Non-vanishing elements of finite groups

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Let $G$ be a finite group, and let $\text{Irr}(G)$ denote the set of irreducible complex characters of $G$. An element $x$ of $G$ is non-vanishing if, for every $\chi \in \text{Irr}(G)$, we have $\chi(x) \neq 0$. We prove that, if $x$ is a non-vanishing element of $G$ and the order of $x$ is coprime to 6, then $x$ lies in the Fitting subgroup of $G$.

1. Introduction

The concept of non-vanishing element of a finite group $G$ was introduced by M. Isaacs, the second author and T. Wolf in [9]: an element $x \in G$ is non-vanishing if $\chi(x) \neq 0$ for every irreducible complex character $\chi$ of $G$. It is a classical theorem of W. Burnside that every non-linear $\chi \in \text{Irr}(G)$ vanishes on some element of $G$. In other words, looking at the character table of $G$, the rows which do not contain the value 0 are precisely those corresponding to linear characters. Somehow violating the standard duality between characters and conjugacy classes, it is in general not true that the columns...
not containing the value 0 are precisely those corresponding to conjugacy classes of central elements, as there are finite groups having non-central non-vanishing elements. (See [9] for general hypotheses guaranteeing the existence of this type of elements.) In fact, a non-vanishing element of \( G \) can even fail to lie in an abelian normal subgroup of \( G \) (Theorem 5.1 in [9] provides a family of solvable examples). However, the main result of [9] was to prove that the non-vanishing odd order elements of a solvable group \( G \) all lie in a nilpotent normal subgroup of \( G \), i.e. they lie in the Fitting subgroup \( \mathbf{F}(G) \). (It remains an open problem to determine whether the odd order hypothesis is really necessary.) Although the authors in [9] were aware of the existence of many non-solvable groups \( G \) having non-vanishing elements outside \( \mathbf{F}(G) \), now we realize, however, that all these are in fact \( \{2, 3\} \)-elements.

**Theorem A.** Let \( G \) be a finite group, and let \( x \in G \) be non-vanishing. If the order of \( x \) is coprime to 6, then \( x \in \mathbf{F}(G) \).

The groups \( A_6, A_7, A_{11}, A_{13}, 2.A_{13}, M_{22}, 2.M_{22} \) all are examples of groups having non-vanishing elements (all of them of order dividing 6). It is tempting to think that a non-vanishing element \( x \in G \) should lie in the generalized Fitting subgroup \( \mathbf{F}^*(G) \). But this is not true (at least not for even order elements): the group \( G = 2^{11} : M_{24} \) has non-vanishing elements of order 2 and 4 outside \( \mathbf{F}^*(G) = \mathbf{F}(G) \).

2. **Theorem A**

If \( x \in G \) is non-vanishing and \( N \triangleleft G \), it is then clear that \( xN \in G/N \) is also non-vanishing. On the other hand, the hypothesis of being non-vanishing does not behave well when restricted to normal subgroups. This is compensated with the following elementary lemma which uses the character induction formula. Recall that \( I_C(\psi) \) is the stabilizer of the character \( \psi \) of \( M \triangleleft G \). In general, we use the notation of [8].

**Lemma 2.1.** Let \( M \) be a normal subgroup of \( G \) and let \( x \) be a non-vanishing element of \( G \). Then \( x \) fixes an element in every \( G \)-orbit on \( \text{Irr}(M) \). In other words, for every \( \psi \in \text{Irr}(M) \) there exists \( g \in G \) such that \( x \in I_C(\psi^g) \).

**Proof.** This is Lemma 2.3 of [9]. \( \square \)

**Theorem 2.2.** Let \( G \) act faithfully and irreducibly on a finite vector space \( V \). Let \( x \in \mathbf{F}(G) \) fix an element in each orbit of \( G \) on \( V \). Then \( x^2 = 1 \).

**Proof.** This is Theorem 4.2 of [9]. \( \square \)

To prove Theorem A we need a new result on almost simple groups whose proof we defer until the next section.

**Theorem 2.3.** Suppose that \( S \leq G \leq \text{Aut}(S) \), where \( S \) is a nonabelian simple group. Let \( x \) be an element of odd order of \( G \) fixing an element in every \( G \)-orbit on \( \text{Irr}(S) \). Then \( x \in S \).

There are several remarks concerning this theorem. First of all, the hypothesis of \( o(x) \) being odd is necessary: a counterexample is \( S = \Omega^+_8(2) \) with \( x \in \text{Aut}(S) \setminus S \) having order 2 in \( \text{Out}(S) \). But, again breaking the symmetry between conjugacy classes and characters, we stress that the corresponding result holds when we replace the action of \( G \) on the set \( \text{Irr}(S) \) with that on the set \( \text{Cl}(S) \) of conjugacy classes of \( S \). In fact, the proof of Theorem C of [3] can be adapted to prove that if \( S \leq G \leq \text{Aut}(S) \) and \( x \in G \) is such that for every \( y \in S \) there is a \( g \in G \) such that \( x \) fixes (setwise) the conjugacy class of \( y^g \) in \( S \), then \( x \in S \). Therefore, \( S = \Omega^+_8(2) \) is an example where the actions of \( \text{Aut}(S) \) on \( \text{Irr}(S) \) and \( \text{Cl}(S) \) are not permutation isomorphic.

Using Theorem 2.3 we can now prove Theorem A, which was stated in the Introduction.
Proof of Theorem A. Working by induction on $|G|$, we have that $xN \in \mathbf{F}(G/N)$ for every nontrivial normal subgroup $N$ of $G$.

Assume that $M_1, M_2$ are minimal normal subgroups of $G$, with $M_1 \neq M_2$. Then $M_1 \cap M_2 = 1$ and the function $\varphi : G \to \hat{G} = G/M_1 \times G/M_2$, defined by $\varphi(g) = (gM_1, gM_2)$ for $g \in G$, is an injective homomorphism. Now,

$$\varphi(x) \in \mathbf{F}(G/M_1) \times \mathbf{F}(G/M_2) = \mathbf{F}(\hat{G})$$

and then $\varphi(x) \in \varphi(G) \cap \mathbf{F}(\hat{G}) \subseteq \mathbf{F}(\varphi(G))$. Since $\varphi$ induces an isomorphism between $G$ and $\varphi(G)$, we see that $x \in \mathbf{F}(G)$.

We can hence assume that $G$ has an unique minimal normal subgroup $M$.

Assume first that $M$ is abelian. Observe that we can also suppose that the Frattini subgroup $\Phi(G)$ of $G$ is trivial, because $\mathbf{F}(G/\Phi(G)) = \mathbf{F}(G)/\Phi(G)$. So, by Lemma 4.4 of [7, III], the abelian normal subgroup $M$ has a complement $H$ in $G$. Observing that $C_H(M)$ is normal in $G$, it hence follows $C_H(M) = 1$ and then $C_G(M) = M$.

Let now $V$ be the group of the irreducible characters of $M$. Then $V$ is a faithful and irreducible $G/M$-module. Moreover, by Lemma 2.1 the element $xM$ fixes some element of each orbit of $G/M$ on $V$. Recalling that $xM \in \mathbf{F}(G/M)$, by Theorem 2.2, it follows that $x^2 \in M$ and, as $x$ is an element of odd order, we conclude that $x \in M \subseteq \mathbf{F}(G)$.

Let us consider the case when $M$ is nonabelian. To finish the proof, we shall show that $x = 1$.

Write $M = S_1 \times S_2 \times \cdots \times S_n$, where $S_i = S_i^{\bar{g}_i}$ for suitable $g_i \in G$, $1 \leq i \leq n$ (set $g_1 = 1$), and $S_1$ is a nonabelian simple group. Let

$$K = \bigcap_{i=1}^{n} N_G(S_i)$$

be the kernel of the permutation action of $G$ on the set $\Omega = \{S_1, S_2, \ldots, S_n\}$. So, $M \leq K \leq G$. Let now $L/K$ be the 2-complement of $\mathbf{F}(G/K)$. By induction, $xK \in L/K$. Recall that by [4] (or [12]), there exists a subset $\Delta$ of $\Omega$ such that the (setwise) stabilizer of $\Delta$ in $L/K$ is trivial. We can assume that $\Delta = \{S_1, \ldots, S_m\}$, for some $m < n$. Let $\theta_1$ be a non-principal irreducible character of $S_1$ and let $\theta_i = \theta_1^{S_i}$ be the corresponding characters of $S_i$, $i = 2, \ldots, m$ (recall that $\theta_1^{S_i}$ is defined by $\theta_1^{S_i}(x_1^{\bar{s}_i}) = \theta_1(s_1)$, for all $s_1$ in $S_1$). Consider the irreducible character of $M$

$$\psi = \theta_1 \times \cdots \times \theta_m \times 1_{S_{m+1}} \times \cdots \times 1_{S_n}.$$ 

By Lemma 2.1, there exists some $g \in G$ such that $x \in I_G(\psi^g)$. So, $y = x^{g^{-1}} \in I_G(\psi)$ and then $yK$ stabilizes the subset $\Delta$. Since $yK \in L/K$ as $L/K \triangleleft G/K$, by the choice of $\Delta$ it follows that $y \in K$ and hence that $x \in K$.

We shall next prove that $x \in M$. As the first step, we show that $x$ lies in $S_1 C_G(S_1)$ for all $i \in \{1, \ldots, n\}$. Without loss of generality, we show this for $i = 1$. Let $\theta_1$ be in $\text{Irr}(S_1)$, and let $\psi = \theta_1 \times \cdots \times \theta_n$, where $\theta_i = \theta_1^{S_i}$. By Lemma 2.1, we have that $x$ fixes $\psi^g$ for some $g \in G$. Write

$$S_i^{g^{-1}} = S_{\sigma(i)} = S_1^{\bar{g}_{\sigma(i)}}.$$ 

Hence

$$S_1^{\bar{g}_{\sigma(i)} g} = S_i.$$ 

Then

$$\psi^g = \theta_1^{\bar{g}_{\sigma(1)} g} \times \cdots \times \theta_1^{\bar{g}_{\sigma(n)} g}.$$
(This is easily seen by evaluating both sides on an arbitrary element of \( S_i \), for all \( i \in \{1, \ldots, n\} \).) Since \( x \in K \) fixes \( \psi^G \), we have that it fixes each of the factors of \( \psi^G \). Hence

\[
\theta_1^G(1)x^G = \theta_1^G(1)\]

and therefore \( \theta_1^{G_u^{-1}} = \theta_1 \), where \( u = g_{\sigma(1)}g \in N_G(S_1) \). We are now in a position to apply Theorem 2.3 with \( N_G(S_1)/C_G(S_1) \) in place of \( G \), and \( S_1C_G(S_1)/C_G(S_1) \) in place of \( S \), to conclude that \( x \) lies in \( S_1C_G(S_1) \).

Now, for all \( i \in \{1, \ldots, n\} \), write \( x = s_1c_i \) with \( s_1 \in S_i \) and \( c_i \in C_G(S_i) \). On the other hand, we can certainly write \( x = s_1s_2 \cdots s_n \cdot y \) for some \( y \in G \), and we work to show that \( y = 1 \). We get \( s_1c_i = s_1(s_2 \cdots s_n) \cdot y \), whence \( y = c_1(s_2 \cdots s_n)^{-1} \in C_G(S_1) \). Similarly, we see that \( y \) lies in \( C_G(S_1) \) also for every \( i \in \{2, \ldots, n\} \), and therefore \( y \) is in \( C_G(M) = 1 \).

We conclude that \( x = s_1 \cdots s_n \) lies in \( M \).

Assume now, working by contradiction, that \( x \neq 1 \). Since \( x \) is a \([2, 3]^-\)-element, there exists a prime divisor \( p \geq 5 \) of the order of \( x \). So, there exists a character \( \theta_1 \in \text{Irr}(S_1) \) of \( p\)-defect zero (see [6, Corollary 1]). Let \( \theta_1 = \theta_1^G \in \text{Irr}(S_1) \), and consider \( \psi = \theta_1 \times \cdots \times \theta_n \in \text{Irr}(M) \). Observe that \( \psi \) is a character of \( p\)-defect zero of \( M \). Let now \( x \in \text{Irr}(G) \) be a constituent of the induced character \( \psi^G \). By Frobenius reciprocity and Clifford’s theorem, we see that \( x_M \) is sum of characters \( \psi_j = \psi^G_j \), for suitable elements \( y_j \in G \). Since all the \( \psi_j \) are characters of \( p\)-defect zero of \( M \) and \( x \in M \) is an element of order multiple of \( p \), by a classical result of R. Brauer [8, (8.17)] we have that all the characters \( \psi_j \) vanish on \( x \). We conclude that \( x = 0 \), against the assumption on \( x \). This final contradiction yields \( x = 1 \), and the proof is complete. \( \square \)

3. Almost simple groups

The aim of this section is to prove Theorem 2.3:

Proof of Theorem 2.3. We will assume that \( x \notin S \) and aim to produce a \( G \)-orbit \( O \) on \( \text{Irr}(S) \) such that \( x \) moves every character in \( O \). Consider the subgroup \( J := \langle xS \rangle \) in \( A := G/S \leq \text{Out}(S) \).

1) First we show that the theorem holds in the case \( J < A \). Indeed, by Theorem C of [3], we have that in the action of \( J \) on the conjugacy classes of \( S \) there is some orbit of length \( > 1 \). Since \( J \) is cyclic, this action of \( J \) is permutation isomorphic to its action on \( \text{Irr}(S) \). In particular, \( J \) has an orbit \( O_1 \) of length \( > 1 \) on \( \text{Irr}(S) \). Now let \( O \) be the \( G \)-orbit on \( \text{Irr}(S) \) that contains \( O_1 \). Since \( J < A \), \( J \) acts semi-transitively on \( O \), i.e. all \( J \)-orbits on \( O \) have the same length. Hence we are done as \( |O_1| > 1 \).

2) The structure of the outer automorphism group \( \text{Out}(S) \) is described for instance in [5]. By the result of 1), we are done if \( \text{Out}(S) \) is abelian. Thus we are left with the cases, where \( S = \text{PSL}_n(q) \) with \( n \geq 3 \), \( P\Omega_2^+(q) \) with odd \( q \) and \( n \geq 4 \), or \( E_6^F(q) \). Here, \( q = p^j \), and \( \epsilon = + \) in the untwisted case and \( \epsilon = - \) in the twisted case.

Next we consider the case where, modulo the inner-diagonal and field automorphisms of \( S \), \( x \) induces a graph automorphism of order \( t > 1 \). Since \( \sigma(x) \) is odd, this implies that \( t = 3 \) and \( S = P\Omega_2^+(q) \). In this case, [11, Theorem 2.5] explicitly describes two subsets of \( \text{Irr}(S) \), each containing three irreducible unipotent characters of \( S \) such that they are permuted cyclically by graph automorphisms of order \( 3 \) of \( S \), but every diagonal or field automorphism of \( S \) acts trivially on each of these two sets. Now we can just choose \( O \) to be any of these two sets.

3) Here we consider the case where \( x \) induces an inner-diagonal automorphism of \( S \): \( xS \in I := \text{Outdiag}(S) \) in the notation of [5]. Thus \( x \) belongs to \( O_2^+(I) \). Notice that \( I \) is either cyclic, or elementary abelian of order \( 4 \); in particular, \( O_2^+(I) \) is cyclic. It follows that \( J \text{char} O_2^+(I) \text{char} I \leq \text{Out}(S) \) and so we are done again.

4) Now we may assume that, modulo \( \text{Inndiag}(S) \), \( x \) induces a field automorphism \( \sigma \) of prime order \( t > 2 \). We can find a simple, simply connected, algebraic group \( G \) in characteristic \( p \) and a Frobenius endomorphism \( F \) on \( G \) such that \( S = L/Z(L) \) for \( L := G^F \). We will also consider the pair \((G^*, F^*)\)
where the latter case occurs only when $nf$ divides $\ell$, and so $x$ is irreducible and trivial at $Z(L)$, hence can be viewed as an irreducible character $\chi \in \operatorname{Irr}(S)$. Notice that, in the cases under consideration, the inner-diagonal automorphisms of $S$ are induced by conjugation using elements in $H$ (when we embed $S$ in $H$), and so they preserve $s^H$; also, we may write $H = \text{Inn} \operatorname{diag}(S)$.

As a result, $I$ fixes $\chi$, cf. [14, §2]. Since $\sigma$ moves $s^H$, [14, Corollary 2.4] and the disjointness of Lusztig series imply that $\chi^\sigma \neq \chi$ and so $\chi^\sigma \neq \chi$.

Now let $O$ be the $G$-orbit of $\chi$. Observe that, in our cases, $\text{Out}(S)/I$ is either abelian, or $C_f \times S_3$, where the latter case occurs only when $S = P\Omega^+_8(q)$ (and $q = p^f$). In either case, since $x$ induces the field automorphism $\sigma$ modulo $H$, we see that $(xH)$ is a normal subgroup of $\text{Out}(S)/I$, and so $(x(G \cap H)) < G/(G \cap H)$. Recall that $G \cap H$ fixes $\chi$. Now arguing as in [1], we see that $x$ moves every character in $O$, and so we are done.

The rest of the proof is to construct the desired element $s$. This construction will follow some arguments given in [13]. In what follows, once the prime $\ell$ is chosen, we will fix $x \in F^*_q$ of order $\ell$.

5) Let $S = \text{PSL}_n(q)$ with $n \geq 3$. Then $H = \text{PGL}_n(q)$. We may assume $q > 2$ as otherwise $\text{Out}(S)$ is abelian and we are done. Hence, by [15] there is a primitive prime divisor (p.p.d.) for short $\ell$ of $p^2f - 1$, that is, a prime divisor of $p^2f - 1$ which does not divide $\prod_{j=1}^{nf} (p^j - 1)$. Next, choose $s \in G_L(q)$ represented by the diagonal matrix $\operatorname{diag}(\alpha, \alpha^q, \ldots, \alpha^{q^{nf-1}})$ over $F_q$. Abusing the notation, we will denote the image of $s$ in $H$ also by $s$ (and we will do the same in subsequent parts of the proof). Notice that $\ell \geq nf + 1$, and so $C_{G^o}(s)$ is connected and $s \in [H, H]$ (as $o(s)$ is coprime to $|Z(G)|$ and $|H/[H, H]|$). It remains to show that $s$ and $s^{\sigma}$ are not conjugate in $H$. We may assume that $s^{\sigma}$ is represented by the diagonal matrix $\operatorname{diag}(\alpha^r, \alpha_r^q, \ldots, \alpha_r^{q^{nf-1}})$ over $F_q$, with $r := p^2f/\ell$. Hence it suffices to show that there is no $\lambda \in F_q^*$ and $0 \leq j \leq n - 1$ such that $\alpha^r = \lambda \alpha^{q^j}$.

Now assume that $n \geq 4$ is even. Since $\text{Out}(\text{PSU}_4(2))$ is abelian, we may assume that $(n, q) \neq (4, 2)$, whence there exists a p.p.d. $\ell$ of $p^{2(n-1)f} - 1$. Next, choose $s \in G_U_n(q)$ represented by the matrix $\operatorname{diag}(1, \alpha, \alpha^{-q}, \alpha^q, \ldots, \alpha^{q^{n-2}})$ over $F_q$. Then $\ell \geq (n + 3)f + 1$, and so $C_{G^o}(s)$ is connected and $s \in [H, H]$. Arguing as above, we see that $s$ and $s^{\sigma}$ are not conjugate in $H$.

7) Assume $S = P\Omega^+_{2n}(q)$ and $n \geq 4$. Here we choose $\ell$ to be a p.p.d. of $p^{2nf} - 1$. Next, choose $s \in G_{O^+_{2n}}(q)$ represented by the matrix $\operatorname{diag}(\alpha, \alpha^q, \alpha^{q^2}, \ldots, \alpha^{q^{2n-1}})$ over $F_q$. Since $\ell$ is odd, $C_{G^o}(s)$ is connected and $s \in [H, H]$. It remains to show that $s$ and $s^{\sigma}$ are not conjugate in $H$. Here, if $q$ is odd, we cannot choose $\gamma$ to be a nonzero non-square element in $F_q^{1/\ell}$ (whence $\gamma$ is also a non-square in $F_q^*$ as $t$ is odd), and define $G_{O^+_{2n}}(q)$ as the group of linear transformations of $F_{q^{1/\ell}}$ that preserve the quadratic form $\sum_{i=1}^{n-1}(x_i^2 - y_i^2) + (x_n^2 - y_n^2)$. If $2|q$, we choose $0 \neq \gamma \in F_q^{1/\ell}$ such that the polynomial $\nu^2 + \nu + \gamma$ is irreducible in $F_q^{1/\ell}[v]$ (and so in $F_q[v]$ as $t$ is odd), and define $G_{O^+_{2n}}(q)$ as the group of linear transformations of $F_{q^{1/\ell}}$ that preserve the quadratic form $\sum_{i=1}^{n-1} x_i y_i + (x_n^2 + \gamma y_n^2)$. Then we can define $\sigma$ as induced by the field automorphism $\lambda \mapsto \lambda^{t}$, and so $s^{\sigma}$ is represented by the diagonal matrix $\operatorname{diag}(\alpha^r, \alpha_r^q, \ldots, \alpha_r^{2q^{nf}-1})$ over $F_q$, with $r := p^{2f/\ell}$. Hence it suffices to show that there exists no $\lambda \in F_q^*$ and $0 \leq i \leq 2n - 1$ such that $\alpha^r = \lambda \alpha^{q^j}$.
Now assume that $S = P \Omega_{2n}^+(q)$ with $n \geq 4$. Since $\text{Out}(\Omega_{2n}^+(2)) \cong S_3$ consists only of graph automorphisms, we may assume that $(n, q) \neq (4, 2)$. Hence there exists a p.p.d. $\ell$ of $p^{2(n-1)f} - 1$. Next, choose $s \in G \Omega_{2n}^+(q)$ represented by the matrix $\text{diag}(1, 1, \alpha, \alpha^q, \alpha^{q^2}, \ldots, \alpha^{q^{2n-3}})$ over $\mathbb{F}_q$. Again, $C_{G^*}(s)$ is connected and $s \in [H, H]$. Arguing as above, we see that $s$ and $s^\sigma$ are not conjugate in $H$.

8) Finally, we consider the case $S = E_6^*(q)$. It is easy to see that $J < A$ in the cases where $o(x)$ is coprime to 3 or $(3, q - 1) = 1$, so we are done in these cases. Assume $3 | (q - 1)$. In the notation of [1], we have $H = E_6^*(q)_{ad}$ and $S = [H, H]$. Next, the proof of [3, Theorem 3.1] yields a maximal torus $T$ of $H$ (of order $(q^8 - q^4 + 1)(q^2 + q + 1)$, such that $T \cap S = \langle s \rangle$ is cyclic and $C_H(s) = T = C_{\text{Aut}(S)}(s)$. Now assume that $s^\sigma$ is conjugate to $s$ in $H$: $s^\sigma = s^h$ for some $h \in H$. Then $h^{-1} \sigma \in C_{\text{Aut}(S)}(s) = T < H$ and so $\sigma \in H$, a contradiction.

To complete the proof, we need to show that $C_{G^*}(s)$ is connected. Under our hypotheses, $Z := Z(\mathcal{G}) = \langle z \rangle$ has order 3, $G^* = \mathcal{G}/Z$, and $F$ acts trivially on $Z$. Abusing the notation, we will identify $s$ with an inverse image of it in $L = \mathcal{G}^F$. Furthermore, we can find an $F$-stable maximal torus $T \ni s$ of $\mathcal{G}$ such that $T = (T/Z)^F$. Then $C_{G^*}(s) \supseteq T$. Since $\mathcal{G}$ is simply connected, $C_{G^*}(s)$ is connected. Moreover, $C_{G^*(s)}(T/Z) \leq T \cap S$ consists only of semisimple elements. It follows that $C_{G^*}(s) = T$. Assume that there is some $x \in \mathcal{G}$ such that $xsx^{-1} = zs$. Then

$$xsx^{-1} = zs = F(zs) = F(xsx^{-1}) = F(x)sF(x)^{-1}$$

and so $x^{-1}F(x) \in C_{G^*}(s)$. Since $C_{G^*}(s)$ is connected and $F$-stable, by the Lang–Steinberg Theorem, there is some $c \in C_{G^*}(s)$ such that $xc^{-1}F(x) = c^{-1}F(c)$. Setting $y := xc^{-1}$, we see that $ysy^{-1} = xsx^{-1} = zs$ and $F(y) = y$. Thus $yz \in C_{L/Z}(s) = C_L(s) = T \cap S = \langle s \rangle$ and so $y \in \langle s, z \rangle$. But both $s$ and $z$ centralize $s$, so we obtain $ysy^{-1} = s$, a contradiction. We have shown that $C_{G^*(s)} = C_{G^*}(s)/Z = T/Z$, whence $C_{G^*}(s)$ is connected, as stated. □

References