))$ with $g_i(x) \in \Gamma$ for each $x \in C_{p^\alpha}$;

(ii) $r \cdot \phi = r \cdot \psi$.

From Theorem 4.1 and Proposition 3.4, we obtain the following extension of [2, Prop. 20].

5.1. THEOREM. *If Γ is a nilpotent group then every matrix $X \in SGL_n(\mathbb{Z}\Gamma)$ of prime power order can be stably diagonalized; i.e., there exist complex roots of unity ξ_i such that the matrices $X \oplus \operatorname{diag}(\xi_1, \ldots, \xi_k)$ and $\operatorname{diag}(g_1, \ldots, g_n, \xi_1, \ldots, \xi_k)$ are conjugate in $GL_{n+k}(\mathbb{C}\Gamma)$.*

We will further extend this result in Theorem 7.1. Now, let us say more about the group elements g_i appearing as "eigenvalues" in the diagonal form.

5.2. PROPOSITION. *Suppose that $X \in SGL_n(\mathbb{Z}\Gamma)$ is a matrix of order p^α and that we have $X \oplus \operatorname{diag}(\xi_1, \ldots, \xi_k) \sim \operatorname{diag}(g_1, \ldots, g_n, \xi_1, \ldots, \xi_k)$ in $GL_{n+k}(\mathbb{C}\Gamma)$. Then $g_i^{p^\alpha} = 1$ holds for all $1 \le i \le n$. Moreover, if all $g_i = 1$ then it must be that $X = I$.*

Proof. Of course $X^{p^\alpha} \oplus \operatorname{diag}(\xi_1^{p^\alpha}, \ldots, \xi_k^{p^\alpha}) \sim \operatorname{diag}(g_1^{p^\alpha}, \ldots, g_n^{p^\alpha}, \xi_1^{p^\alpha} \ldots, \xi_k^{p^\alpha})$ and hence the two matrices have equal Bass ranks. As $X^{p^\alpha} = I$, we get $n + \sum \xi_i^{p^\alpha} \equiv \sum g_i^{p^\alpha} + \sum \xi_i^{p^\alpha} \operatorname{mod}[\mathbb{C}\Gamma, \mathbb{C}\Gamma]$. It follows that $\sum g_i^{p^\alpha} - n \in [\mathbb{C}\Gamma, \mathbb{C}\Gamma] \cap \mathbb{Z}\Gamma = [\mathbb{Z}\Gamma, \mathbb{Z}\Gamma]$ which is possible only if $g_i^{p^\alpha} = 1$ for all $1 \le i \le n$.

Suppose now that all $g_i = 1$. By looking at the ranks again, we get $r(X) = n \cdot \{1\}$. Of course the same holds for all powers of X. Now Lemma 2.1 completes the argument. ∎

In Problem 33 of [8], one considers a finite order matrix $X \in SGL_n(\mathbb{Z}\Gamma)$ and asks whether its matrix trace is "non-negative" in the sense that $\operatorname{Tr}(X) = \sum n_g g$ with $n_g \ge 0$ for all $g \in \Gamma$. From Proposition 5.1 we can easily conclude

5.3. COROLLARY. *If Γ is an arbitrary nilpotent group then any matrix $X \in SGL_n(\mathbb{Z}\Gamma)$ of p-power order has non-negative trace.*

6. DIAGONAL EMBEDDINGS OF FINITE MATRIX GROUPS

We now apply the results from Sections 2 and 5 to study finite subgroups of $SGL_n(\mathbb{Z}\Gamma)$. To do so, we need to recall the residual properties of nilpotent group rings.

6.1. LEMMA. *Let* Γ *be a finitely generated nilpotent group.*

(i) *For any finite subset* $Z \subseteq \Gamma$ *there exists a finite index normal subgroup* $N \lhd \Gamma$ *such that the natural homomorphism* $\pi\colon \Gamma \to \Gamma/N$ *is injective on* Z.

(ii) *If* F *is a finite subset in* $\mathbb{Z}\Gamma$ *then there exists a finite index normal subgroup* $N \lhd \Gamma$ *such that the induced ring homomorphism* $\pi_*\colon \mathbb{Z}\Gamma \to \mathbb{Z}[\Gamma/N]$ *is injective on* F.

Proof. (i) Consider the finite set $Z' = \{xy^{-1} | x, y \in Z,\ x \neq y\} \subseteq \Gamma \backslash \{1\}$. From [6, Thm. 1.4], we know that Γ is residually finite. For each element $z \in Z'$ it is then possible to find a finite index normal subgroup $N_z \lhd \Gamma$ such that $z \notin N_z$. We set $N = \bigcap_{z \in Z'} N_z$ and consider $\pi\colon \Gamma \to \Gamma/N$. For any pair $x, y \in Z$ of different elements, we have $xy^{-1} \notin N$. Consequently $\pi(xy^{-1}) \neq 1$ and hence $\pi(x) \neq \pi(y)$.

(ii) Consider the set $F' = \{x - y | x, y \in F,\ x \neq y\}$. Part (i), applied to the finite set $Z = \bigcup\{\operatorname{supp}(s) | s \in F'\}$, provides us with a subgroup $N \lhd \Gamma$. For any $z \in F'$ the map $\pi_*\colon \mathbb{Z}\Gamma \to \mathbb{Z}[\Gamma/N]$ is injective on the support of z and hence $\pi_*(z) \neq 0$. It follows that for any pair of different elements $x, y \in F$ we have $\pi_*(x - y) \neq 0$ and so $\pi_*(x) \neq \pi_*(y)$. ∎

Let $T_p(\Gamma)$ be the p-torsion part of a nilpotent group Γ. We denote by $D_p(\Gamma)$ the diagonal subgroup $\{\operatorname{diag}(g_1, \dots, g_n) | g_i \in T_p(\Gamma)\} < SGL_n(\mathbb{Z}\Gamma)$. Obviously, $D_p(\Gamma)$ is isomorphic to $T_p(\Gamma) \times \cdots \times T_p(\Gamma)$ (n times).

6.2. LEMMA. *Let* Γ *be a finitely generated nilpotent group. For every finite* p*-subgroup* $U < SGL_n(\mathbb{Z}\Gamma)$ *there exists a finite* p*-group* P *and a map* $\phi\colon \Gamma \to P$ *such that*

(i) ϕ *does not fuse the conjugacy classes of* p*-elements in* Γ;

(ii) *the induced map* $\phi_*\colon SGL_n(\mathbb{Z}\Gamma) \to SGL_n(\mathbb{Z}P)$ *is injective on* $U \cup D_p(\Gamma)$. ∎

Proof. Consider the finite subset $\{u_{k,l} | u \in U,\ 1 \le k, l \le n\} \subseteq \mathbb{Z}\Gamma$ consisting of the entries of all matrices from U. Lemma 6.1 gives us a finite index subgroup $N_1 \lhd \Gamma$ such that the natural map $\mathbb{Z}\Gamma \to \mathbb{Z}[\Gamma/N_1]$ is injective on this set. The same lemma gives us a finite index subgroup $N_2 \lhd \Gamma$ such that the projection $\Gamma \to \Gamma/N_2$ is injective on $T_p(\Gamma)$.

From [6, Prop. 4.1], we know that Γ is conjugacy separable. Therefore, if $\mathscr{C}_1, \dots, \mathscr{C}_k$ is the full list of conjugacy classes of p-elements in Γ then there exists a finite index normal subgroup $N_3 \lhd \Gamma$ such that the natural map $\Gamma \to \Gamma/N_3$ does not fuse them.

Let us take $N = N_1 \cap N_2 \cap N_3$. Then the map $\pi\colon \Gamma \to \Gamma/N$ clearly satisfies conditions (i) and (ii). The finite nilpotent group Γ/N can be written as a product $P \times P'$, where P is a p-group and P' has no p-torsion.

We take $\phi = \alpha\pi$, where $\alpha\colon \Gamma/N \to P$ is the natural projection. It is easy to see that the homomorphism α does not fuse the p-conjugacy classes $\pi(\mathscr{C}_i)$ and hence ϕ satisfies condition (i).

Clearly the map $\phi_*\colon SGL_n(\mathbb{Z}\Gamma) \to SGL_n(\mathbb{Z}P)$ is injective on $D_p(\Gamma)$. Because the kernel of α contains no p-elements, it follows from Corollary 1.8 that $\alpha_*\colon \mathbb{Z}[\Gamma/N] \to \mathbb{Z}P$ is injective on $\pi_* U$. Consequently, ϕ_* is injective on U and hence it satisfies (ii). ∎

6.3. THEOREM. *Let Γ be a nilpotent group. For each finite p-subgroup $U < SGL_n(\mathbb{Z}\Gamma)$ there exists an embedding $\varphi\colon U \to D_p(\Gamma)$ into the diagonal of $SGL_n(\mathbb{Z}\Gamma)$ such that for each $X \in U$ the Bass ranks of X and $\varphi(X)$ coincide.*

Proof. Fix a finite p-subgroup $U < SGL_n(\mathbb{Z}\Gamma)$. Let $i: \Gamma_0 \subseteq \Gamma$ be the subgroup generated by the union of the supports of all entries appearing in the matrices of U. Clearly Γ_0 is a finitely generated group and we have $U < SGL_n(\mathbb{Z}\Gamma_0)$. If there exists an embedding $\varphi_0\colon U \to D_p(\Gamma_0)$ such that $r(X) = r(\varphi(X)) \in \mathbb{Z}[Cl(\Gamma_0)]$ for all $X \in U$, then from the diagram

$$\begin{array}{ccc} SGL_n(\mathbb{Z}\Gamma_0) & \xrightarrow{r} & \mathbb{Z}[Cl(\Gamma_0)] \\ \downarrow i_* & & \downarrow i' \\ SGL_n(\mathbb{Z}\Gamma) & \xrightarrow{r} & \mathbb{Z}[Cl(\Gamma)] \end{array}$$

it is clear that $\varphi = i_* \cdot \varphi_0$ is an embedding with the required properties. Therefore, we may assume that Γ is finitely generated.

Let $\phi\colon \Gamma \to P$ be the map given by Lemma 6.2. We write $\overline{U} = \phi_*(U) \subseteq SGL_n(\mathbb{Z}P)$. By Theorem 0.3, we can conjugate $\overline{U}$ with a unit $\gamma \in GL_n(\mathbb{C}P)$ onto a subgroup $H = \gamma\overline{U}\gamma^{-1}$ of the diagonal $D_p(P) < SGL_n(\mathbb{Z}P)$. We now prove that $H < \phi_* D_p(\Gamma) < D_p(P)$.

Take any $h = \operatorname{diag}(h_1, \ldots, h_n) \in H < D_p(P)$. One can find a matrix $X \in U$ such that $\gamma\phi_*(X)\gamma^{-1} = h$. By Theorem 5.1, the matrix X can be stably diagonalized; i.e., $X \oplus \operatorname{diag}(\xi_1, \ldots, \xi_k) \sim \operatorname{diag}(g_1, \ldots, g_n, \xi_1, \ldots, \xi_k)$ in $GL_{n+k}(\mathbb{C}\Gamma)$. Hence the matrices $\operatorname{diag}(\phi(g_1), \ldots, \phi(g_n)) \oplus \operatorname{diag}(\xi_1, \ldots, \xi_k)$ and $\operatorname{diag}(h_1, \ldots, h_n) \oplus \operatorname{diag}(\xi_1, \ldots, \xi_k)$ are conjugate in $GL_{n+k}(\mathbb{C}P)$. By comparing their ranks, we get

$$\sum_{i=1}^{n} \phi(g_i) - \sum_{i=1}^{n} h_i \in [\mathbb{Z}P, \mathbb{Z}P].$$

It can happen only when each h_i is conjugate in P to some $\phi(g_j) \in \phi(T_p(\Gamma))$. But the subgroup $\phi(T_p(\Gamma))$ is normal in P. Hence $h_i \in \phi(T_p(\Gamma))$ for $1 \le i \le n$ and so $h \in \phi_* D_p(\Gamma)$.

We define φ as the following composition of injections:

$$U \xrightarrow{\phi_*} \overline{U} \xrightarrow{\gamma-\gamma^{-1}} H \xrightarrow{\text{incl}} \phi_* D_p(\Gamma) \xrightarrow{\phi_*^{-1}} D_p(\Gamma).$$

To verify the statement about the Bass ranks, take any matrix $X \in U$. Then we have $\varphi(X) = \text{diag}(g_1, \dots, g_n)$ for some elements $g_i \in T_p(\Gamma)$ such that the matrices $\phi_*(X)$ and $\text{diag}(\phi(g_1), \dots, \phi(g_n))$ are conjugate.

Consider the diagram

$$\begin{array}{ccc} SGL_n(\mathbb{Z}\Gamma) & \xrightarrow{r} & \mathbb{Z}[Cl(\Gamma)] \\ \downarrow \phi_* & & \downarrow \phi' \\ SGL_n(\mathbb{Z}P) & \xrightarrow{\bar{r}} & \mathbb{Z}[Cl(P)]. \end{array}$$

We have $\phi_*(X) \sim \phi_* \text{diag}(g_1, \dots, g_n)$ and hence

$$\bar{r}(\phi_*(X)) = \bar{r}(\phi_* \text{diag}(g_1, \dots, g_n)).$$

Using the other way in the diagram, we conclude that

$$\phi'(r(X)) = \phi'(r(\text{diag}(g_1, \dots, g_n))).$$

Obviously $r(\text{diag}(g_1, \dots, g_n))$ is a linear combination of p-conjugacy classes in Γ. From [8, Lemma 48.6], we know that the same is true about $r(X)$. By construction, the map ϕ' is injective on the span of those classes. Hence we obtain the desired equality

$$r(X) = r(\text{diag}(g_1, \dots, g_n)) = r(\varphi(X)). \qquad \blacksquare$$

For a nilpotent group Γ, let us write $D(\Gamma) = \{\text{diag}(g_1, \dots, g_n) | g_i \in T(\Gamma)\} < SGL_n(\mathbb{Z}\Gamma)$. Clearly $D(\Gamma) \approx \bigoplus_p D_p(\Gamma) \approx T(\Gamma) \times \cdots \times T(\Gamma)$ (n times). As a consequence of Theorem 6.3, we obtain:

6.4. THEOREM. *Let Γ be a nilpotent group. Every finite subgroup $U < SGL_n(\mathbb{Z}\Gamma)$ has an embedding $\varphi: U \to D(\Gamma)$. Moreover, if $X \in U$ is of prime power order then the Bass ranks of X and $\varphi(X)$ coincide.*

Proof. We know from Theorem 2.4 that U is nilpotent and hence it is a product of its Sylow p-subgroups: $U \approx \bigoplus_p U_p$. Theorem 6.3 states that for each prime p we have an embedding $\varphi_p: U_p \to D_p(\Gamma)$. Consequently, there is also an embedding

$$\varphi: U \bigoplus_p U_p \xrightarrow{\oplus \varphi_p} \bigoplus_p D_p(\Gamma) \approx D(\Gamma).$$

Moreover, if $X \in U$ is of p-power order, then $\varphi(X) = \varphi_p(X)$ and hence $r(X) = r(\varphi(X))$ by Theorem 6.3. $\blacksquare$

7. STABLE DIAGONALIZATION OF ABELIAN p-GROUPS

Here we prove the promised extension of Theorem 5.1.

7.1. PROPOSITION. *Let Γ be a nilpotent group and let P be a finite p-group. For each representation ϕ: $P \to SGL_n(\mathbb{Z}\Gamma)$ there exists a diagonal representation ψ: $P \to D_p(\Gamma) < SGL_n(\mathbb{Z}\Gamma)$ such that ϕ and ψ are stably conjugate over $\mathbb{C}\Gamma$.*

Proof. Consider the subgroup $U = \phi(P) < SGL_n(\mathbb{Z}\Gamma)$. It is enough to take $\psi = \varphi\phi$ where φ: $U \to D_p(\Gamma)$ is the embedding provided by Theorem 6.3. We have then $r \cdot \phi = r \cdot \psi$ and hence Proposition 3.4 concludes the proof. ∎

7.2. THEOREM. *If Γ is a nilpotent group then any finite abelian p-group $U < SGL_n(\mathbb{Z}\Gamma)$ can be stably diagonalized over $\mathbb{C}\Gamma$. It means that there exist*:

—*an embedding φ: $U \to D_n(\Gamma)$,*

—*a family of linear characters ξ_i: $U \to \mathbb{C}^*$, $1 \le i \le k$,*

—*a matrix $Y \in GL_{n+k}(\mathbb{C}\Gamma)$,*

such that for all $X \in U$, it holds that

$$Y \cdot \left(X \oplus \operatorname{diag}(\xi_1(X), \dots, \xi_k(X))\right) \cdot Y^{-1} = \varphi(X) \oplus \operatorname{diag}(\xi_1(X), \dots, \xi_k(X)).$$

Proof. It follows from Proposition 7.1 and the observation that over the field $\mathbb{C}$ of complex numbers the regular representation of a finite abelian group can be diagonalized. ∎

8. APPLICATIONS TO UNITS IN $\mathbb{Z}\Gamma$

When we set $n = 1$ in Theorem 6.4, we obtain:

8.1. COROLLARY. *If Γ is a nilpotent group then every finite subgroup $U \subseteq \mathscr{U}_1\mathbb{Z}\Gamma$ is isomorphic to a subgroup of $T(\Gamma)$.*

Now we can partially reconstruct the structure of Γ from the group ring data.

8.2. THEOREM. *Let Γ be a finitely generated nilpotent group. From $\mathbb{Z}\Gamma \approx \mathbb{Z}H$, it follows that H is also a finitely generated nilpotent group and*

(i) $T(\Gamma) \approx T(H)$;

(ii) $\Gamma/T(\Gamma) \approx H/T(H)$.

Proof. The statement that H must be a nilpotent group has been proved in [5]. If Γ is generated by $\{g_i | 1 \leq i \leq k\}$ then H is generated by the union of supports of the images of all g_i's, which is a finite set.

(i) We can assume that the ring isomorphism $\varphi\colon \mathbb{Z}\Gamma \to \mathbb{Z}H$ preserves the augmentation. Then it maps $T(\Gamma)$ to a finite subgroup $U \subseteq \mathscr{U}_1(\mathbb{Z}H)$ which is, by Corollary 8.1, isomorphic to a subgroup of $T(H)$. Hence $T(\Gamma)$ is isomorphic to a subgroup of $T(H)$ and, by symmetry, $T(H)$ is isomorphic to a subgroup of $T(\Gamma)$. Because both groups are finite it follows that they are isomorphic.

(ii) We know that the group $\Gamma/T(\Gamma)$ is torsion free nilpotent and hence orderable [4, Lemma 13.1.6]. Therefore its group ring $\mathbb{Z}[\Gamma/T(\Gamma)]$ is a domain. Let $\Delta(\Gamma, T(\Gamma))$ be the kernel of the natural homomorphism $\pi\colon \mathbb{Z}[\Gamma] \to \mathbb{Z}[\Gamma/T(\Gamma)]$ and let $\varphi(\Delta(\Gamma, T(\Gamma)) = I \subseteq \mathbb{Z}[H/T(H)]$. Then the quotient ring $\mathbb{Z}H/I$ is a domain.

Take any $h \in T(H)$. If $h^n = 1$ then $(h-1)\hat{h} = 0$, where $\hat{h} = 1 + h + \cdots + h^{n-1}$. Moreover, $\hat{h} \notin I$ as $I \subseteq \Delta_Z(H)$ while $\hat{h}$ has augmentation n. Hence $h - 1$ must belong to the ideal I. The elements of the form $h-1$, $h \in T(H)$ generate the ideal $\Delta(H, T(H))$. Hence we have proved that $\Delta(H, T(H)) \subseteq \varphi(\Delta(\Gamma, T(\Gamma))$. By symmetry, we also have $\Delta(\Gamma, T(\Gamma)) \subseteq \varphi^{-1}(\Delta(H, T(H))$. Putting those two inclusions together, we obtain that φ restricts to an isomorphism $\Delta(H, T(H)) \approx \Delta(\Gamma, T(\Gamma))$.

It induces then an isomorphism $\overline{\varphi}\colon \mathbb{Z}[\Gamma/T(\Gamma)] \to \mathbb{Z}[H/T(H)]$. Because both group rings possess trivial units only, $\overline{\varphi}$ establishes an isomorphism between $\Gamma/T(\Gamma)$ and $H/T(H)$. ∎

As an application of our investigations about matrices over $\mathbb{Z}\Gamma$, we obtain an extension of Corollary 8.1 to a larger class of groups.

8.3. THEOREM. *Consider an extension $\Gamma \to \tilde{\Gamma} \to G$ where Γ is a nilpotent group and G is any finite Π'-group. Then every finite Π-subgroup $U < \mathscr{U}_1\mathbb{Z}\tilde{\Gamma}$ can be embedded into $T(\Gamma)$.*

Proof. Let us fix a system of right coset representatives $\{g_1, \dots, g_n\}$ of Γ in $\tilde{\Gamma}$. In this way, we have a fixed basis for $\mathbb{Z}\tilde{\Gamma}$ as a left free $\mathbb{Z}\Gamma$-module. Right multiplication by an element of $\mathbb{Z}\tilde{\Gamma}$ gives us an $\mathbb{Z}\Gamma$-endomorphism and hence a matrix. In this way we obtain a regular coset representation $\rho\colon \mathbb{Z}\tilde{\Gamma} \to M_n(\mathbb{Z}\Gamma)$, $n = |G|$. Moreover, we have a commuting diagram

$$\begin{array}{ccc} \mathbb{Z}\tilde{\Gamma} & \xrightarrow{\rho} & M_n(\mathbb{Z}\Gamma) \\ \downarrow \pi & & \downarrow \epsilon_* \\ \mathbb{Z}G & \xrightarrow{\rho'} & M_n(\mathbb{Z}) \end{array}$$

where $\pi: \mathbb{Z}\tilde{\Gamma} \to \mathbb{Z}G$ is the natural map, $\epsilon: \mathbb{Z}\Gamma \to \mathbb{Z}$ is the augmentation homomorphism and ρ' is the regular representation of G. Because G has no Π-torsion, Theorem 0.1 implies that $\pi(U) = \langle 1 \rangle$. From the diagram, it then follows that ρ embeds U into $SGL_n(\mathbb{Z}\Gamma)$.

Let us consider a single unit $x \in U$. We can write $x = x_\Gamma + x'$ with $x_\Gamma \in \mathbb{Z}\Gamma$ and $\operatorname{supp}(x') \cap \Gamma = \varnothing$. It is easy to see that

$$\operatorname{Tr}(\rho(x)) = \sum_{i=1}^{n} x_\Gamma^{g_i} \in \mathbb{Z}\Gamma.$$

In particular, $t_{\{1\}}\operatorname{Tr}(\rho(x)) = n \cdot t_{\{1\}}(x_\Gamma)$; i.e., it is an integer divisible by n.

Assume now that the unit $x \in U$ is of p-power order and let $X = \rho(x)$. Recall that Theorem 6.4 gives an embedding $\varphi: \rho(U) \to D(\Gamma) < SGL_n(\mathbb{Z}\Gamma)$ with the property that the Bass ranks of X and $\varphi(X)$ coincide. Let $\varphi(X) = (h_1, \dots, h_n) \in D(\Gamma)$. Then

$$t_{\{1\}}\operatorname{Tr}(X) = t_{\{1\}}\operatorname{Tr}\big(\operatorname{diag}(h_1, \dots, h_n)\big) = t_{\{1\}}\left(\sum_{i=1}^{n} h_i\right) = k$$

for some integer $0 \le k \le n$. But we have seen before that k must be divisible by n and hence either $k = 0$ or $k = n$.

Consider now the composition of $\varphi \cdot \rho$ with the projection π_1 on the first coordinate:

$$U \xrightarrow{\rho} \rho(U) \xrightarrow{\varphi} D(\Gamma) \approx T(\Gamma) \times \cdots \times T(\Gamma) \xrightarrow{\pi_1} T(\Gamma).$$

We complete the proof by showing that the composite map is injective. Suppose that it is not. Then we can find in its kernel a unit x of prime order p. Obviously we must have $\varphi\rho(x) = (1, h_2, \dots, h_n) \in D(\Gamma)$. But then $t_{\{1\}}\operatorname{Tr}(\rho(x))$ is positive, and hence it must be equal to n. It can happen only when all $h_i = 1$. But we have proved in Proposition 5.2 that then $\rho(x) = 1$. It implies that $x = 1$, a contradiction. ∎

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